

Chapter 5

Representations of Compact Lie Groups

In this chapter we deal with the classical representation theory of a compact group. We shall see that compact Lie groups play a special role although the case of finite or abelian groups is also of great interest. The central object of study here is the set of all finite dimensional, continuous, irreducible unitary representations. Although we will not need this, actually every continuous, irreducible, unitary representation of a compact group on a Hilbert space is automatically finite dimensional. In Section 1 we introduce the players, in Section 2 we prove the Schur orthogonality relations. In Section 3 we develop what we need from functional analysis and in Section 4 we prove the Peter-Weyl theorem and its many consequences. Section 5 deals with characters and class functions. In our final section we study induced representations and the Frobenius reciprocity theorem as well as a number of related ideas which have proven to be quite useful in geometric questions (such as the Mostow-Palais equivariant embedding theorem) and spherical harmonics. It might also be mentioned that the results on representations and harmonic analysis have been generalized from compact to other classes of groups. The most direct analogies have been found in the case of *central groups*, those which are compact modulo their center (see [27] and [28]).

5.1 Introduction

Unless otherwise stated, throughout this chapter G will denote a compact topological group and dg will be the normalized Haar measure on G (recall G is unimodular by Corollary 2.2.2), $L_1(G)$ and $L_2(G)$ will denote the integrable, respectively square integrable, complex valued measurable functions on G with respect to the Haar measure and $C(G)$ the continuous complex valued functions on G . Likewise if G operates on a space X with a measure dx preserved by G we denote by $L_1(X)$ and $L_2(X)$ the integrable, respectively square integrable, complex valued measurable functions on X and $C(X)$ the continuous ones.

Even though compact groups have many interesting infinite dimensional representations, we shall also see why we concentrate on finite dimensional representations.

For the readers convenience we list the following definitions. These are fundamental and actually do not require compactness of G , but merely that the representations are continuous and finite dimensional.

- Definition 5.1.1.** (1) Given two such representations of ρ and σ of G we shall call an operator $T : V_\rho \rightarrow V_\sigma$ an *intertwining operator* if $T\rho_g = \sigma_g T$, $g \in G$.
- (2) ρ and σ are said to be *equivalent* if there exists an invertible intertwining operator between them.
- (3) We shall say ρ is a *unitary representation* if $\rho(G) \subseteq U(n, \mathbb{C})$ for some n .
- (4) A Hermitian inner product $\langle \cdot, \cdot \rangle$ on V_ρ is called *invariant* if $\langle \rho_g(v), \rho_g(w) \rangle = \langle v, w \rangle$ for all $g \in G$ and $v, w \in V_\rho$.
- (5) A subspace W of V_ρ is called an *invariant subspace* if $\rho_g(W) \subseteq W$ for all $g \in G$.
- (6) ρ is called *completely reducible* if every invariant subspace has a complementary invariant subspace.
- (7) ρ is called *irreducible* if it has no nontrivial invariant subspaces.
- (8) Finally, we denote by $\mathcal{R}(G)$ the *equivalence classes of finite dimensional, continuous, irreducible unitary representations* of G . As we shall see this set is quite interesting even when G is finite

or abelian.

The following proposition and its corollary was proved in Chapter 2 (see the proof of Theorem 2.5.1).

Proposition 5.1.2. *Any finite dimensional representation of a compact group G has a G -invariant inner product and hence is equivalent to a unitary representation.*

In particular,

Corollary 5.1.3. *Any finite dimensional continuous representation of a compact group G is completely reducible.*

We leave the following important exercise to the reader.

Exercise 5.1.4. (1) Show that equivalence of representations is an equivalence relation.

- (2) Given a single representation ρ , show the set of intertwining operators forms a subalgebra of $\text{End}(V_\rho)$.
- (3) Give an example to show that the proposition and corollary above is false if G is not compact e.g. consider a unipotent representation of \mathbb{R} .
- (4) Show that two representations are equivalent if and only if the modules (G, V_ρ, ρ) and (G, V_σ, σ) are isomorphic. Thus they share all module theoretic properties such as a composition series for one corresponds to such a series for the other etc.
- (5) Show that a finite dimensional continuous representation ρ is completely reducible if and only if the corresponding module is semisimple.
- (6) Show that a finite dimensional continuous representation ρ is irreducible if and only if the corresponding module is simple.
- (7) Define a unitary representation (not necessarily continuous) of a group (not necessarily compact) on a Hilbert space V and show that it is completely reducible in the sense that any closed G invariant subspace of V has a complimentary closed invariant subspace.

We conclude this section with an important example. We shall find all the finite dimensional irreducible unitary representation of $SU(2, \mathbb{C})$. Now we know all the complex irreducible representations of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ (see Section 3.1.5). Since the Lie group $SL(2, \mathbb{C})$ is simply connected (Corollary 6.3.7) and has $\mathfrak{sl}(2, \mathbb{C})$ as its Lie algebra its irreducible representations are in bijective correspondence with those of the Lie algebra by $\rho \mapsto \rho'$ (Corollary 1.4.15). Similarly, $SU(2, \mathbb{C})$ is also simply connected (Corollary 1.5.2), so its real continuous (smooth) irreducible representations are in bijective correspondence with those of its Lie algebra, $\mathfrak{su}(2, \mathbb{C})$. Finally, as we will see in Chapter 7, $\mathfrak{su}(2, \mathbb{C})$ is a compact real form of $\mathfrak{sl}(2, \mathbb{C})$. Hence complex irreducibles of the latter bijectively correspond with the real irreducibles of the former.

Corollary 5.1.5. *There are an infinite number of continuous, finite dimensional, irreducible, unitary representations of $SU(2, \mathbb{C})$, one for each degree.*

Exercise 5.1.6. Show that, within these, the representations of $SO(3, \mathbb{R})$ are the ones of odd degree.

5.2 The Schur Orthogonality Relations

The Schur orthogonality relations are the following.

Theorem 5.2.1. *Let G be a compact group, dg be the normalized Haar measure and ρ and σ finite dimensional continuous irreducible unitary representations of G . Then*

- (1) *If ρ and σ are inequivalent, then $\int_G \rho_{ij}(g) \overline{\sigma_{lk}(g)} dg = 0$ for all $i, j = 1, \dots, d_\rho$ and $k, l = 1, \dots, d_\sigma$.*
- (2) $\int_G \rho_{ij}(g) \overline{\rho_{kl}(g)} dg = \frac{\delta_{ik} \delta_{jl}}{d_\rho}$.

Proof. Let V_ρ and V_σ be the respective representation spaces and $B(V_\sigma, V_\rho)$ be the (finite dimensional) \mathbb{C} -vector space of linear operators between them. Let $T \in B(V_\sigma, V_\rho)$ and consider the map $G \rightarrow B(V_\sigma, V_\rho)$ given by $g \mapsto \rho(g)T\sigma(g^{-1})$. This is a continuous operator valued function on G and so $\int_G \rho(g)T\sigma(g^{-1})dg$ is also an operator in $B(V_\sigma, V_\rho)$. For

$h \in G$ we have

$$\begin{aligned} \rho(h) \int_G \rho(g) T \sigma(g^{-1}) dg \sigma(h)^{-1} &= \int_G \rho(h) \rho(g) T \sigma(g^{-1}) \sigma(h)^{-1} dg \\ &= \int_G \rho(hg) T \sigma(hg)^{-1} dg \\ &= \int_G \rho(g) T \sigma(g)^{-1} dg \end{aligned}$$

Letting $T_0 = \int_G \rho(g) T \sigma(g)^{-1} dg$ we see $\rho(g) T_0 = T_0 \sigma(g)$ for every $g \in G$. That is, T_0 is an intertwining operator. By Schur's lemma, Lemma 3.1.52, there are only two possibilities. Either ρ and σ are equivalent and T_0 is invertible (and implements the equivalence), or ρ and σ are inequivalent and $T_0 = 0$. In the latter case $\int_G \rho(g) T \sigma(g)^{-1} dg = 0$, where T is arbitrary. Let $T = (t_{jk})$. Then $(\rho(g) T \sigma(g)^{-1})_{il} = \sum_{jk} \rho_{ij}(g) t_{jk} \sigma_{kl}(g^{-1})$. Since (t_{jk}) are arbitrary we get $\int_G \rho_{ij}(g) \sigma_{kl}(g^{-1}) dg = 0$ for all $i, j = 1, \dots, d_\rho$ and $k, l = 1, \dots, d_\sigma$. Because σ is a unitary representation $\sigma(g^{-1}) = \sigma(g)^{-1} = \sigma(g)^*$. Thus $\int_G \rho_{ij}(g) \overline{\sigma_{lk}(g)} dg = 0$ for all $i, j = 1, \dots, d_\rho$ and $k, l = 1, \dots, d_\sigma$.

