

Chapter 1

The concept of null curves

In this chapter we review on semi-Riemannian manifolds, covariant and exterior derivatives, followed by the concept of a null curve C in a semi-Riemannian manifold (M, g) , where g is its non-degenerate metric tensor of a constant index. We show that there exist three types of quasi-orthonormal basis along C .

1.1 Smooth manifolds

Let V be a real n -dimensional vector space with a symmetric bilinear mapping $g : V \times V \rightarrow \mathbf{R}$. We say that g is positive (negative) definite on V if $g(v, v) > 0$ ($g(v, v) < 0$) for any non-zero $v \in V$. On the other hand, if $g(v, v) \geq 0$ ($g(v, v) \leq 0$) for any $v \in V$ and there exists a non-zero $u \in V$ with $g(u, u) = 0$, we say that g is positive (negative) semi-definite on V .

Let $B = \{u_1, \dots, u_n\}$ be an arbitrary basis of V . Then, g can be expressed by an $n \times n$ symmetric matrix $G = (g_{ij})$, where

$$g_{ij} = g(u_i, u_j), \quad (1 \leq i, j \leq n).$$

G is called the *associated matrix* of g with respect to the basis B . We assume that $\text{rank } G = n \iff g$ is non-degenerate on V . The non-degenerate g on V is called a *semi-Euclidean metric (scalar product)*. Then (V, g) is called a *semi-Euclidean vector space*. For a semi-Euclidean $V \neq \{0\}$, there exists an orthonormal basis $E = \{e_1, \dots, e_n\}$ such that

$$g(v, v) = - \sum_{i=1}^q (v^i)^2 + \sum_{a=q+1}^{q+p} (v^a)^2, \quad (1.1.1)$$

where $q + p = n$ and (v^i) are the coordinate components of v with respect to E . Thus, with respect to (1.1.1), G is a diagonal matrix of canonical form:

$$\text{diag}(- \dots - + \dots +). \quad (1.1.2)$$

The sum of these diagonal elements (also called the *trace* of the canonical form) is called the *signature* of g and the number of negative signs in (1.1.2) is called the *index* of V . Throughout this book, we set the form of the signature of g as given by (1.1.2), unless otherwise stated. Also, we denote by \mathbf{R}_q^n a semi-Euclidean space of constant index $q > 0$ and by \mathbf{R}^n a Euclidean space.

Smooth manifolds. Given a set M , a *topology* on M is a family \mathcal{T} of subsets of M such that

1. the empty set \emptyset and M are in \mathcal{T} ,
2. the intersection of any two members of \mathcal{T} is in \mathcal{T} ,
3. the union of an arbitrary collection of members of \mathcal{T} is in \mathcal{T} .

In the above case, (M, \mathcal{T}) is called a *topological space* whose elements are the open sets of \mathcal{T} . As M depends on the choice of \mathcal{T} , M can have many topologies. In the sequel, we assume that M is a topological space with a given \mathcal{T} . M is a *Hausdorff topological space* if for every $p, q \in M$, $p \neq q$, there exist non intersecting neighborhoods \mathcal{U}_1 and \mathcal{U}_2 respectively. A neighborhood of p in M is an open set that contains p . A system of open sets of \mathcal{T} is called a *basis* if its every open set is a union of the set of the system.

Definition 1.1. An n -dimensional manifold M is a topological Hausdorff space whose each point has a neighborhood homeomorphic to an open set in \mathbf{R}^n .

A manifold is simply understood to be a set M with the property that each point of M can serve as the origin of local coordinates valid in an open neighborhood which is homeomorphic to an open set in \mathbf{R}^n .

The Hausdorff condition is not necessary, although is assumed most often. The open neighborhood of each point admits a *coordinate system* which determines the position of points and the topology of that neighborhood. For a smooth transformation of two such coordinate systems and also taking care of the intersecting neighborhoods, we need the concept of *differentiable manifolds* as follows:

A homeomorphism $\varphi : M \rightarrow \mathbf{R}^n$, mapping an open set \mathcal{U} of M onto an open set $\varphi(\mathcal{U})$ of \mathbf{R}^n , is called a *chart*. By assigning to each point x in \mathcal{U} the n local coordinates x^1, \dots, x^n , we call \mathcal{U} a local coordinate neighborhood. Let x be the point of the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ of two local coordinate neighborhoods \mathcal{U}_1 and \mathcal{U}_2 with respect to charts φ_1 and φ_2 . We say that φ_1 and φ_2 are C^k -compatible if $\mathcal{U}_1 \cap \mathcal{U}_2$ is non-empty and $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) \rightarrow \varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$ and its inverse are C^k .

Definition 1.2. An n -dimensional differentiable manifold M is a set of points together with a family $\mathcal{A} = \{\mathcal{U}_\alpha, \varphi_\alpha\}$ of local coordinate neighborhoods such that

1. The union of \mathcal{U}_α 's is M ,
2. Any two charts of M are C^k -compatible,
3. Any two members of \mathcal{A} are C^k -compatible.

The family \mathcal{A} is called a maximal atlas on M . M is called a *smooth manifold* if M is C^∞ . An atlas $\mathcal{A} = \{\mathcal{U}_\alpha, \phi_\alpha\}$ of M is said to be locally finite if for each p in M , there is a local coordinate neighborhood \mathcal{U} which intersects with only finitely many out of \mathcal{U}_α 's. Another atlas $\mathcal{B} = \{\mathcal{V}_\beta, \psi_\beta\}$ of M is called a refinement of the atlas \mathcal{A} , if each \mathcal{V}_β is contained in some \mathcal{U}_α . M is *paracompact* if for every atlas \mathcal{A} there is a locally finite refined atlas \mathcal{B} of \mathcal{A} . In this book all manifolds will be assumed smooth and paracompact.

A trivial example of a manifold is \mathbf{R}^n . Other examples are 2-sphere \mathbf{S}^2 , cylinder, tori, and Minkowski spacetime. To illustrate this, consider \mathbf{S}^2 in \mathbf{R}^3 , with coordinates (y^i) , centered at $(0, 0, 0)$ having radius a . We need only two charts, with respect to the rectangular coordinates (x, y) , as follows:

$$\begin{aligned} y^1 &= \frac{2a^2x}{x^2 + y^2 + a^2}, & y^2 &= \frac{2a^2y}{x^2 + y^2 + a^2}, \\ y^3 &= \epsilon a \frac{x^2 + y^2 - a^2}{x^2 + y^2 + a^2}, & (\epsilon &= \pm 1). \end{aligned}$$

Note that the usual spherical coordinates (ϕ, θ) fail to give a one-one mapping at the poles, where θ is undetermined. It is left as an exercise to show that \mathbf{S}^2 is a smooth manifold in \mathbf{R}^3 . Similarly, one can show that \mathbf{S}^n in \mathbf{R}^{n+1} is a smooth manifold. See at the end of this chapter some exercises on manifolds.

