1

What Makes the World Tick?

1.1. Motion

We see motion all around us. Leaves fall; waves break; heavenly bodies move.

What causes motion?

The answer is interaction. Interaction makes the world tick.

If there were no interactions, bodies would stand still, or move with unchanging velocity. Any change requires force, and that means interaction. Newton's law, the foundation of classical mechanics, states

$$\mathbf{F} = m\mathbf{a}\,.$$

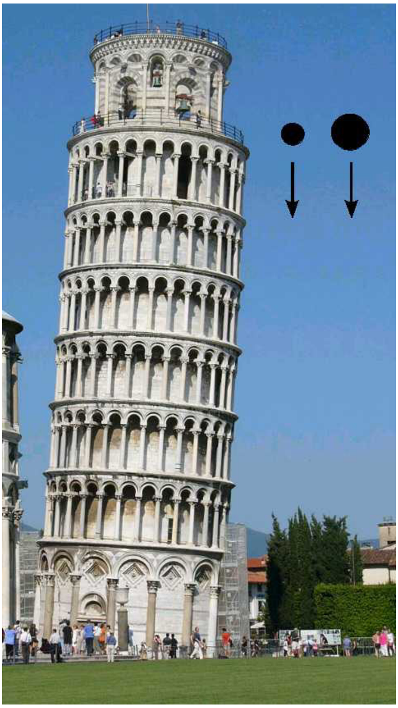
Here, $\mathbf{F}$ is the force acting on a body, m is the inertial mass of the body, and $\mathbf{a}$ is its acceleration — the rate of change of the velocity. We can use this equation in two ways:

- as definition of force;
- as equation of motion.

In the first instance, we obtain the force $\mathbf{F}(\mathbf{x})$ by measuring the acceleration of the body at position $\mathbf{x}$. The force can be represented by a table of data, or by a force law we deduce from the data.

When the force is given, Newton's equation takes the form of a differential equation that can be solved, either analytically using calculus, or through numerical integration on a computer:

$$\ddot{\mathbf{x}} = \frac{\mathbf{F}(\mathbf{x})}{m}\,.$$



Galileo Galilei
(1564–1642)

Fig. 1.1 Galileo dropped two balls from the top of the leaning tower of Pisa, one light, the other heavy. They hit the ground simultaneously, showing that the acceleration due to gravity is independent of mass.

An overhead dot denotes time derivative. Thus, $\dot{\mathbf{x}}$ denotes velocity, and $\ddot{\mathbf{x}}$ is acceleration. Time has entered the picture, and the equation describes dynamical evolution.

1.2. Gravitation

The earliest known interaction is gravity. As legend has it, Galileo dropped two balls from the top of the Leaning Tower of Pisa, one heavy, the other light. They hit the ground simultaneously, showing that the acceleration due to gravity is independent of the mass. That

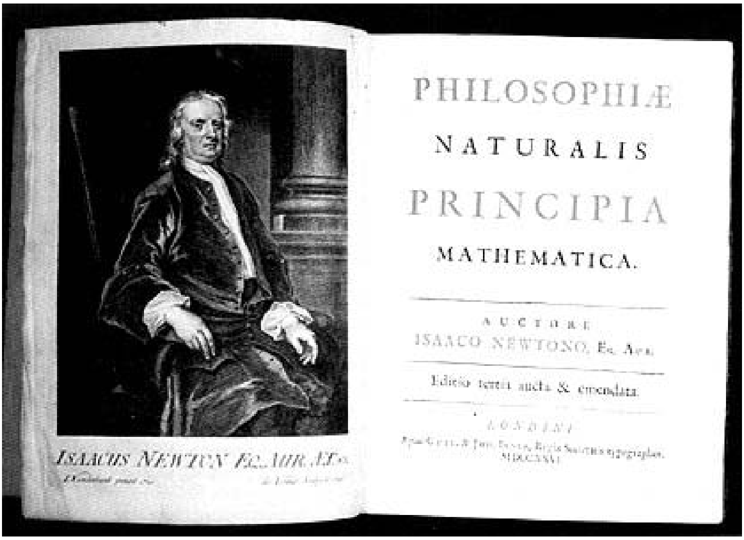


Fig. 1.2 Isaac Newton (1643–1727) laid the foundation of theoretical physics in his *Principia Mathematica* (1637).

is, $a = g$.[1] Newton's law then identifies mg as the force of gravity acting on a body. When this is substituted into the second form, the mass m cancels, and we get $\ddot{\mathbf{x}} = g$. We can use this equation to calculate the path of a projectile, such as a golf ball.