We now consider the case when we have equivalence. Here we may as well just take σ to be ρ . In this case Schur's lemma tells us T_0 is a scalar multiple of the identity. Thus $\int_G \rho(g) T \rho(g)^{-1} dg = \lambda(T) I$. Taking the trace of each side yields

$$\operatorname{tr} \left(\int_G \rho(g) T \rho(g)^{-1} dg \right) = \int_G \operatorname{tr}(\rho(g) T \rho(g)^{-1}) dg = \int_G \operatorname{tr}(T) dg = \operatorname{tr}(T),$$

while $\operatorname{tr}(\lambda(T) I) = \lambda(T) d_\rho$. We conclude $\lambda(T) = \frac{\operatorname{tr}(T)}{d_\rho}$ and so $\int_G \rho(g) T \rho(g)^{-1} dg = \frac{\operatorname{tr}(T)}{d_\rho} I$. Using reasoning similar to the earlier case one finds $\int_G \rho_{ij}(g) \overline{\rho_{lk}(g)} dg = 0$ for all $i, j, k, l = 1, \dots, d_\rho$ whenever $i \neq l$, or $j \neq k$. Now we consider the case when $i = l$ and $j = k$. By taking T to be diagonal with all zero entries except for one we get $i, j = 1, \dots, d_\rho$. Hence in general one has $\int_G \rho_{ij}(g) \overline{\rho_{kl}(g)} dg = \frac{\delta_{ik} \delta_{jl}}{d_\rho}$. \square

5.3 Compact Integral Operators on a Hilbert Space

Before proceeding further we must now prove the spectral theorem for compact self-adjoint operators on a Hilbert space. Then we will apply this result to compact self-adjoint integral (Fredholm) operators to conclude that the range of such an operator always has an eigenfunction expansion. It is this fact which is behind the Peter-Weyl theorem.

Definition 5.3.1. Let V and W be real or complex Hilbert spaces. A bounded linear operator $T : V \rightarrow W$ is called a *compact operator* if $T(B_1(0))$ is compact where $B_1(0)$ is the unit ball in V .

For an operator T , the *norm*, if it exists, is defined to

$$\|T\| = \sup\{\|Tv\| : \|v\| = 1\}.$$

Exercise 5.3.2. Evidently, if this were so it would be true of every ball about 0. In fact, T is compact if and only if it takes bounded sets to compact ones. Notice that when $V = W$ and is infinite dimensional, then the identity map I or, more generally λI , $\lambda \neq 0$ is not compact while if W is finite dimensional all bounded linear operators are compact. Such operators are called *finite rank operators*. Observe also that the restriction of a compact operator to a closed invariant subspace is again compact.

What would be a nontrivial example of a compact operator? Suppose $V = L_2(X)$, where X is a compact Hausdorff space, dx is a (finite which we may as well normalize to have total mass 1) regular measure on X and k is a continuous function on $X \times X$. We can use k to define an operator $T_k : V \rightarrow V$, by $T_k(f)(x) = \int_X k(x, y)f(y)dy$. T_k is well defined since by compactness k is bounded and by the Schwarz inequality together with compactness tells us $L_2(X) \subseteq L_1(X)$. T_k is evidently linear. Applying the Schwarz inequality again shows $\int_X |k(x, y)f(y)|^2 dy \leq \|k\|_{X \times X}^2 \int_X |f(y)|^2 dy \leq \|k\|_{X \times X}^2 \|f\|_2^2$. Hence T_k is a bounded operator (whose operator norm is $\leq \|k\|_{X \times X}^2$).

Definition 5.3.3. In this context the k above is called a *kernel function* and T_k a Fredholm operator.

So for example, we can let ϕ_i and ψ_i be two sets of n continuous functions on X , where n is any integer and $k(x, y) = \sum_{i=1}^n \phi_i(x)\psi_i(y)$. Then T_k is a compact operator for a very simple reason. $T_k(V) \subseteq W$, where W is the linear span of ϕ_i and hence is finite dimensional.

Now this conclusion actually holds for any jointly continuous k . We shall see that it will be sufficient for our purposes to understand Fredholm integral operators.

Theorem 5.3.4. *For jointly continuous k , T_k is a compact operator. Furthermore, $T_k(L_2(X)) \subseteq C(X)$.*

Proof. We first prove the second statement. Since $X \times X$ is compact, k is uniformly continuous. That is given $x_0 \in X$ and $\epsilon > 0$ there exists a neighborhood U_{x_0} of x_0 so that $|k(x, y) - k(x_0, y)| < \epsilon$, whenever $y \in X$ and $x \in U_{x_0}$. Therefore $|T_k(f)(x) - T_k(f)(x_0)| \leq \int_X |k(x, y) - k(x_0, y)| |f(y)| dy \leq \epsilon \|f\|_1$ and since $L_2 \subseteq L_1$, $\|f\|_1 < \infty$. Thus $T_k(f)$ is always a continuous function.

We now show that T_k is a compact operator. Notice that on $C(X)$ we have two norms, the sup norm $\|\cdot\|_X$ and the restricted L_2 norm. But since we have normalized the measure, $\|f\|_2 \leq \|f\|_X$. Let B be a bounded set in L_2 . If we can prove $T_k(B)$ is compact in $C(X)$, then by continuity of the injection $C(X) \rightarrow L_2(X)$ we will be done. To do this we apply the Ascoli theorem. Now again by Schwarz, for $x \in X$, $|T_k f(x)| \leq \int_X |k(x, y)| |f(y)| dy \leq \|k\|_{X \times X}^2 \|f\|_2^2$. Thus $\|T_k f\|_X < \infty$ and $T_k(B)$ is uniformly bounded. Moreover $|T_k(f)(x) - T_k(f)(x_0)|^2 \leq \int_X |k(x, y) - k(x_0, y)|^2 |f(y)|^2 dy \leq \epsilon^2 \|f\|_2^2$, if $x \in U_{x_0}$. Thus $T_k(B)$ is equicontinuous at every point of X . By Ascoli, $T_k(B)$ has compact closure in $C(X)$. \square

Definition 5.3.5. (1) A linear operator $T : V \rightarrow V$ on a Hilbert space is called *self adjoint* if for all $v, w \in V$, $\langle Tv, w \rangle = \langle v, Tw \rangle$.

(2) For such an operator if $\lambda \in \mathbb{C}$, we define $V_\lambda = \{v \in V : Tv = \lambda v\}$. Here λ is called an eigenvalue of T and V_λ the corresponding eigenspace.

Exercise 5.3.6. (1) So for example, a Fredholm operator T_k is self-adjoint if and only if $k(x, y) = \overline{k(y, x)}$ for all $x, y \in X$.

(2) If a linear operator $T : V \rightarrow V$ on a Hilbert space is self adjoint, then all its eigenvalues are real.

Now we wish to prove the following *spectral theorem* for compact self adjoint operators on a Hilbert space.

Theorem 5.3.7. *Let T be a compact self-adjoint operator on a Hilbert space V . Then*

- (1) *These are all real.*
- (2) *If $\lambda \neq \mu$ are distinct eigenvalues then V_λ and V_μ are orthogonal.*
- (3) *If $\lambda \neq 0$ then V_λ is finite dimensional.*
- (4) *T has at most a countable number of nonzero eigenvalues.*
- (5) $\text{Ker } T = V_0$.
- (6) $V = V_0 \oplus \overline{(\sum_{\lambda_i \neq 0} V_{\lambda_i})}$ (orthogonal direct sum).