Let $F(x)$ denote the set of all real-valued smooth functions defined on some neighborhood of a point x of M . A *tangent vector* of M at x is a linear mapping $X_x : F(x) \rightarrow \mathbf{R}$. The set of all tangent vectors, at x , forms a vector space, called the *tangent space* denoted by $T_x M$, where $\dim(T_x M) = \dim(M) = n$, such that n vectors $\{\partial_{x^1}, \dots, \partial_{x^n}\}$ form the basis (called a coordinate basis) of $T_x(M)$ with respect to a local coordinate system (x^i) , $i = 1, \dots, n$, on a neighborhood \mathcal{U} of x . Let $f \in F(x)$. Then, the derivative $X_x f$ is given by

$$X_x f = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(x) X_x(x^i) = X_x(x^i) \left(\frac{\partial}{\partial x^i} \right) f.$$

Therefore, it follows that $X_x = X^i(x)(\partial_{x^i})_x$, where $X^i(x) = X_x(x^i)$ are the components of X_x . The *tangent bundle* of M is defined by

$$TM = \cup_{x \in M} T_x M, \quad (1.1.3)$$

which is a real $2n$ -dimensional smooth manifold. A *vector bundle* is a smooth manifold M together with a vector space attached at each point of M . A *vector field* on M is called a *smooth section* of its respective vector bundle. For example a 2-sphere \mathbf{S}^2 in \mathbf{R}^3 together with the collection of its normals is a vector bundle. Another example is the tangent bundle, defined by (1.1.3) since it has a collection of all the tangent vector spaces $T_x M$ at each x of M . The local coordinates on any vector bundle are given by a pair $(x^i; y^a)$, where (x^i) and (y^a) are the local coordinates of M and its attached vector space respectively. We denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth

sections of a vector bundle E (same notation for any other vector bundle) over M .

The *Lie bracket* $[X, Y]$ of any two vector fields X and Y is given by

$$[X, Y]f = X(Yf) - Y(Xf), \quad \forall f \in F(M). \quad (1.1.4)$$

Using the local expressions $X = X^i \partial_i$ and $Y = Y^j \partial_j$, it is easy to show that

$$[X, Y] = (X^i \partial_i Y^j - Y^i \partial_i X^j) \partial_j$$

is also a vector field on M , with the following properties:

1. $[aX + bY, Z] = a[X, Z] + b[Y, Z]$; $a, b \in R$, (bilinear)
2. $[X, Y] = -[Y, X]$, (skew-symmetry)
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, (Jacobi identity)
4. $[fX, hY] = fh[X, Y] + f(Xh)Y - h(Yf)X$, $\forall f, h \in F(M)$.

It follows that the set of all tangent vector fields on M , denoted by $\Gamma(M)$, has a *Lie-algebra* structure with respect to the Lie-bracket operation.

Every $T_p(M)$ has a *dual vector space* or *cotangent space* $T_p^*(M)$ of the same dimension and its elements, called *differential 1-forms*, are linear maps

$$\omega : T_x(M) \rightarrow \mathbf{R}.$$

Let $f \in F(M)$ and $X \in \Gamma(M)$. We know from above that $X(f)$ is a derivative on $F(M)$ and conversely, every derivative on $F(M)$ comes from some vector field on M . In other words, $X(f) = df(X)$ where d is the symbol for ordinary derivative. Relating this with ω , we conclude that

$$\omega = df : T_x(M) \rightarrow \mathbf{R}, \quad \omega = \omega_i dx^i,$$

where the differentials dx^i form a *dual basis* of $T_x^*(M)$, satisfying $(dx^i)(\frac{\partial}{\partial x^j}) = \delta_j^i$ with respect to a coordinate system (x^i) .

A *linear connection* on M is a map $\nabla : \Gamma(M) \times \Gamma(M) \rightarrow \Gamma(M)$ such that

$$\begin{aligned} \nabla_{fX+hY}Z &= f(\nabla_X Z) + h(\nabla_Y Z), & \nabla_X f &= Xf, \\ \nabla_X(fY + hZ) &= f\nabla_X Y + h\nabla_X Z + (Xf)Y + (Xh)Z, \end{aligned}$$

for arbitrary vector fields X, Y, Z and smooth functions f, h on M . ∇_X is called *covariant derivative operator* and $\nabla_X Y$ is called *covariant derivative* of Y with respect to X . Define a tensor field ∇Y , of type $(1, 1)$, and given by $(\nabla Y)(X) = \nabla_X Y$, for any X . Also, $\nabla_X f = Xf$ is the covariant derivative of f long X . The covariant derivative of a 1-form ω is given by

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y). \quad (1.1.5)$$

For local expressions, we consider the natural basis $\{\partial_i\}$, $i \in \{1, \dots, n\}$, on a coordinate neighborhood \mathcal{U} and set $\nabla_{\partial_j}\partial_i = \Gamma_{ji}^k\partial_k$, where Γ_{ji}^k are n^3 local components of ∇ on M . For $X = X^i\partial_i$, $Y = Y^j\partial_j$ and $\omega = \omega_i dx^i$ we have

$$\nabla_X f = X^i\partial_i f, \quad \nabla_X Y = Y^i_{;k} X^k\partial_i, \tag{1.1.6}$$

$$Y^i_{;k} = \partial_k Y^i + \Gamma_{kj}^i Y^j, \quad \omega_{i;j} = \partial_j \omega_i - \Gamma_{ji}^k \omega_k, \tag{1.1.7}$$

where $;$ is a symbol for the covariant derivative. A vector field Y on M is said to be parallel with respect to a linear connection ∇ if for any vector field X on M it is covariant constant, i.e., $\nabla_X Y = 0$. With respect to a natural basis $\{\partial_i\}$, we say that Y is parallel on M if and only if its local components Y^i , satisfy the following differential equation:

$$\partial_j Y^i + \Gamma_{kj}^i Y^k = 0, \tag{1.1.8}$$

where Γ_{kj}^i are the connection coefficients of ∇ on M . More details on the above material may be seen, for example, in Kobayashi and Nomizu [75].