The force due to gravity is approximately constant only near the surface of Earth. When you leave the surface, the force decreases inversely as the square of the distance from the center of Earth.

Newton's law of universal gravitation gives the force of attraction between any two bodies:

$$\text{Gravitational force} = \frac{\gamma m m'}{r^2},$$

where r is the distance between their centers, m and m' are their respective masses, and γ is the *gravitational constant*.[2]

[1]The constant g is called "acceleration due to gravity", or simply but misleadingly "g-force". Its value is 9.8 m s^{-2}, or 32 ft s^{-2}.
[2]The value of the gravitational constant is $\gamma = 6.670 \times 10^{-11}$ m^3 kg^{-1} s^{-2}.

The Earth's pull on a person can be obtained by putting

$$m = \text{Earth's mass},$$
$$m' = \text{Person's mass},$$
$$r = \text{Distance between person and center of Earth.}$$

Thus, r is very nearly the radius of Earth, even for a high jumper; whence the approximate constancy of the acceleration of gravity:

$$g = \frac{\gamma m}{R^2},$$

where R is Earth's radius.

The same inverse-square law gives the force between Jupiter and Mars, the force acting on a comet by the Sun, and indeed on any two masses in the universe. This is why it is called *universal* gravitation.

1.3. The force field

A mass m exerts a gravitational force on any other mass, proportional to the latter's mass. The force per unit mass is called the gravitational field:

$$\text{Gravitational field} = \frac{\gamma m}{r^2}.$$

Any other mass at a distance r from it will feel a force equal to this field times its mass.

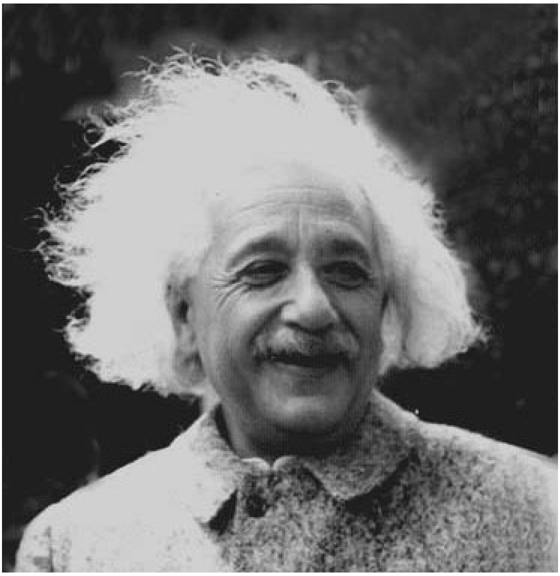
In a sense the mass alters the property of space, for it creates a force field permeating all space. The *field* is to become a central concept in modern physics.

1.4. Equivalence principle

The mass m appears both as a measure of inertia, and a measure of field strength. These two roles are conceptually distinct, and we should really denote them with different symbols:

- The inertial mass m_{inertia} is the quantity appearing in

$$F = m_{\text{inertia}}\, a\,.$$



Albert Einstein (1879–1955)

Fig. 1.3 Some three hundred years after Gallileo's Pisa experiment, Einstein explained it in terms of the geometry of space-time, in his theory of general relativity.

It measures the body's response to an external force.

- The gravitational mass m_{grav} appears in $\gamma m_{\text{grav}}/r^2$, and measures the field strength it produces.

Experimentally, they have the same numerical value:

$$m_{\text{inertia}} = m_{\text{grav}} \,.$$

This is known as the *equivalence principle*, and appears to be accidental.

Einstein could not accept the accidental explanation. He held that the two masses can be considered equivalent only when their defining concepts are shown to be equivalent. In 1917, nearly three hundred years after Galileo's experiment, he turned the accident into an imperative through the theory of general relativity.

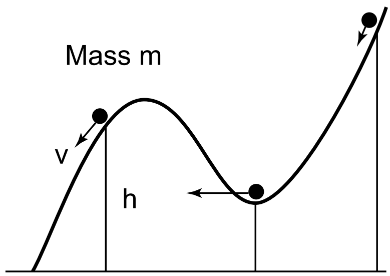


Fig. 1.4 In a roller coaster, kinetic energy $\frac{1}{2}mv^2$ and potential energy mgh convert into each other during the ride, but their sum remains constant.