The main point being the last item which says, in particular, that the range of T can be expanded in a convergent series of eigenvectors *i.e.* given $\epsilon > 0$, for any $v \in V$, there exists a positive integer $n(v)$ so that $\|T(v) - \sum_{i=1}^n \lambda_i T(v_i)\| < \epsilon$.

Here are some consequences of the spectral theorem.

Exercise 5.3.8. Prove that:

$$\sum_{i=1}^{\infty} \lambda_i^2 \dim V_{\lambda_i} = \|T\|.$$

In particular, for any nonzero eigenvalue, $\lambda_i^2 \dim V_{\lambda_i} \leq \|T\|$. Hence $\dim V_{\lambda_i} \leq \frac{\|k\|_2^2}{\lambda_i^2}$.

Exercise 5.3.9. Notice that in the case of a finite dimensional operator this just amounts to the fact that a self-adjoint operator is unitarily diagonalizable with real eigenvalues.

Before turning to the proof of Theorem 5.3.7 we need some preparatory results.

Lemma 5.3.10. *Let $T : V \rightarrow V$ be a compact operator and $\delta > 0$, then the number of eigenvectors of norm 1 with eigenvalues $\lambda > \delta$ is finite.*

In particular the number of such distinct (*i.e.* orthogonal) eigenspaces is finite. In particular, the total number of such distinct eigenspaces associated with positive eigenvalues is countable (even if V has an uncountable orthonormal basis!). Moreover, (using the positive integers) if we order the positive eigenvalues, λ_n , in decreasing order, then $\lambda_n \rightarrow 0$.

Proof. Let λ and μ be distinct eigenvalues of T both bigger than δ and v and w be the respective eigenvectors of norm 1. Then since these eigenspaces are orthogonal (see item 2), $\|Tv - Tw\|^2 = \|\lambda v - \mu w\|^2 = \lambda^2 + \mu^2 \geq 2\delta^2$. Thus $\|Tv - Tw\| \geq \sqrt{2}\delta$. Clearly if there were an infinite number of such eigenvalues there could be no convergent subsequence contradicting the fact that T is compact. \square

Lemma 5.3.11. *Let $T : V \rightarrow V$ be a bounded self-adjoint operator and W a T -invariant subspace of V . Then W^\perp is also T -invariant.*

In particular if T is compact self-adjoint and W is a closed T -invariant subspace then T restricted to W^\perp is again a compact self adjoint operator.

Proof. Let $w \in W$ and $w^\perp \in W^\perp$. Then $\langle Tw^\perp, w \rangle = \langle w^\perp, Tw \rangle = 0$ since W is T -invariant. Thus since w is arbitrary $Tw^\perp \in W^\perp$. \square

Lemma 5.3.12. *For a self-adjoint operator T ,*

$$\|T\| = \sup\{|\langle T(v), v \rangle| : \|v\| = 1\}.$$

Proof. Let $M = \sup\{|\langle T(v), v \rangle| : \|v\| = 1\}$. By the Cauchy-Schwarz inequality it is obvious that

$$\|\langle Tv, v \rangle\| \leq \|T(v)\| \cdot \|v\| \leq \|v\| = \|T\|$$

if $\|v\| = 1$, therefore M exists and $M \leq \|T\|$. It remains to prove that $\|T\| \leq M$ for which it suffices to show that $\|T(v)\| \leq M$ if $\|v\| = 1$. We

assume that $T(v) \neq 0$ and let $w = Tv/\|Tv\|$. Then $\langle Tv, w \rangle = \langle v, Tw \rangle = \|Tv\|$ and

$$\begin{aligned} 4\|Tv\| &= \langle T(v+w), v+w \rangle - \langle T(v-w), v-w \rangle \\ &\leq M\|v+w\|^2 + M\|v-w\|^2 = 4M. \end{aligned}$$

□

Proposition 5.3.13. *Let $T : V \rightarrow V$ be a compact self-adjoint operator on a Hilbert space. Then there is some w of norm 1 with $T(w) = \pm\|T\|w$.*

Proof. By previous result there is sequence of vectors v_n of norm 1 such that $\|T\| = \lim_{n \rightarrow \infty} |\langle T(v_n), v_n \rangle|$. By passing to a subsequence, we may assume that $\langle T(v_n), v_n \rangle$ converges to r which is $\|T\|$ or $-\|T\|$, and $T(v_n)$ converges to some vector v , as T is compact. Then

$$\begin{aligned} 0 \leq \|T(v_n) - rv_n\|^2 &= \|T(v_n)\|^2 - 2r\langle T(v_n), v_n \rangle + r^2\|v_n\|^2 \\ &\leq 2r^2 - 2r\langle T(v_n), v_n \rangle. \end{aligned}$$

As the right side of the inequality converges to zero therefore $\lim_{n \rightarrow \infty} \|T(v_n) - rv_n\| = 0$. On the other hand, $w = \lim_{n \rightarrow \infty} T(v_n)$ hence $\lim_{n \rightarrow \infty} rv_n = w$. For $w = r^{-1}v$, we have $T(w) = rw$.

□

Proof of the spectral theorem.

- (1) This is the exercise above.
- (2) Suppose $Tv = \lambda v$ and $Tw = \mu w$. Then $\langle Tv, w \rangle = \lambda \langle v, w \rangle$. But its also $\langle v, Tw \rangle = \bar{\mu} \langle v, w \rangle$. Therefore either $\langle v, w \rangle = 0$ or $\lambda = \bar{\mu}$, but since μ is real this would mean $\lambda = \mu$.
If $\lambda \neq 0$, then since T acts on V_λ as a nonzero scalar multiple of the identity, V_λ is finite dimensional by a remark made earlier. By Lemma 5.3.10 T has at most a countable number of positive eigenvalues. Since $-T$ is also a compact operator, T must also have at most a countable number of negative eigenvalues hence a countable number of nonzero eigenvalues.
- (3) Clearly $V_0 = \text{Ker } T$.

- (4) Finally, let $W = \overline{(\sum_{\lambda_i \neq 0} V_{\lambda_i})}$. Then we have $V = W \oplus W^\perp$. We prove that $V_0 = W^\perp$ which complete the proof. Since each $V_{\lambda_i} \subseteq V_0^\perp$, hence $V_0 \subseteq W^\perp$. Since W is clearly T -invariant, the same is true of W^\perp by Lemma 5.3.11 and moreover T restricted to W^\perp is a compact self-adjoint operator. By Proposition 5.3.13 choose w_0^\perp of norm 1 in W^\perp so that $T(w_0^\perp) = \pm \|S\| w_0^\perp$, where S is the restriction of T to W^\perp . If $\|S\| > 0$, then $w_0^\perp \in V_{\lambda_i}$ for some i . Since $V_{\lambda_i} \subseteq W$ this means $w_0^\perp = 0$ which is impossible as its norm is 1. Thus $\|S\| = 0$ so T restricted to W^\perp is zero or in other words $W^\perp \subseteq \text{Ker } T = V_0$.

Since we were within $C(X)$ and estimated by the sup norm we get

Corollary 5.3.14. *The range, $T_k(L_2(X))$, can be expanded in a uniformly convergent series of eigenfunctions of T_k with $\lambda_i \neq 0$.*

$$T_k(f) = \sum_{i=1}^{\infty} f_{\lambda_i}.$$

Moreover each eigenfunction ϕ of T_k associated with a nonzero eigenvalue is continuous because $T(\phi) = \lambda\phi$ and $T(\phi)$ is continuous, hence so is ϕ .

This completes our study of compact operators. We remark that using the spectral theorem for compact integral operators proven above, one can also get the following theorem which is important in the study of compact Riemann surfaces or, more generally, compact hyperbolic manifolds of higher dimension. Here G is a non-compact simple Lie group and H is the (discrete) fundamental group of the compact manifold. For the details the reader is referred to Representation Theory and Automorphic Functions by I.M. Gelfand et al. The definition of induced representations is given in the last section of this chapter.