1.2 Semi-Riemannian manifolds

Let M be a real n -dimensional smooth manifold with a type $(0, 2)$ non-degenerate symmetric tensor field g . Thus g assigns smoothly, to each point x of M , a symmetric bilinear form g_x on the tangent space $T_x M$. Suppose g_x is non-degenerate on $T_x M$ and of constant index q at all points $x \in M$ so that $T_x M$ becomes an n -dimensional semi-Euclidean space. Then, (M, g) is called a *semi-Riemannian manifold* endowed with a *semi-Riemannian metric* g . In general, there are three categories of vectors fields, namely,

- Spacelike** if $g(X, X) > 0$ or $X = 0$,
- Timelike** if $g(X, X) < 0$,
- Lightlike** if $g(X, X) = 0$ and $X \neq 0$.

In particular, M is *Riemannian* or *Lorentzian* according as $q = 0$ or 1 respectively. In case, $0 < q < n$, then, M is called a *proper semi-Riemannian manifold*. In general, the type into which a given vector field X falls is called the *causal character* of X . A simple way of understanding causal character of curves is as follows:

A null curve C in M has a causal structure if all its tangent vectors are null (lightlike); similarly for non-null curves. In case of Lorentz manifolds, we refer [10] for an important roll of the causal character of spacetime manifolds in relativity.

It is well-known that, although Riemannian metrics always exist on a paracompact manifold, but, in general, the existence of non-degenerate metrics can not be assured. This is why we assume that each semi-Riemannian manifold is paracompact, with a non-degenerate metric tensor field. Indeed, let (M, g) be a real n -dimensional smooth manifold with a symmetric tensor field g of type $(0, 2)$. Thus g assigns smoothly, to each point x of M , a symmetric bilinear form g_x on the tangent space $T_x M$. Suppose g_x is non-degenerate on $T_x M$ and the index of g_x (see

equation (1.1.2)) is constant for all $x \in M$ so that $T_x M$ becomes an n -dimensional semi-Euclidean space. The tensor field g satisfying the above conditions is called a *semi-Riemannian metric* or a *metric tensor field* and (M, g) is called a *semi-Riemannian manifold*.

Observe that Riemannian metrics always exist on a paracompact manifold M . In fact, suppose $\{\mathcal{U}_\alpha, \phi_\alpha\}_{\alpha \in I}$ is a smooth atlas of M such that $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ is a local finite open cover of M . Consider g_α as a Riemannian metric on \mathcal{U}_α given by

$$g_\alpha(x)(u, v) = \sum_{i=1}^m u^i v^i,$$

with respect to the natural frames field $\{\partial_{x^i}\}$. Then $g = \sum_{\alpha \in I} f_\alpha g_\alpha$ is the desired Riemannian metric where $\{f_\alpha\}_{\alpha \in I}$ is the partition of unity subordinated to the covering $\{\mathcal{U}_\alpha, \phi_\alpha\}_{\alpha \in I}$. The proof of this result is based on the positive definiteness of g_α and thus does not hold, in general, for a non-degenerate metric. However, the existence of a Riemannian metric g on M and a unit vector field E_o on M enables one to construct a Lorentz metric on M (see O'Neill [82, page 148]). In fact, consider the associate 1-form ω_o to E_o with respect to g , that is,

$$\omega_o(X) = g(X, E_o), \quad \forall X \in \Gamma(TM),$$

and define \bar{g} by

$$\bar{g}(X, Y) = g(X, Y) - 2\omega_o(X)\omega_o(Y), \quad \forall X, Y \in \Gamma(TM).$$

Then, it is easy to check that \bar{g} is a Lorentz metric on M .

The following result is very important for the study of null subspaces.

Proposition ([82]). *Let g be a proper semi-Euclidean metric on an n -dimensional vector space V of index q . Then, there exists a subspace \bar{W} of V of dimension $\min\{q, n - q\}$ and no larger, such that $g|_{\bar{W}} = 0$.*

Proof. Let $E = \{e_1, \dots, e_n\}$ be an orthonormal basis of V . Define g by

$$g(x, y) = - \sum_{i=1}^q x^i y^i + \sum_{a=p+1}^n x^a y^a, \quad \forall x, y \in \mathbf{R}_q^n,$$

where (x^i) and (y^i) are the coordinates of x and y . Suppose $2q < n$. Now define a q -dimensional subspace

$$\bar{W} = \text{Span}\{u_1 = e_1 + e_{q+1}, \dots, u_q = e_q + e_2\}.$$

It follows that $g|_{\bar{W}} = 0$. Choose a null vector $N = \sum_{i=1}^n N^i e_i$ such that $g(N, u_a) = 0, \forall a \in \{1, \dots, q\}$. Thus, $N^1 = N^{q+1}, \dots, N^q = N^{2q}$. Since $\|N\| = 0$ and $\{e_1, \dots, e_{2q}\}$ and $\{e_{q+1}, \dots, e_n\}$ are timelike and spacelike respectively, we conclude that $N^{2q+1} = \dots = N^n = 0$. Hence, $N = \sum_{a=1}^q N^a u_a$.

Thus, there is no subspace larger than \bar{W} on which g vanishes. Similarly, for $2q \geq n$.

See chapter 7 (section 1) for some more information on curvature tensors of a semi-Riemannian manifold which we need for the study of lightlike hypersurfaces. The reader may find more information on semi-Riemannian manifolds in [28, chapter 2] or any other standard book on differential geometry.

1.3 Introduction to null curves

Let C be a smooth curve immersed in an $(m + 2)$ -dimensional proper semi-Riemannian manifold $(M = M_q^{m+2}, g)$ of a constant index $q \geq 1$. Then, with respect to a local coordinate neighborhood \mathcal{U} on C and a local parameter t , C is given by

$$x^i = x^i(t), \quad i \in \{0, \dots, m+1\}, \quad \text{rank}(dx_t^0 \dots dx_t^{m+1}) = 1, \quad \forall t \in I,$$

where I is an open interval of a real line and we denote each $\frac{dx^i}{dt}$ by dx_t^i . The non-zero tangent vector field on \mathcal{U} is given by

$$\frac{d}{dt} \equiv (dx_t^0, \dots, dx_t^{m+1}).$$

In particular, the curve C is called a *regular curve* if $\frac{dC}{dt} \neq 0$ holds everywhere. A non-null regular curve C can be parameterized by arc length in the sense that $g(\frac{dC}{dt}, \frac{dC}{dt}) = \pm 1$ is valid everywhere. On the other hand, since each null vector has zero length, the usual arc length parameterization is not possible for a null curve.