In Einstein's general relativity, mass generates curvature in space-time. A body in its neighborhood simply rolls along a groove in curved space, following the shortest possible path (a geodesic). Thus, the mass has no bearing on motion in a gravitational field.

The actual curvature of space-time is very slight, and discernible only over cosmic distances. At relatively small scales, such as in the solar system, or even in galaxies, ordinary Netwonian mechanics is quite adequate.

1.5. Energy

A body has more "motion" when it goes faster, and a measure of the vigor is the kinetic energy

$$\text{Kinetic energy} = \frac{1}{2}mv^2,$$

where v is the velocity. When the body moves in a force field, the velocity changes from point to point.

For example, a roller coaster moves under gravity, at varying heights constrained by the track. The velocity is small near the top, and large near the bottom, as illustrated in Fig. 1.4.

We can define a potential energy mgh, where h is the height above ground. When added to the kinetic energy, we obtain a constant total energy, when friction is neglected:

$$\text{Total energy} = \text{Kinetic energy} + \text{Potential energy}.$$

This relation is known as the conservation of energy. The speedup and slowdown of a roller coaster signifies the conversion of potential energy to kinetic energy and *vice versa*.

1.6. Momentum

Momentum is defined as mass times velocity:

$$\text{Momentum} = m\mathbf{v}\,.$$

Newton's law says force is the rate of change of momentum. Thus, the momentum remains constant in the absence of force. This underlies the intuitive notion that momentum is what keep things on the move.

If a system is composed of more than one body, then each body has an individual momentum, and their sum is called the total momentum:

$$\text{Total momentum} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \cdots\,.$$

When there is no overall external force acting on the system, the sum of the internal forces must be zero, and the total momentum is conserved. If two particles collide in free space, their individual momenta will suffer changes, but the sum of the momenta must be the same before and after the collision.

1.7. Least action

The magic formula $F = ma$ explains the classical world.

Why is it true?

To properly pose the question, consider the motion depicted schematically in Fig. 1.5. The solid line represents a particle's actual path, which is governed by Newton's equation. The dotted lines represent other "virtual" paths with the same endpoints. How does

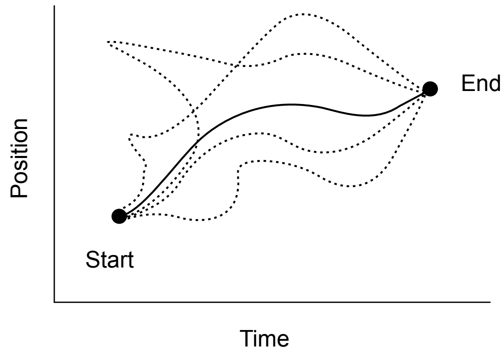


Fig. 1.5 A particle travels between two fixed endpoints. The solid curve is the correct path dictated by Newton's equation. It is singled out of all "virtual" paths (dotted curves) as one with the least "action".

the particle pick the correct path from the infinite number of virtual paths?

Joseph-Louis Lagrange answered this question with the *principle of least action*,[3] as follows. First, consider the quantity now known as the "Lagrangian":

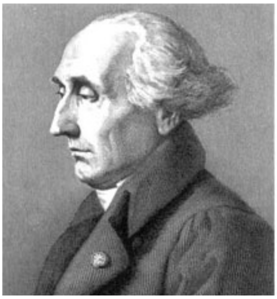
$$\text{Lagrangian} \; = \text{Kinetic energy} - \text{Potential energy.}$$

We can calculate it along any virtual path. The "action" of the path is the Lagrangian accumulated over the entire path:

$$\text{Action of path} = \int_{\text{Path}} dt \text{ Lagrangian.}$$

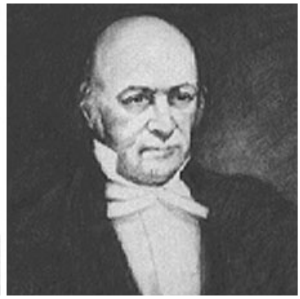
As we vary the path, the corresponding action changes. The correct path is that which minimizes the action.[4]

[3]The principle of least action had been proposed in various forms by Pierre Fermat (1601–1665), Pierre-Louis Moreau de Maupertuis (1698–1759), and Leonhard Euler (1707–1783).
[4]Actually, the sign of the action is immaterial, and the action could be maximal instead of minimal. For this reason purists prefer the name "principle of stationary action".