Theorem 5.3.15. *Let G be a locally compact group, H a closed subgroup with G/H compact and having a finite G -invariant measure. Let σ be a finite dimensional, continuous, unitary representation of H . Then the induced representation, $\text{Ind}(H \uparrow G, \sigma)$, decomposes into a countable orthogonal direct sum of irreducible unitary representations, each of finite multiplicity (but usually of infinite dimension).*

5.4 The Peter-Weyl Theorem and its Consequences

In order to prove the Peter-Weyl theorem it will be necessary to study a certain infinite dimensional representation called the left regular representation L which is defined as follows.

The representation space of L is $L_2(G)$ and the action is given by left translation, $L_g(f)(x) = f(g^{-1}x)$, where $g, x \in G$ and $f \in L_2(G)$. We leave it to the reader to check that this is well-defined on L_2 and is a linear action. It is actually a unitary representation; that is each L_g is a unitary operator since $\langle L_g(f_1), L_g(f_2) \rangle = \langle f_1, f_2 \rangle$ for all $g \in G$, $f_1, f_2 \in L_2(G)$ because of the invariance of Haar measure.

L has another important feature. Namely it is continuous in the following sense (called strong continuity). If $g_\nu \rightarrow g$ in G and $f \in L_2(G)$, then $L_{g_\nu}(f) \rightarrow L_g(f)$. First let $f \in C(G)$. Then by compactness, f is uniformly continuous. So if $\epsilon > 0$ then $|f(g_\nu^{-1}x) - f(g^{-1}x)| < \epsilon$ whenever $g_\nu^{-1}x(g^{-1}x)^{-1} = g_\nu^{-1}xx^{-1}g = g_\nu^{-1}g \in U$, where U is a sufficiently small neighborhood of 1 in G . Hence $\|L_{g_\nu}(f) - L_g(f)\|_G \leq \epsilon$ and so also $\|L_{g_\nu}(f) - L_g(f)\|_2 \leq \epsilon$. Now since Haar measure is regular, $C(G)$ is dense in L_2 . So if $f \in L_2$ we can choose $f_1 \in C(G)$ with $\|f - f_1\| < \epsilon$. Then

$$\begin{aligned} \|L_{g_\nu}(f) - L_g(f)\|_2 &\leq \|L_{g_\nu}(f) - L_{g_\nu}(f_1)\|_2 + \|L_{g_\nu}(f_1) - L_g(f_1)\|_2 \\ &\quad + \|L_g(f_1) - L_g(f)\|_2 \leq 3\epsilon \end{aligned}$$

if $g_\nu^{-1}g \in U$.

Now let ϕ be a continuous non-negative function on G which is not identically zero. We can make it symmetric (that is $\phi(x) = \overline{\phi(x^{-1})}$) by replacing ϕ by $\phi(x) + \overline{\phi(x^{-1})}$. We can also have $\int \phi dx = 1$ by normalizing *i.e.* replacing ϕ by $\frac{\phi}{\int \phi dx}$. Let $k(x, y) = \phi(x^{-1}y)$. Then k is continuous and since ϕ is symmetric $k(x, y) = k(y, x)$. Because ϕ is real we see T_k is a compact self-adjoint Fredholm operator. In fact here $T_k(f)$ is called the convolution of ϕ and f and this is precisely the type of integral operator we are interested in. Let Ω be the set of all the eigenfunctions associated with nonzero eigenvalues of all such T_k . Then Ω and therefore also its complex linear span, *l.s.* $_{\mathbb{C}}(\Omega)$, is contained

in $C(G)$. We will now show that any continuous function on G is the uniform limit of some complex linear combination of such T_k . To do so requires a lemma, sometimes called the *approximate identity* lemma.

Lemma 5.4.1. *Let $f \in C(G)$, $\epsilon > 0$ and U be a symmetric neighborhood of 1 in G so that $|f(x) - f(y)| < \epsilon$, if $x^{-1}y \in U$. Let ϕ be a function as described above with a support contained in U . Then for all $x \in G$ $|f(x) - T_k(f)(x)| < \epsilon$.*

Proof. $|f(x) - T_k(f)(x)| = |f(x) \int_G \phi(y) - \int_G \phi(x^{-1}y)f(y)|$. But by invariance of Haar measure and the fact that ϕ non-negative this is $|f(x) \int_G \phi(x^{-1}y)dy - \int_G \phi(x^{-1}y)f(y)dy| \leq \int_G \phi(x^{-1}y)|f(x) - f(y)|dy$. Now because $\text{Supp } \phi_{x^{-1}} \subseteq xU$, we see $|f(x) - T_k(f)(x)| \leq \epsilon \int_{\text{Supp } \phi_x} \phi(x^{-1}y) = \epsilon \int_G \phi(x^{-1}y) = \epsilon$. \square

Proposition 5.4.2. *$l.s._{\mathbb{C}}(\Omega)$ is dense in $C(G)$.*

Proof. Let $f \in C(G)$, $\epsilon > 0$ and U be a symmetric neighborhood of 1 in G sufficiently small so that $|f(x) - f(y)| < \epsilon$, if $x^{-1}y \in U$. Choose a neighborhood U_1 of 1 so that $\overline{U_1} \subseteq U$ and by Urysohn's lemma a continuous function $h : G \rightarrow [0, 1]$ with $\text{Supp } h \subseteq U$ and $h \equiv 1$ on U_1 . Then this gives rise to a function ϕ as above with $\text{Supp } \phi \subseteq U$. By Lemma 5.4.1 $|f(x) - T_k(f)(x)| < \epsilon$ for all $x \in G$ so that $\|f - T_k(f)\|_G \leq \epsilon$. This proves the proposition since by spectral theorem $T_k(f)$ itself is the uniform limit of a finite linear combination of nonzero eigenfunctions associated with T_k . \square

Our next lemma shows that L restricted to V_λ gives a finite dimensional continuous unitary representation of G .

Lemma 5.4.3. *If V_λ is an eigenspace of such a T_k , where $\lambda \neq 0$. Then V_λ (which we know is a finite dimensional $\subseteq C(G) \subseteq L_2$) is invariant under L .*

Proof. We know $\int_G \phi(x^{-1}y)\psi(y)dy = \lambda\psi(x)$, $x \in G$, we apply L and replace x by $g^{-1}x$. Then

$$\int_G \phi((g^{-1}x)^{-1}y)\psi(y)dy = \lambda\psi(g^{-1}x).$$

That is

$$\int_G \phi(x^{-1}gy)\psi(y)dy = \lambda\psi(g^{-1}x).$$

Applying left invariance, the first term is

$$\int_G \phi(x^{-1}gy)\psi(y)dy = \int_G \phi(x^{-1}y)\psi(g^{-1}y)dy,$$

therefore

$$\int_G \phi(x^{-1}y)\psi_g(y)dy = \lambda\psi_g(x).$$

□

Now let Δ be the set of all matrix coefficients of all finite dimensional continuous unitary representations of G and $R(G)$ be $l.s._{\mathbb{C}}\Delta$. $R(G)$ is called the space of *representative functions* on G . We leave it to the reader to check that $R(G)$ is intrinsic to G and does not depend on the choice of basis needed to get these matrices. Notice that $R(G)$ is stable under conjugation since if ρ is a finite dimensional continuous unitary representations of G so is $\bar{\rho}$, its conjugate. If ρ is irreducible so is $\bar{\rho}$.

Exercise 5.4.4. The reader should verify these statements. Particularly the irreducibility of $\bar{\rho}$. Hint use Schur's lemma.

We now show that $l.s._{\mathbb{C}}\Omega \subseteq R(G)$ and $R(G)$ is uniformly dense in $C(G)$. To do so only requires the following.

Lemma 5.4.5. $\Omega \subseteq R(G)$.