(\mathbf{R}_1^{m+2}, g) is called *Minkowski space* defined by a *Minkowski metric*

$$g(x, y) = -x^0 y^0 + \sum_{a=1}^{m+1} x^a y^a.$$

The set of all null vectors of \mathbf{R}_1^{m+2} forms what is called the *light cone* given by

$$(x^0)^2 = \sum_{a=1}^{m+1} (x^a)^2, \quad x^0 \neq 0.$$

Physically, \mathbf{R}_1^4 and \mathbf{R}_1^3 are important Minkowski spaces studied in general relativity. Here are two examples of non-null curves in \mathbf{R}_1^3 .

Example 1. The hyperbola $(x^0)^2 = (x^1)^2 + 1$, $x^2 = 0$ is a spacelike curve. This can be parameterized by arc length using the parameter $C(t) = (\cosh t, \sinh t, 0)$ since $g(\frac{dC}{dt}, \frac{dC}{dt}) = 1$.

Example 2. The hyperbola $(x^0)^2 = (x^1)^2 - 1$, $x^2 = 0$ is timelike with an arc length parameter $C(t) = (\sinh t, \cosh t, 0)$ since $g(\frac{dC}{dt}, \frac{dC}{dt}) = -1$. The line

$C(t) = (t, t, 0)$, $t \neq 0$ lies entirely on the light cone.

We first assume that C is a non-null curve in M . Let TC be the 2-dimensional tangent bundle manifold of C . Then, its normal bundle manifold, defined by,

$$TC^\perp = \{X \in \Gamma(TM) : g(X, V) = 0\}, \quad V \equiv \frac{d}{dt}, \tag{1.3.1}$$

is a $2(m + 1)$ -dimensional non-null bundle subspace of TM satisfying

$$TM = TC \perp TC^\perp, \quad TC \cap TC^\perp = \{\emptyset\}. \tag{1.3.2}$$

Along C , a vector field Y is said to be parallel, with respect to ∇ , if $\nabla_V Y = 0$. Using this and (1.1.8), we conclude that Y is parallel along C if and only if

$$\frac{dY^k}{dt} + \Gamma_{ij}^k Y^i \frac{dx^j}{dt} = 0. \tag{1.3.3}$$

The curve C is called a *geodesic* if V is parallel along C , i.e., if $\nabla_V V = fV$ for some smooth function f along C . For a non-null curve C , it is possible to find an arc-length parameter s of C such that f is zero along C and then the geodesic equation $\nabla_V V = 0$ can be expressed, in local coordinate system (x^i) , as

$$\frac{d^2 x^k}{ds^2} + \Gamma_{ji}^k \frac{dx^j}{ds} \frac{dx^i}{ds} = 0. \tag{1.3.4}$$

Two arc length parameters s_1 and s_2 are related by $s_2 = as_1 + b$, where a and b are constants. If the connection ∇ is smooth (or C^∞), then the theory of differential equations certifies that, given a point x of M and a tangent vector X_x , there is a *maximal geodesic* $C(s)$ such that $C(0) = x$ and $\frac{dx^i}{ds}|_{s=0} = X_x^i$.

It is important to note that an arbitrary curve need not have causal structure, but, a geodesic always does since parallel translations preserve causal character of vectors. This is a very useful geometric property of geodesic curves.

Now we let the curve C be a *null curve* which preserves its causal character. Then, all its tangent vectors are null. Thus, C is a null curve if and only if at each point x of C we have

$$g\left(\frac{d}{dt}, \frac{d}{dt}\right) = 0.$$

The normal bundle manifold of TC , defined exactly as in (1.3.1), is given by

$$TC^\perp = \left\{ X \in \Gamma(TM) : g\left(X, \frac{d}{dt}\right) = 0 \right\}, \quad \dim(TC^\perp)_x = m + 1. \tag{1.3.5}$$

However, null curves behave differently than the non-null curves as follows:

- (1) TC^\perp is also a null bundle subspace of TM .
- (2) $TC \cap TC^\perp = TC \rightarrow TC \oplus TC^\perp \neq TM$.

Thus, contrary to the case of non-null curves, since the normal bundle manifold TC^\perp contains the tangent bundle TC of C , (1.3.2) does not hold for any null curve as their sum is not the whole of the tangent bundle space TM . In other words, a vector of T_xM cannot be decomposed uniquely into a component tangent to C and a component perpendicular to C . Moreover, since the length of any arc of a null curve is zero, arc-length parameter makes no sense for null curves. Later on, the reader will see several other differences between non-null and null geometries. Thus, one can not use, in the usual way, the standard theory of non-null curves (in general, non-null submanifolds) in the study of the geometry of null curves (in general, lightlike submanifolds). Because of this anomaly, null curves (and, in general, lightlike submanifolds) have been studied by several ways corresponding to their use in a given problem. In this book we follow the technique first introduced by Bejancu [11] and then presented in a 1996 Duggal-Bejancu's book [28]. We also review the works of Honda-Inoguchi [57], Inoguchi-Lee [64] on null curves in \mathbf{R}_1^3 , Ferrández-Giménez-Lucas [35, 36, 37, 38, 39, 40, 41] on null curves and its applications. In general, throughout the book, we suggest many references on null curves (such as Barros [6], Bonnor [18], Cartan [20], Graves [45], Ikawa [61] etc.), discuss the works of Gutiérrez et al. [49, 50, 51, 52] on conjugate points along null geodesics and highlight the significance of null geodesics in geometry and physics.