Joseph-Louis Lagrange (1732–1813)

Pierre-Simon Laplace (1749–1827)



William Rowan Hamilton (1805–1865)

Fig. 1.6 Unlocking the power and beauty of Newtonian mechanics.

An early philosophical underpinning of the principle came from Laozi[5]:

Least action achieves all actions.

1.8. Newton canonized

Lagrange paved the way for William Hamiltonian, who based his approach on what we now call the "Hamiltonian":

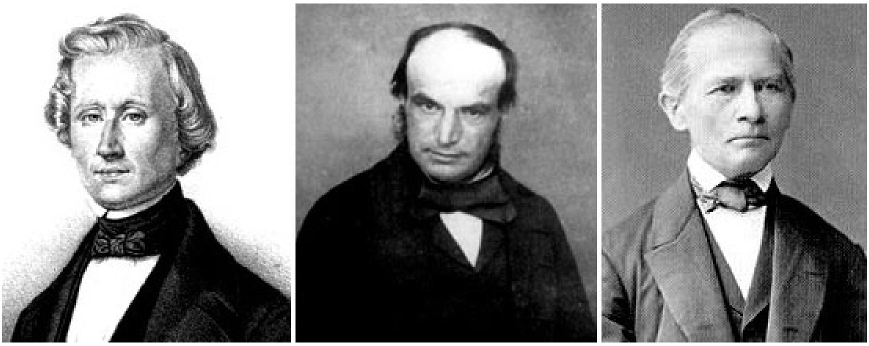
$$\text{Hamiltonian} = \text{Kinetic energy} + \text{Potential energy}.$$

Its value is none other than the total energy, but the formalism requires that the Hamiltonian be expressed in terms of "canonical variables" — the coordinate q and its "canonically conjugate" momentum p. Accordingly we write it as $H(p,q)$. Newton's law is now recast in Hamilton's *canonical equations*:

$$\dot{q} = \frac{\partial}{\partial p} H(p,q),$$

$$\dot{p} = -\frac{\partial}{\partial q} H(p,q).$$

[5] 老子 道德经 (*Dao De Jing*, ca. 500 B.C.): "无为而无不为."



Urbain Le Verrier (1811–1877) | John Couch Adams (1819–1892) | Johann G. Galle (1812–1910)

Fig. 1.7 Truimph of Newtonian mechanics: prediction and discovery of the planet Neptune.

The Lagrangian and Hamiltonian formulations are equivalent. The most succinct way to specify a system is to give its Lagrangian or Hamiltonian.

1.9. The mechanical universe

The correctness of Newtonian mechanics had been confirmed over and over in celestial mechanics, through the effort of Pierre Simon Laplace and others. The crowning moment was surely the prediction and discovery of a heretofore unknown planet — Neptune. Its existence was deduced independently by Urbain Le Verrier and John Couch Adams, from perturbations in the orbit of Uranus. A letter from Le Verrier containing the predicted planet's coordinates reached Johann Galle on September 23, 1846. The same evening, Galle wangled observation time on the Berlin telescope. Pointing it to the predicted position, he found Neptune.

The understanding of the the universe seemed complete. Laplace said that, given the positions and velocities of all the stars at any one instant, he will be able to calculate, in principle, the history of the universe for all times. The ability to quantitatively understand natural phenomena led to profound philosophical shifts.

The following exchange reportedly took place during a meeting of Laplace and Lagrange with Napoleon Bonaparte (1769–1821)[6]:

Napoleon: How is it that, although you say so much about the Universe, you say nothing about its Creator?
Laplace: No, Sire, I had no need of that hypothesis.
Lagrange: Ah, but it is such a good hypothesis: it explains so many things!
Laplace: Indeed, Sire, Monsieur Lagrange has, with his usual sagacity, put his finger on the precise difficulty with the hypothesis: it explains everything, but predicts nothing.

Laplace may think that he was able to predict everything; but his was a mechanical universe. An essential ingredient of the real universe was not yet considered: electromagnetism.

[6] A. De Morgan, *Budget of Paradoxes* (Longmans, Green, London, 1872).