Proof. Let $f \in \Omega$. Then for some appropriate k , $T_k(f) = \lambda f$, where $\lambda \neq 0$. Choose an orthonormal basis, ϕ_1, \dots, ϕ_n for V_λ . Then $f = \sum_{i=1}^n c_i \phi_i$. Since V_λ is invariant under L we get $L_g(\phi_i) = \sum_{j=1}^n \rho_{ij}(g)\phi_j$. That is $\phi_i(g^{-1}x) = \sum_{j=1}^n \rho_{ij}(g)\phi_j(x)$ for all $g, x \in G$. Taking $x = 1$ and replacing g by its inverse tell us $\phi_i(g) = \sum_{j=1}^n \rho_{ij}(g^{-1})\phi_j(1)$ for all $g \in G$. But since $\rho(g^{-1}) = \rho(g)^{-1} = \rho(g)^*$ and $\rho_{ij}^* = \overline{\rho_{ji}}$ we see $\phi_i(g) = \sum_{j=1}^n \overline{\rho_{ji}(g)}\phi_j(1)$. Thus each $\phi_i \in R(G)$ and since this is a linear space so is f . □

Corollary 5.4.6. *For a compact topological group G , $R(G)$ is uniformly dense in $C(G)$. Also $R(G)$ is dense in L_2 (with respect to $\|\cdot\|$ norm).*

This yields the following which is also called the *Peter-Weyl theorem*.

Corollary 5.4.7. *For a compact topological group G , $R(G)$ separates the points of G .*

Proof. Let $g \neq h \in G$ and suppose $r(g) = r(h)$ for all $r \in R(G)$. Choose a continuous real valued function f such that $f(g) \neq f(h)$. Since $R(G)$ is uniformly dense in $C(G)$ we can choose a representative function r so that $\|r - f\|_G < \frac{1}{2}|f(g) - f(h)|$. Since $|r(g) - f(g)|$ and $|r(h) - f(h)| \leq \frac{1}{2}|f(g) - f(h)|$, it follows that $|f(g) - f(h)| \leq |f(g) - r(g)| + |r(g) - r(h)| + |r(h) - f(h)| < |f(g) - f(h)|$, a contradiction. Now there must be an irreducible representation $\rho \in \mathcal{R}(G)$ satisfying $\rho(g) \neq \rho(h)$. For otherwise by complete reducibility all continuous finite dimensional unitary representations would take the same value on g and h . Hence $r(g) = r(h)$ for all $r \in R(G)$, a contradiction. \square

Corollary 5.4.8. *A compact topological group G is isomorphic to a closed subgroup of a product of unitary groups. Conversely, such a group is compact.*

Proof. For each $\rho \in \mathcal{R}(G)$ we get a unitary representation $\rho : G \rightarrow U_\rho$. Putting them together gives a continuous homomorphism $G \rightarrow \prod_{\rho \in \mathcal{R}(G)} U_\rho$, a product of unitary groups. Since $R(G)$ separates the points of G this map is injective. By compactness G is homeomorphic (and isomorphic) to its image which is closed. The converse is obvious. \square

Corollary 5.4.9. *Given a compact topological group G and a neighborhood U of 1 there is a closed normal subgroup H_U of G contained in U such that G/H_U is isomorphic to a closed subgroup of some $U(n, \mathbb{C})$.*

Proof. Now $G \setminus U$ is closed and therefore compact. By Corollary 5.4.7 each $g \in G \setminus U$ has $\rho_g \in \mathcal{R}(G)$ so that $\rho_g(g) \neq I$. By continuity of ρ_g there is a neighborhood, V_g of g where ρ_g is never the identity anywhere on V_g . Since these V_g cover $G \setminus U$ we have by throwing in U an open covering

of G itself. By compactness $G = U \cup V_{g_i}$, the union of a finite number of these. Consider the corresponding ρ_{g_i} . Let $H_U = \cap_i \text{Ker } \rho_{g_i}$. Then H_U is a closed normal subgroup of G . Let $\rho = \oplus_i \rho_{g_i}$. Then ρ is a finite dimensional unitary representation, $\text{Ker } \rho = H_U$ and $G/H_U = \rho(G)$ is a closed subgroup of some unitary group. Let $g \in H_U$. If g is not in U then $g \in V_{g_i}$ for some i . But then $\rho_{g_i}(g) \neq I_{\rho_{g_i}}$. On the other hand since $g \in H_U$, $\rho(g) = I_\rho$ and hence $\rho_{g_i}(g) = I_{\rho_{g_i}}$, a contradiction. \square

Corollary 5.4.10. *A compact Lie group G is isomorphic to a closed subgroup of some $U(n, \mathbb{C})$ and conversely.*

Proof. This follows immediately from Corollary 5.4.9 since G has no small subgroups. That is, there is some U containing only the subgroup $\{1\}$. Hence the H_U is trivial and so G is isomorphic to a closed subgroup of some $U(n, \mathbb{C})$. The converse follows from Cartan's theorem. \square

We make a few final remarks concerning the abelian case. Here the irreducibles are all 1-dimensional. That is, they are multiplicative characters $\chi : G \rightarrow \mathbb{T}$ and enables us to sharpen the conclusions of the Peter-Weyl theorem in this case.

The next corollary shows that if a compact group has only 1-dimensional irreducible representations it must be abelian since it is embedded in an abelian group.

Corollary 5.4.11. *For a compact abelian topological group G , the characters separate the points and the linear span of the characters is uniformly dense in $C(G)$. G is isomorphic to a closed subgroup of a product of tori and if G is a Lie group it is isomorphic to a closed subgroup of a torus.*

We can now study the (in general infinite dimensional) left regular representation, L .

Definition 5.4.12. If ρ is a finite dimensional irreducible representation of the compact group G we denote by $R(\rho)$ the linear span of the coefficients of ρ i.e. the representative functions associated with ρ .

$R(\rho)$ is a subspace of $R(G) \subseteq C(G) \subseteq L_2(G)$. By Section 5.2 dimension is d_ρ^2 . If ρ and σ are distinct in $R(G)$ then $R(\rho)$ and $R(\sigma)$ are orthogonal (see Section 5.2). Now the linear span of all $R(\rho)$, as $\rho \in \mathcal{R}(G)$, is $R(G)$. Hence by the Peter-Weyl theorem we have

Corollary 5.4.13. $L_2(G) = \overline{\bigoplus_{\rho \in R(G)} R(\rho)}$ with $\{d_\rho^{\frac{1}{2}} \rho_{ij}\}$ as an orthonormal basis.

We can look at $R(\rho)$ in another way as follows.

Proposition 5.4.14. $R(\rho)$ is both a left and right invariant subspace of L_2 . In particular, $R(\rho)$ is also invariant under inner automorphisms.

Proof. We prove left invariance. Right invariance is done similarly. Let $r(x) = \sum c_{ij} \rho_{ij}(x)$, $c_{ij} \in \mathbb{C}$, be a generic element of $R(\rho)$. Since L is a linear representation it is sufficient to show $L_g \rho_{ij} \in R(\rho)$. But $\rho(g^{-1}x) = \rho(g^{-1})\rho(x)$ so $\rho_{ij}(g^{-1}x) = \sum_k \rho_{ik}(g^{-1})\rho_{k,j}(x) \in R(\rho)$. \square

Thus we have decomposed L_2 into the orthogonal direct sum of perhaps a large number of finite dimensional (closed) *left invariant subspaces*. In order to completely analyze L we merely need to know which irreducibles occur in each of the $R(\rho)$.