1.4 Screen and null transversal bundles

In order to develop the geometry of null curves (in line with the case of non-null curves) our objective of this section is to change the equation (1.3.2) such that the tangent bundle TM can be split into three non-intersecting complementary (but non-orthogonal) vector bundles. For this purpose consider a complementary vector bundle $S(TC^\perp)$ to TC in TC^\perp . This means that

$$TC^\perp = TC \oplus S(TC^\perp).$$

Following [28, chapter 1] we call $S(TC^\perp)$ a *screen vector bundle* of C in M which obviously is non-degenerate. Since we assume that M is paracompact, there always exist a screen bundle. Thus, along C we have the following decomposition

$$TM|_C = S(TC^\perp) \oplus_{orth} S(TC^\perp)^\perp, \quad (1.4.1)$$

where $S(TC^\perp)^\perp$ is a complementary orthogonal vector bundle to $S(TC^\perp)$ in $TM|_C$. Note that $S(TC^\perp)^\perp$ is of rank 2 and contains TC . Clearly, contrary to the non-null case, the equation (1.4.1) is not unique as it depends on the choice of a screen vector bundle which is not unique. We will examine such a dependence later on. On the other hand, given a $S(TC^\perp)$ for a null curve C , there exists a unique null vector bundle of rank 1 which plays a roll similar to the roll of unique normal vector bundle of a non-null curve and TM splits into three non-intersecting sub bundles. For this we state and prove the following theorem:

Theorem 4.1. *Let C be a null curve of a proper semi-Riemannian manifold (M, g) and $\pi : ntr(C) \rightarrow M$ be a sub bundle of a screen vector bundle $S(TC^\perp)^\perp$*

of C such that $S(TC^\perp)^\perp = TC \oplus ntr(C)$. Let $V \in \Gamma^\infty(\mathcal{U}, ntr(C))$ be a locally defined nowhere zero section, defined on the open subset $\mathcal{U} \subseteq M$. Then

- (i) $g(\frac{d}{dt}, V) \neq 0$ everywhere on $\mathcal{U} \subseteq M$.
- (ii) If we consider $N_V \in \Gamma^\infty(\mathcal{U}, S(TC^\perp)^\perp)$ given by

$$N_V = \frac{1}{g(\frac{d}{dt}, V)} \left\{ V - \frac{g(V, V)}{2g(\frac{d}{dt}, V)} \frac{d}{dt} \right\}, \tag{1.4.2}$$

then $ntr(C)$ is a unique vector bundle over C of rank 1 such that on each $\mathcal{U} \subset C$ there is a unique vector field $N \in \Gamma(ntr(TC)|_{\mathcal{U}})$ satisfying

$$g(N_V, N_V) = 0, \quad g\left(\frac{d}{dt}, N_V\right) = 1. \tag{1.4.3}$$

- (iii) The tangent bundle TM splits into the following three bundle spaces:

$$TM|_C = TC \oplus ntr(C) \oplus_{orth} S(TC^\perp) = TC \oplus tr(TC). \tag{1.4.4}$$

Proof. Suppose $g(\frac{d}{dt}, V) = 0$ for some $x_0 \in \mathcal{U}$. Then g would be degenerate on TM at least at $x_0 \in \mathcal{U}$ which is a contradiction. Thus, (i) holds. Let V' be another nowhere zero section in ntr on \mathcal{U}' with $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$. Then $V' = \alpha V$ for some smooth function $\alpha \in \mathcal{U} \cap \mathcal{U}'$, and it follows from (1.4.2) that $N_V = N_{V'}$ on $\mathcal{U} \cap \mathcal{U}'$. This also shows that N depends neither on $ntr(C)$ nor on its local section V . Now, the relations of (1.4.3) easily follow from (1.3.2), which proves (ii). Finally, (1.4.4) of (iii) holds if we set $tr(C) = ntr(C) \oplus S(TC^\perp)$, which completes the proof.

According to the terminology in [28] we call $tr(C)$ and $ntr(C)$ the *transversal bundle* and the *null transversal bundle* with respect to $S(TC^\perp)$ and with respect to $\frac{d}{dt}$ of C in that order. It is easy to see that if C is a null curve of a semi-Riemannian manifold (M, g) of index q , then, its $S(TC^\perp)$ is also semi-Riemannian but of index $q - 1$. Thus, (1.4.4) replaces (1.3.2) such that TM splits into a sum of three non-intersecting complementary (but non-orthogonal) vector bundles.

Note. The statement and the proof of above theorem is an improved version of the one appeared in [28, page 53].

1.5 Quasi-orthonormal basis along a null curve

In this section, we show that, based on the decomposition equation (1.3.4) there exists a quasi-orthonormal basis along a null curve C of a proper semi-Riemannian manifold (M_q^{m+2}, g) . Since g is of constant index q , at each point $x \in M$ there is an associated semi-Euclidean space $T_x M$ whose quadratic form is of type $(0, p, q)$ with $p \cdot q \neq 0$ and $p + q = m + 2$. Based on the convention (1.1.2), let $\{e_1, \dots, e_{m+2}\}$ be an orthonormal basis of $T_x M$ such that $\{e_1, \dots, e_q\}$ and $\{e_{q+1}, \dots, e_{q+p}\}$ are

unit timelike and spacelike vectors, respectively. To construct a basis including some null vectors we consider the following types.

Type I ($q < p$). Construct vectors

$$\xi_i = \frac{1}{\sqrt{2}} \{e_{q+i} + e_i\}; \quad \xi_i^* = \frac{1}{\sqrt{2}} \{e_{q+i} - e_i\}, \quad i \in \{1, \dots, q\},$$

which satisfy

$$g(\xi_i, \xi_j) = g(\xi_i^*, \xi_j^*) = 0,$$

and

$$g(\xi_i, \xi_j^*) = \delta_{ij}, \quad i, j \in \{1, \dots, q\}.$$

Thus

$$\{\xi_1, \dots, \xi_q, \xi_1^*, \dots, \xi_q^*, e_{2q+1}, \dots, e_{q+p}\}$$

is a basis of $T_x M$ with $2q$ null vectors and $p - q$ spacelike vectors. One can also take at least 2 null base vectors and all others non-null.

Type II ($p < q$). In this case define

$$\xi_a = \frac{1}{\sqrt{2}} \{e_{q+a} + e_a\}, \quad \xi_a^* = \frac{1}{\sqrt{2}} \{e_{q+a} - e_a\}, \quad a \in \{1, \dots, p\},$$

and, in a similar way, obtain a basis

$$\{\xi_1, \dots, \xi_p, \xi_1^*, \dots, \xi_p^*, e_{p+1}, \dots, e_q\}$$

which contains $2p$ null vectors and $q - p$ timelike vectors.

Type III ($p = q$). Similarly, we obtain a totally null basis

$$\{\xi_1, \dots, \xi_q, \xi_1^*, \dots, \xi_q^*\}.$$

The construction of above three types shows that, in general, there exists a basis

$$B = \{\xi_1, \dots, \xi_r, \xi_1^*, \dots, \xi_r^*, u_1, \dots, u_t\}$$

of a proper semi-Euclidean space $T_x M$, at any point $x \in M$ if

$$\begin{aligned} g(\xi_i, \xi_j) &= g(\xi_i^*, \xi_j^*) = 0; & g(\xi_i, \xi_j^*) &= \delta_{ij}, \\ g(u_\alpha, \xi_i) &= g(u_\alpha, \xi_i^*) = 0; & g(u_\alpha, u_\beta) &= \epsilon_\alpha \delta_{\alpha\beta}, \end{aligned} \tag{1.5.1}$$

for any $i, j \in \{1, \dots, r\}$ and $\alpha, \beta \in \{1, \dots, t\}$. The basis B , satisfying above, is called a *quasi-orthonormal basis* of $T_x M$. In particular, let C be a null curve of a Lorentz space M_1^{m+2} , then, a quasi-orthonormal basis of $T_x M$, along C , is given by

$$B = \{\xi, \xi^*, u_1, \dots, u_m\}$$

such that $T_x C = Span\{\xi\}$. Also, based on the decomposition equation (1.4.4), we have $T_x(ntr(C)) = Span\{\xi^*\}$ and $T_x(S(TC^\perp)) = Span\{u_1, \dots, u_m\}$.

Next, consider an n -dimensional lightlike subspace W of an $(m+2)$ -dimensional proper semi-Euclidean space V . Then a quasi-orthonormal basis

$$B = \{f_1, \dots, f_r, f_1^*, \dots, f_r^*, u_1, \dots, u_t\}$$

such that $W = Span\{f_1, \dots, f_r, u_1, \dots, u_s\}$, if $n = r + s$, $1 \leq s \leq t$, or

$$W = Span\{f_1, \dots, f_n\}, \quad \text{if } n \leq r,$$

is called a quasi-orthonormal basis of V along W .

Proposition 5.1 ([28]). *There exists a quasi-orthonormal basis of V along W .*

Proof. First, suppose $null W = r < \min\{n, m+2-n\}$. Then we have

$$W = Rad W \perp W',$$

and

$$W^\perp = Rad W \perp W'',$$

where W' and W'' are some screen subspaces. We decompose V as follows

$$V = W' \perp (W')^\perp. \tag{1.5.2}$$

As W'' is a non-degenerate subspace of $(W')^\perp$ we obtain

$$(W')^\perp = W'' \perp (W'')^\perp, \tag{1.5.3}$$

where $(W'')^\perp$ is the complementary orthogonal subspace to W'' in $(W')^\perp$. It is easy to see that $Rad W$ is a subspace of $(W'')^\perp$. Denote by U a complementary subspace to $Rad W$ in $(W'')^\perp$. As $(W'')^\perp$ is of dimension $2r$ we may consider the basis $\{f_1, \dots, f_r\}$ and $\{v_1, \dots, v_r\}$ of $Rad W$ and U , respectively. Now, we look for $\{f_i^*, \dots, f_r^*\}$ given by

$$f_i^* = A_i^j f_j + B_i^j u_j, \tag{1.5.4}$$

and satisfying the relations in the first line of (1.5.1). By direct calculations, one obtains that $g(f_i, f_k^*) = \delta_{ik}$ if and only if

$$B_k^j g(f_i, v_j) = \delta_{ik}. \tag{1.5.5}$$

As $\det[g(f_i, v_j)] \neq 0$, (otherwise $(W'')^\perp$ would be degenerate), the system (1.5.5) has a unique solution (B_k^j) . Next, by using (1.5.4) and (1.5.5) one obtains $g(f_i^*, f_j^*) = 0$ if and only if

$$A_j^i + A_i^j + B_i^h B_j^k g(v_h, v_k) = 0$$

which proves the existence of A_i^j from (1.5.2). Finally, from (1.5.2) and (1.5.3), and taking into account of the above construction, we obtain the following decomposition

$$V = W' \perp W'' \perp (Rad W \oplus \{f_1^*, \dots, f_r^*\}).$$

Hence we have a quasi-orthonormal basis of V along W given by

$$\{f_1, \dots, f_r, f_1^*, \dots, f_r^*, u_1, \dots, u_{n-r}, w_1, \dots, w_{m+2-n-r}\},$$

where $\{u_1, \dots, u_{n-r}\}$ and $\{w_1, \dots, w_{m-n-r}\}$ are two orthonormal basis of W' and W'' , respectively. In this case

$$W = \text{Span} \{f_1, \dots, f_r, u_1, \dots, u_{n-r}\}.$$

In case $r = n < m + 2 - n$, it follows that $\text{Rad} W = W \subset W^\perp$. We put

$$W^\perp = W \perp W'',$$

where W'' is an arbitrary screen subspace of W^\perp . One obtains for V the orthogonal decomposition

$$V = W'' \perp (W'')^\perp,$$

where $(W'')^\perp$ is the complementary orthogonal subspace to W'' in V . Moreover $(W'')^\perp$ is of dimension $2n$ and contains W . In a similar way as in the first case we find the quasi-orthonormal basis of V along W

$$\{f_1, \dots, f_n, f_1^*, \dots, f_n^*, w_1, \dots, w_{m-2n}\},$$

where $\{w_1, \dots, w_{m-2n}\}$ is an orthonormal basis of W'' and

$$W = \text{Span} \{f_1, \dots, f_n\}.$$

In case $r = m + 2 - n < n$, it follows that $\text{Rad} W = W^\perp \subset W$. Then we set

$$W = W^\perp \perp W',$$

where W' is a screen subspace of W . Thus V is decomposed as follows

$$V = W' \perp (W')^\perp,$$

where $(W')^\perp$ is the complementary orthogonal subspace to W' in V . It follows that $(W')^\perp$ is of dimension $2(m+2-n)$ and it contains W^\perp . Then the quasi-orthonormal basis of V along W is given by

$$\{f_1, \dots, f_{m+2-n}, f_1^*, \dots, f_{m-n}^*, u_1, \dots, u_{2n-m+2}\},$$

where $\{u_1, \dots, u_{2n-m+2}\}$ is an orthonormal basis of W' . In this case

$$W = \text{Span} \{f_1, \dots, f_{m-n}, u_1, \dots, u_{2n-m+2}\}.$$

Finally, if $r = n = \frac{m+2}{2}$, we get $\text{Rad} W = W = W^\perp$ and the decomposition

$$V = W \oplus \text{Span} \{f_1^*, \dots, f_n^*\}.$$

Then the quasi-orthonormal basis of V along W is given by

$$\{f_1, \dots, f_n, f_1^*, \dots, f_n^*\},$$

where $\{f_1, \dots, f_n\}$ is a basis of W .