Now if τ is a finite dimensional representation of G on V and ρ is an irreducible representation of G , then $[\tau : \rho]$, the multiplicity that ρ occurs in τ , is given in Corollary 5.5.5 below by $\langle \chi_\tau, \chi_\rho \rangle = \int_G \chi_\tau(x) \overline{\chi_\rho(x)} dx$. In our case $\tau = L|_{R(\rho)}$. It can be easily checked that $\chi_{L|_{R(\rho)}}(g) = d_{\bar{\rho}} \chi_{\bar{\rho}}$. We also saw that $\bar{\rho} \in \mathcal{R}(G)$ if ρ is. Thus the multiplicity of $\bar{\rho}$ in $L|_{R(\rho)}$ is $d_{\bar{\rho}} = d_\rho$. Since $\dim_{\mathbb{C}} R(\rho) = d_\rho^2$ it follows that $L|_{R(\rho)}$ contains only $\bar{\rho}$ with multiplicity $d_{\bar{\rho}}$ and nothing else. Since ρ then occurs in $L|_{R(\bar{\rho})}$ with multiplicity $d_{\bar{\rho}} = d_\rho$ we have

Corollary 5.4.15. Each irreducible of G occurs in L with a multiplicity equal to its degree.

In particular, if the group is finite one has

Corollary 5.4.16. A finite group has a finite number of inequivalent finite dimensional irreducible representation ρ_1, \dots, ρ_r . These are constrained by the requirement $\sum_{i=1}^r d_{\rho_i}^2 = |G|$.

Example 5.4.17. Let $G = S_3$, the symmetric group on 3 letters. This group has two 1-dimensional characters. These are the characters of $S_3/A_3 = \mathbb{Z}_2$ lifted to G . It has no others since $[S_3, S_3] = A_3$. S_3 must have an irreducible of degree $d > 1$ for otherwise it would be abelian see 1.4.20. Since $1^2 + 1^2 + 2^2 = 6$, the order of S_3 , we see $|\mathcal{R}(S_3)| = 3$ and the higher dimensional representation has degree 2.

If we consider the two-sided regular representation of G on L_2 (the Haar measure is both left and right invariant and left and right translations commute) then a similar analysis shows that this representation on $R(\rho)$ is now actually irreducible and equivalent to $\bar{\rho} \otimes \rho$. This can be done by calculating the character of this representation (see beginning of the next section). Here $R(\rho)$ is identified with $V_{\bar{\rho}} \otimes V_{\rho}$ and $\chi_{\bar{\rho} \otimes \rho}(g, h) = \chi_{\bar{\rho}}(g)\chi_{\rho}(h)$, $g, h \in G$. Applying Proposition 5.5.4 shows that these representations are equivalent and Corollary 5.5.6 that they are irreducible. We leave this verification to the reader as an exercise.

We now turn to the *Plancherel theorem*. Let $\rho \in \mathcal{R}(G)$ and ϕ be an L_1 function. We define $T_{\phi}(\rho) = \int_G \phi(g)\rho(g)dg$. Thus $T_{\phi}(\rho)$ is a linear operator on V_{ρ} . It is called the *Fourier transform* of ϕ at ρ and so each fixed ϕ gives an operator valued function on $\mathcal{R}(G)$, but always taking its value in a different space of operators. Since $\phi \in L_1$ and the coefficients of ρ are bounded, $T_{\phi}(\rho)$ always exists.

Now let ϕ and $\psi \in L_2(G)$. We want to calculate $\langle \phi, \psi \rangle_{L_2}$ by Fourier analysis. This is exactly what the Plancherel theorem does

$$\langle \phi, \psi \rangle_{L_2} = \sum_{\rho \in \mathcal{R}(G)} d_{\rho} \operatorname{tr}(T_{\phi}(\rho)T_{\psi}(\rho)^*),$$

where $*$ means the adjoint operator. Since $L_2 \subseteq L_1$ the Fourier transform applies. To prove this using polarization we may take $\psi = \phi$. Then we get the following formula involving the Hilbert Schmidt norm of an operator.

$$\|\phi\|_2^2 = \sum_{\rho \in \mathcal{R}(G)} d_{\rho} \operatorname{tr}(T_{\phi}(\rho)T_{\phi}(\rho)^*).$$

Matrix calculations similar to those involved in the orthogonality relations themselves yield

$$d_\rho \operatorname{tr}(T_\phi(\rho)T_\phi(\rho)^*) = \sum_{i,j=1}^{d_\rho} |d_\rho^{\frac{1}{2}} \int_G \phi(g)\rho_{ij}(g)dg|^2.$$

Since by Corollary 5.4.13 $\{d_\rho^{\frac{1}{2}}\rho_{ij}\}$ form an orthonormal basis, the claim follows from the identity

$$\|\phi\|_2^2 = \sum \langle \phi, d_\rho^{\frac{1}{2}}\rho_{ij} \rangle^2.$$

Corollary 5.4.18. *Let G be a compact Lie group and ρ_0 be a faithful finite dimensional unitary representation as guaranteed by Corollary 5.4.10. Then each irreducible representation ρ of G is an irreducible component of $\otimes^n \rho_0 \otimes^m \bar{\rho}_0$ for some n and m non-negative integers.*

Proof. Consider the representative functions \mathcal{F} associated with irreducible subrepresentations of $\otimes^n \rho_0 \otimes^m \bar{\rho}_0$ as n and m vary. Since ρ_0 is faithful, \mathcal{F} separates the points. It is clearly stable under conjugation, contains the constants and is a subalgebra of $C(G)$. By the Stone-Weierstrass theorem \mathcal{F} is dense in $C(G)$. Let $\sigma \in \mathcal{R}(G)$. If σ is not equivalent to some irreducible component of $\otimes^n \rho_0 \otimes^m \bar{\rho}_0$ for some n and m then $R(\sigma) \perp \mathcal{F}$. On the other hand given σ_{ij} we can choose $f_\nu \rightarrow \sigma_{ij}$ uniformly on G . Therefore $\langle f_\nu, \sigma_{ij} \rangle = 0 \rightarrow \langle \sigma_{ij}, \sigma_{ij} \rangle \neq 0$, a contradiction. \square

We conclude this section with the following result concerning infinite dimensional representations of a compact group.

Theorem 5.4.19. *Let γ be a strongly continuous unitary representation of a compact group, G , on a complex Hilbert space, V . Then γ is the direct sum of finite dimensional, continuous, irreducible unitary subrepresentations.*

Notice that here the multiplicities need not be finite. Also observe that it follows from Theorem 5.4.19 that irreducible unitary representations of a compact group on a Hilbert space are finite dimensional.

Before turning to the proof of this result we extend the definition of the Fourier transform to the case of a strongly continuous unitary representation, γ on a complex Hilbert space V . For $f \in L_1(G)$ define $T_f(\gamma) = \int_G f(x)\gamma_x dx$. This is the integral of an operator valued function. Hence if the integral exists, the result is an operator. This integral does indeed exist since $f \in L_1$ and the coefficients, $x \mapsto \langle \gamma_x(v), w \rangle$, ($v, w \in V$) are all bounded. Hence this operator has the property that $\langle T_f(\gamma)(v), w \rangle = \int_G f(x)\langle \gamma_x(v), w \rangle dx$. Also, since γ is unitary and $T_f(\gamma)(v) = \int_G f(x)\gamma_x(v) dx$. It follows that for $v \in V$,

$$\|T_f(\gamma)(v)\| \leq \|f\|_1 \|v\|. \quad (5.1)$$

Lemma 5.4.20. *For each $f \in L_1$, T_f preserves all γ -invariant subspaces of V .*

Proof. Let W be an invariant subspace and W^\perp be its orthocomplement. We want to prove that if $w \in W$, then $T_f(\gamma)(w) \in W$. That is, $\langle T_f(\gamma)(w), w^\perp \rangle = 0$ for all $w^\perp \in W^\perp$. But $\langle T_f(\gamma)(w), w^\perp \rangle = \int_G f(x)\langle \gamma_x(w), w^\perp \rangle dx$. Since $\gamma_x(w) \in W$ for all $x \in G$ and $w^\perp \in W^\perp$ the integrand is zero. \square

Proof of Theorem 5.4.19. Ordering the set of all orthonormal, finite dimensional, irreducible, γ -invariant subspaces of V by inclusion and applying Zorn's lemma shows there is such a maximal set. Let W be the closure of the subspace generated by the subspaces in this maximal set. We want to show that $W = V$. In any case W is a γ -invariant subspace. Hence since γ is unitary the orthocomplement W^\perp is also a γ -invariant subspace (by the same argument we used for the finite dimensional case). Choose a family of functions (approximate identity), f_U , consisting of continuous non-negative functions on G with $\text{Supp } f_U \subseteq U$ and $\int_G f_U dx = 1$, as in the beginning of Section 5.4. By Lemma 5.4.20 for each U , T_{f_U} preserves W^\perp . Hence if $v \in W^\perp$, $T_{f_U}v \in W^\perp$ for all U . Let us assume there is such a nonzero v .