For more details on the quasi-orthonormal basis, see [28, Chapter 1].

1.6 Brief notes and exercises

Serret-Frenet equations of a curve in \mathbf{R}^3 . Let $C = C(s)$ be a curve in \mathbf{R}^3 with its Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ consisting of unit tangent, normal and binormal vectors respectively. Then, the differential equations

$$\begin{aligned} \mathbf{t}' &= \kappa \mathbf{n} \\ \mathbf{n}' &= -\kappa \mathbf{t} + \tau \mathbf{b} \\ \mathbf{b}' &= -\tau \mathbf{n} \end{aligned}$$

are called *Serret-Frenet* (or briefly Frenet) equations of C . These equations are basic in the development of the theory of curves in \mathbf{R}^3 . The functions κ and τ are called curvature and torsion functions respectively. The fundamental theorem of space curves establishes the existence and uniqueness of a curve with given Frenet frame. Unfortunately, in general, the unique existence of a null curve can not be established.

Null curves of \mathbf{R}_1^3 . Consider a 3-dimensional *Minkowski space* \mathbf{R}_1^3 defined as a space to be the usual 3-dimensional vector space consisting of vectors $\{(x^0, x^1, x^2) \mid x^0, x^1, x^2 \in \mathbf{R}\}$, but with a linear connection ∇ corresponding to its Minkowski metric \langle , \rangle given by

$$\langle x, y \rangle = -x^0y^0 + x^1y^1 + x^2y^2.$$

The set of all null vectors of \mathbf{R}_1^3 forms the light cone in coordinates;

$$\{(x^0, x^1, x^2) \mid (x^0)^2 = (x^1)^2 + (x^2)^2, x^0 \neq 0\}$$

in \mathbf{R}_1^3 , the rules of calculus are the same as in Euclidean space. Thus, we speak of immersions or regular or smooth curves just as in the Euclidean case. However, since any \mathbf{R}_1^3 can have a null curve (for example, any null vector can generate a null curve) as stated before the calculus of null curves is quite different than the non-null curves.

To derive the Frenet type equations for a null curve C , defined by $C : [a, b] \rightarrow \mathbf{R}_1^3$, Cartan [20] has shown that with respect to an affine parameter, say p , and a positively oriented set $\{C'(p), C''(p), C'''(p)\}, \forall p \in [a, b]$, there exists a local frame $F = \{\xi = C', N, W, \}$, called *Cartan frame* satisfying

$$\begin{aligned} \langle \xi, \xi \rangle &= \langle N, N \rangle = 0, & \langle \xi, N \rangle &= 1, \\ \langle W, \xi \rangle &= \langle W, N \rangle = 0, & \langle W, W \rangle &= 1, \end{aligned}$$

with the vector product \times given by $\xi \times W = -\xi, \xi \times N = -W$ and $W \times N = -N$. The Cartan equations are given by

$$\begin{aligned} \xi' &= \nabla_\xi \xi = kW, \\ N' &= \nabla_\xi N = -\tau W, \\ W' &= \nabla_\xi W = -\tau \xi + kN, \end{aligned} \tag{1.6.1}$$

where k and τ are the curvature and torsion functions of C with respect to F . If $k = 0$ then the first Cartan equation reduces to

$$\frac{d^2 x^i}{dp^2} + \Gamma_{jk}^i \frac{dx^j}{dp} \frac{dx^k}{dp} = 0, \quad i \in \{0, 1, 2\},$$

where Γ_{jk}^i are the Christoffel symbols induced by ∇ . Thus C is a *null geodesic* of \mathbf{R}_1^3 if and only if its curvature function k vanishes identically on C . This characterization result also holds for any null curve of a semi-Riemannian manifold. For ($k \neq 0$), Honda-Inoguchi [57] have recently proved that there exists a Cartan frame for any non-geodesic null curve $C = C(t)$ in \mathbf{R}_1^3 such that the Cartan equations (1.6.1) hold for the original parameter t . In chapter 2 we show that one can take a canonical affine parameter for every non-geodesic null curve in a Lorentzian manifold M_1^{m+2} . Here we show that it is possible to express the Cartan frame F in terms of ξ and its covariant derivatives up to the order 3 as follows:

Suppose $k = \frac{1}{\sigma} \neq 0$ on \mathcal{U} and denote $\sigma_1 = \xi(\sigma)$. From (1.6.1) we obtain

$$W = \sigma \xi', \quad N = -\sigma(\tau \xi + \sigma_1 \xi' + \sigma \xi'').$$

Again using (1.6.1) and above two equations, we obtain

$$k = \pm \|\xi'\|, \quad \tau = \frac{\sigma}{2} \left\{ \sigma^2 \langle \xi'', \xi'' \rangle - \sigma^{-2} (\sigma_1)^2 \right\}.$$

Null tetrad of \mathbf{R}_1^4 . Consider a Euclidean space \mathbf{R}^4 and a canonical basis $E = \{e_1 = (1, 0, 0, 0), \dots, e_4 = (0, 0, 0, 1)\}$. Define on \mathbf{R}^4 the Minkowski metric:

$$ds^2 = -(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2,$$

with respect to a local coordinate system (x^1, x^2, x^3, x^4) . Then, we say that \mathbf{R}_1^4 is a *Minkowski* vector space with the above metric of Special Relativity, where the coordinate x^1 represents the time component. Using Type 1, we construct a quasi-orthonormal basis $B = \{\xi, N, e_3, e_4\}$ by the following transformation:

$$\begin{aligned} \xi &= \frac{1}{\sqrt{2}}\{e_2 + e_1\}; & N &= \frac{1}{\sqrt{2}}\{e_2 - e_1\}, \\ g(\xi, \xi) &= g(N, N) = 0, & g(\xi, N) &= 1. \end{aligned}$$

Let C be a null curve of \mathbf{R}_1^4 , with the basis B , along C , such $T_x C = \text{Span}\{\xi\}$, $T_x(\text{ntc}(C)) = \text{Span}\{N\}$ and $T_x(S(TC^\perp)) = \text{Span}\{e_3, e_4\}$, for any $x \in \mathbf{R}_1^4$.

The *complexified vector space* of \mathbf{R}_1^4 is the complex vector space $(\mathbf{R}_1^4)^c$ of vectors $x + iy$, $x, y \in \mathbf{R}_1^4$, $i = \sqrt{-1}$. The scalar product g on \mathbf{R}_1^4 induces a scalar product g^c on $(\mathbf{R}_1^4)^c$ which is a symmetric non-degenerate **C**-bilinear mapping on $(\mathbf{R}_1^4)^c$. Then the quasi-orthonormal basis B of \mathbf{R}_1^4 induces the so called *null tetrad* of $(\mathbf{R}_1^4)^c$ (see Trautman et al. [107, Page 57])

$$T = \{\ell, n, m, \bar{m}\}; \quad \ell = \xi, \quad n = -N, \quad m = \frac{1}{\sqrt{2}}(e_3 + ie_4), \quad \bar{m} = \frac{1}{\sqrt{2}}(e_3 - ie_4)$$

consisting of two real null vectors and two conjugate complex null vectors with respect to g^c such that the only surviving scalar products are

$$g^c(\ell, n) = -1; \quad g^c(m, \bar{m}) = 1.$$

Denoting the dual basis of T by $T^* = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ of the null tetrad T and obtain the associated quadratic form h^c of g^c by $h^c = 2\{\eta_3 \eta_4 - \eta_1 \eta_2\}$.

Using above, one can obtain a local null tetrad with respect to a coordinate neighborhood on a spacetime manifold, which has been a useful tool in relativity. Moreover, it is possible to express $\{n, e_3, e_4\}$ in terms of ℓ and its covariant derivatives up to the order 3. For details, see [28, pages 62, 63].

Using Cartan’s approach [20], Bonnor [18], Cöken and Ciftci [22] and Ikawa [61] have done some work on null curves in a 4-dimensional Minkowski spacetime. See chapter 5 for fundamental existence and uniqueness theorems on null Cartan curves in Minkowski spaces and pseudo-Euclidean spaces of index two.

Exercises

- (1) (a) Show that the circle \mathbf{S}^1 in \mathbf{R}^3 can be made into a smooth 1-manifold by constructing an atlas with two charts.
 (b) Show that a one-chart does not exist. Thus, a circle is not homeomorphic to a line or interval.
- (2) The n -sphere \mathbf{S}^n is defined by the set of points (x^i) in \mathbf{R}^{n+1} such that $\sum_{i=1}^{n+1} (x^i)^2 = a^2$ centered at the origin with radius a . A coordinate patch for a neighborhood of origin is:

$$x^1 = y^1, \quad x^2 = y^2, \quad \dots, \quad x^n = y^n, \quad x^{n+1} = \sqrt{a^2 - (y^1)^2 - (y^2)^2 - \dots - (y^n)^2}$$

where the mapping is into the n -dimensional neighborhood $\sum_{i=1}^n (y^i)^2 < a^2$ (the interior of \mathbf{S}^{n-1}). Establish the analogous patches around the other [diametrically opposite] endpoints, and obtain an atlas of $2n + 2$ charts to see \mathbf{S}^n as a manifold.

- (3) Show that the hyperboloid of one sheet $4x^2 + 4y^2 + z^2 = 16$ is a smooth 2-manifold M , by the coordinatization $(k = 1, 2)$, where $((k - 2)\pi < \theta < h\pi)$ and

$$\mathbf{r}_k : x = 2 \cos \theta \cosh \phi, \quad y = 2 \sin \theta \cosh \phi, \quad z = 4 \sinh \phi$$

with the neighborhoods $\mathcal{U}_{(1)} = \mathcal{U}_a$ and $a = (2, 0, 0)$, $\mathcal{U}_{(2)} = \mathcal{U}_b$ and $b = (-2, 0, 0)$.

- (4) Find a convenient atlas showing that the set in \mathbf{R}^4 given by the equation

$$x^2 + y^2 + z^2 - t^2 = a^2$$

can be made into a smooth 3-manifold. [Hint: six charts will suffice.]

- (5) Determine the curve whose curvature and torsion are given by

$$\kappa = (1/2as)^{\frac{1}{2}}, \quad \tau = 0.$$

- (6) Show that the curve with $\kappa = \sqrt{2} \frac{1}{s^2+4} = \tau$ is a general helix on a cylinder whose cross section is a catenary.
- (7) Consider a null Cartan curve C of \mathbf{R}_1^3 given by the equations

$$x^0 = \sinh p, \quad x^1 = p, \quad x^2 = \cosh p,$$

with a Cartan frame $F = \left\{ \xi = \frac{d}{dp}, W, N \right\}$ as follows:

$$\xi = (\cosh p, 1, \sinh p), \quad W = (\sinh p, 0, \cosh p), \quad N = \frac{1}{2}(-\cosh p, 1, -\sinh p).$$

Then, using Cartan equations show that $k_1 = 1$ and $k_2 = -\frac{1}{2}$.

- (8) Consider a curve C of \mathbf{R}_1^4 defined by

$$x^0 = 5t, \quad x^1 = 3 \cos t, \quad x^2 = 3 \sin t, \quad x^3 = 4t,$$

and $0 \leq t \leq 1$ is an affine parameter. Prove that C is a null curve.

- (9) Show that a curve C given by

$$x^1 = \frac{1}{3}\{2t - 1\}, \quad x^2 = \frac{1}{2}t^2 - t, \quad x^3 = t \sin t + \cos t, \quad x^4 = \sin t - t \cos t,$$

with $t > 1$ is a null curve of a semi-Euclidean space \mathbf{R}_2^4 of index 2.

- (10) Consider a curve in a semi-Euclidean space \mathbf{R}_2^6 given by

$$C : (\alpha t^2 - 2t, \beta t^2, \alpha t^3, \sqrt{2}t, \cos \sqrt{2}t, \sin \sqrt{2}t),$$

where $\alpha, \beta, t \in \mathbf{R}$. Find a relation between α and β such that C is null.