Now let $v_1 \in V$. By the Schwarz inequality $|\langle T_{f_U}(v) - v, v_1 \rangle| \leq \int_G |f_U \gamma_x(v) - v| dx \|v_1\|$. Since f_U is non-negative, is supported on U and has integral 1 we get $|\langle T_{f_U}v - v, v_1 \rangle| \leq \sup_{x \in U} \|\gamma_x(v) - v\| \|v_1\|$.

Taking the sup over all v_1 with $\|v_1\| \leq 1$ we conclude that $\|T_{f_U}v - v\| \leq \sup_{x \in U} \|\gamma_x(v) - v\|$. Hence by strong continuity of γ , $\|T_{f_U}v - v\| \rightarrow 0$ as U shrinks to 1. (The reader will notice the similarity with Lemma 5.4.1.) Finally, since $v \neq 0$, $T_{f_U}v \neq 0$ for some small U . Next we apply the Peter-Weyl theorem to uniformly approximate f_U by a representative function $r \in \mathcal{R}(G)$ to within $\frac{\epsilon}{\|v\|}$. Since this is also an approximation in L_2 we see $\|T_{f_U}v - T_r(v)\| = \|T_{f_U-r}v\|$. The latter is $\leq \|f_U - r\|_1 \|v\| \leq \|f_U - r\|_2 \|v\| < \epsilon$. Thus also $T_r(v) \neq 0$.

Now the linear span of all left translates by $x \in G$ of r lies in a finite dimensional subspace \mathcal{F} of $L_2(G)$ and so gives a finite dimensional continuous unitary representation of G . Let f_1, \dots, f_n be an orthonormal basis of \mathcal{F} . Then $L_g(f_i) = \sum_{j=1}^n c_{ij}(g)f_j$. These functions all being in $L_2(G)$ and hence also in $L_1(G)$. Therefore for $g \in G$ and $i = 1, \dots, n$,

$$\gamma_g T_{f_i}(v) = \gamma_g \int_G f_i(x) \gamma_x(v) dx = \int_G f_i(x) \gamma_g \gamma_x(v) dx = \int_G f_i(x) \gamma_{gx}(v) dx.$$

Now by invariance of the integral under translation this is

$$\int_G f_i(g^{-1}x) \gamma_x(v) dx = T_{L_g(f_i)}(v) = T_{\sum_{j=1}^n c_{ij}(g)f_j}(v) = \sum_{j=1}^n c_{ij}(g) T_{f_j}(v).$$

Hence, $T_{\mathcal{F}}(v)$ is a finite dimensional γ -invariant subspace of V which lies in W^\perp by Lemma 5.4.20. Moreover, it is nontrivial since $T_r(v) \neq 0$. This means it lies in W , a contradiction.

5.5 Characters and Central Functions

Definition 5.5.1. Let ρ be a finite dimensional, continuous, unitary representation of G . We shall call $\chi_\rho(g) = \text{tr}(\rho(g))$, $g \in G$ the *character* of ρ . Then $\chi_\rho : G \rightarrow \mathbb{C}$ is a bounded continuous function on G .

Exercise 5.5.2. Why is χ_ρ bounded? Where does it take its largest absolute value?

Corollary 5.5.3. Let ρ and σ be a finite dimensional, continuous, irreducible, unitary representation of G . Then $\langle \chi_\rho, \chi_\sigma \rangle = 0$ if ρ and σ are inequivalent and $\langle \chi_\rho, \chi_\rho \rangle = 1$.

Thus the set $\mathcal{X}(G)$ consisting of the characters of the irreducible unitary representations of G form an orthonormal family of functions in $L_2(G)$. In particular they are linearly independent.

Proof. We have,

$$\sum_{i=1}^{d_\rho} \rho_{ii}(g) \left(\sum_{j=1}^{d_\sigma} \overline{\sigma_{jj}(g)} \right) = \sum_{i=1}^{d_\rho} \sum_{j=1}^{d_\sigma} \rho_{ii}(g) \overline{\sigma_{jj}(g)}.$$

Hence

$$\int_G \chi_\rho \overline{\chi_\sigma} dg = \sum_{i=1}^{d_\rho} \sum_{j=1}^{d_\sigma} \int_G \rho_{ii}(g) \overline{\sigma_{jj}(g)} dg.$$

This is clearly 0 if ρ and σ are inequivalent. Now if $\sigma = \rho$, then

$$\|\chi_\rho\|_2^2 = \sum_{i=1}^{d_\rho} \sum_{j=1}^{d_\rho} \int_G \rho_{ii}(g) \overline{\rho_{jj}(g)} dg.$$

If $i \neq j$ we get 0. Hence $\|\chi_\rho\|_2^2 = \sum_{i=1}^{d_\rho} \int_G \rho_{ii}(g) \overline{\rho_{ii}(g)} dg = 1$ by Theorem 5.2.1.

□

Proposition 5.5.4. *Let ρ and σ be finite dimensional, continuous, unitary representation of G . Then ρ and σ are equivalent if and only if they have the same character.*

Proof. Evidently, equivalent representations have the same character. We now suppose $\chi_\rho \equiv \chi_\sigma$. Decompose ρ and σ into irreducibles. $\rho = \sum_{i=1}^r n_i \rho_i$, $n_i > 0$ and $\sigma = \sum_{i=k}^s m_i \rho_i$, $m_i > 0$.

After renumbering we can consider the overlap to be from ρ_k, \dots, ρ_r . Then

$$0 = \chi_\rho - \chi_\sigma = \sum_{i=1}^{k-1} n_i \chi_{\rho_i} + \sum_{i=k}^r (n_i - m_i) \chi_{\rho_i} + \sum_{i=r+1}^s -m_i \chi_{\rho_i}.$$

Since the χ_{ρ_i} are linearly independent we conclude $n_i = 0$ for $i = 1, \dots, k-1$, $n_i = m_i$ for $i = k, \dots, r$ and $-m_i = 0, i = r+1, \dots, s$.

But since the n_i and m_i are positive $k = 1$, $r = s$ and $n_i = m_i$ for all $i = 1, \dots, r$. That is, ρ and σ are equivalent. \square

The next result follows from the orthonormality of $\mathcal{X}(G)$ in a similar manner. We leave its proof to the reader as an exercise.

Corollary 5.5.5. *Let ρ be a finite dimensional, continuous, unitary representation of G whose decomposition into irreducibles is $\rho = \sum_{i=1}^r n_i \rho_i$, $n_i > 0$. Then $\|\chi_\rho\|_2^2 = \sum_{i=1}^r n_i^2$. In particular ρ is irreducible if and only if $\|\chi_\rho\|_2^2 = 1$. Moreover, the multiplicity of an irreducible ρ_i in ρ is $\langle \chi_\rho, \chi_{\rho_i} \rangle$.*

We can use our irreducibility criterion to study tensor product representations. Let G and H be compact groups and ρ and σ are finite dimensional continuous representations of G and H respectively. Form the representation $\rho \otimes \sigma$ of $G \times H$ on $V_\rho \otimes V_\sigma$ by defining $\rho \otimes \sigma(g, h) = \rho_g \otimes \sigma_h$.

We leave it to the reader to check that this is a continuous finite dimensional representation of $G \times H$. It is actually unitary, but this does not matter since everything is equivalent to a unitary representation anyway.

Corollary 5.5.6. *If $\rho \in \mathcal{R}(G)$ and $\sigma \in \mathcal{R}(H)$, then $\rho \otimes \sigma \in \mathcal{R}(G \times H)$. Conversely, all irreducibles of $G \times H$ arise in this way.*

Proof. If dg and dh are normalized Haar measures on G and H respectively then $dgdh$ is normalized Haar measure on the compact group $G \times H$. Now $\chi_{\rho \otimes \sigma}(g, h) = \chi_\rho(g)\chi_\sigma(h)$. Hence

$$\|\chi_{\rho \otimes \sigma}\|_2^2 = \int_G \int_H \chi_\rho(g)\chi_\sigma(h)\overline{\chi_\rho(g)\chi_\sigma(h)}dgdh = \|\chi_\rho\|_2^2\|\chi_\sigma\|_2^2 = 1 \cdot 1 = 1,$$

proving the irreducibility.

Now let $\tau \in \mathcal{R}(G \times H)$ and consider, as before, all $\rho \otimes \sigma$ where $\rho \in \mathcal{R}(G)$ and $\sigma \in \mathcal{R}(H)$. Let $f \in C(G \times H)$. By the Stone-Weierstrass theorem f can be uniformly approximated by the functions of the form $\sum_{i=1}^n g_i(x)h_i(y)$, where $g_i \in C(G)$ and $h_i \in C(H)$. But by the Peter-Weyl theorem these in turn can be uniformly approximated

by $\phi(x, y) = \sum_{i=1}^n r_i(x)s_i(y)$, where $r_i \in R(G)$ and $s_i \in R(H)$. Hence Φ the collection of these ϕ 's are the representative functions of the irreducible representations of $G \times H$ which are of the form $\rho \otimes \sigma$, form a uniformly dense linear subspace of $C(G \times H)$. If an irreducible representation τ is not of the form $\rho \otimes \sigma$, then its coefficients must be perpendicular to Φ and therefore to all of $L_2(G \times H)$. In particular it must be orthogonal to itself, a contradiction. \square

Definition 5.5.7. A function $f : G \rightarrow \mathbb{C}$ is called a *central function* or a *class function* if $f(xy) = f(yx)$ for all $x, y \in G$. Equivalently, f is a class function if and only if $f(gxg^{-1}) = f(x)$ for all $g, x \in G$. That is, f is constant on conjugacy classes of G . We denote the central functions by $C(G)^G$.

Exercise 5.5.8. Show that these two definitions are equivalent.

Obvious examples of class functions are characters of finite dimensional, continuous representations ρ of G and since a linear combination of a class function is again such a function, the linear span of all characters is a class function. Pursuing this idea somewhat further, it is quite clear that the uniform limit (even the pointwise limit if the limiting function is continuous) of class functions is again a class function. Thus we know that the elements of $\overline{\mathcal{X}(G)}$ are class functions. It turns out that the converse is also true. Namely,

Theorem 5.5.9. *Every central function is a uniform limit of functions in $\mathcal{X}(G)$ and conversely.*

Before turning to the proof we need a pair of lemmas.

Lemma 5.5.10. *Let $\rho \in \mathcal{R}(G)$ and $r \in R(\rho)$. If r is central, then $r = \lambda\chi_\rho$, where $\lambda \in \mathbb{C}$.*

Proof. Here $r(x) = \sum_{i,j=1}^{d_\rho} c_{ij}\rho_{ij}(x)$. Since $r(x) = r(gxg^{-1})$ we conclude upon substituting and taking into account the linear independence of ρ_{ij} that $C = \rho(g)^t C \overline{\rho(g)}$. Alternatively, $\rho(g)\bar{C} = \bar{C}\rho(g)$ for all $g \in G$, where C is the matrix of c_{ij} . By Schur's lemma \bar{C} is a scalar multiple of the identity and hence so is C . \square

Lemma 5.5.11. *Suppose that $f \in R(G)$ is central then f is in the linear span of $\mathcal{X}(G)$.*

Proof. $f = \sum_{i=1}^n r_i$, where $r_i \in R(\rho_i)$ and the ρ_i are distinct in $\mathcal{R}(G)$. Applying the assumption $f(gxg^{-1}) = f(x)$ and taking into account the linear independence of the r_i and Proposition 5.4.14 tells us each r_i is itself a class function. Hence by Lemma 5.5.10 each $r_i = \lambda_i \chi_{\rho_i}$ and $f = \sum_{i=1}^n \lambda_i \chi_{\rho_i}$. \square

Proof of Theorem 5.5.9. We first define a projection operator $\# : C(G) \rightarrow C(G)^G$ via the formula $f^\#(x) = \int_G f(gxg^{-1})dg$. It is easy to see that this gives a continuous function $f^\#$, the operator is norm decreasing $\|f^\#\|_G \leq \|f\|_G$ and f is central if and only if $f = f^\#$. We leave these details to the reader to check.

Let $f \in C(G)^G$. Then by the Peter-Weyl theorem f can be uniformly approximated on all of G by representative functions, ϕ , $\|f - \phi\|_G < \epsilon$. Apply the $\#$ operator and get $\|f^\# - \phi^\#\|_G = \|(f - \phi)^\#\|_G \leq \|f - \phi\|_G < \epsilon$. On the other hand $f = f^\#$ and $\phi^\#$ is a central representative function which by Lemma 5.5.11 is a linear combination of characters of $\mathcal{R}(G)$. Thus f is the uniform limit of a linear combination of irreducible characters.

Exercise 5.5.12. The reader should verify the various properties of $\#$ mentioned above as these are necessary to complete the proof of Theorem 5.5.9.

Corollary 5.5.13. $\mathcal{X}(G)$ separates the conjugacy classes of G .

Proof. Let C_x and C_y be disjoint conjugacy classes. Since these are disjoint compact sets Urysohn's lemma tells us there is $f \in C(G)$ with $f|_{C_x} = 0$ and $f|_{C_y} = 1$. Applying $\#$ yields $f^\#|_{C_x} = 0$ and $f^\#|_{C_y} = 1$. Now approximate $f^\#$ by a linear combination of characters to within $\frac{1}{2}$ by Theorem 5.5.9. If $\chi_\rho(x) = \chi_\rho(y)$ for every $\rho \in \mathcal{R}(G)$ this would give a contradiction. Hence the conclusion. \square

Thus the irreducible representations of a compact group are in bijective correspondence with the irreducible characters and, if the group

is finite, the characters are in bijective correspondence with the set of conjugacy classes. This is the basis of the so-called character tables of a finite group. Vertically the characters are listed and horizontally the conjugacy classes are listed. Then the table must be filled in with the value of that character on that particular conjugacy class. For example as we saw above, S_3 has exactly 3 characters and therefore also 3 conjugacy classes.

We conclude this section with the functional equation for a character of a representation in $\mathcal{R}(G)$.

Theorem 5.5.14. *Let $f = \chi_\rho$ be the character of a finite dimensional, continuous, irreducible, unitary representation ρ . Then for all $x, y \in G$,*

$$f(x)f(y) = f(1) \int_G f(gxg^{-1}y)dg.$$

Conversely, if f is a continuous function $G \rightarrow \mathbb{C}$, not identically zero satisfying this equation, then $\frac{f}{f(1)} = \frac{\chi_\rho}{\chi_\rho(1)}$, for a unique $\rho \in \mathcal{R}(G)$.

Proof. We extend the $\#$ operator defined earlier on functions to representations. For any finite dimensional continuous unitary representation, ρ , let $\rho^\#(x) = \int_G \rho(gxg^{-1})dg$, giving an operator valued function on G . For $y \in G$ using invariance of dg we get

$$\begin{aligned} \rho(y)\rho^\#(x)\rho(y)^{-1} &= \int_G \rho(y)\rho(gxg^{-1})\rho(y)^{-1}dg = \int_G \rho((gy)x(gy)^{-1})dg \\ &= \int_G \rho(gxg^{-1})dg = \rho^\#(x). \end{aligned}$$

Thus $\rho^\#(x)$ is an intertwining operator. If ρ is irreducible $\rho^\#(x) = \lambda(x)I$ and taking traces shows $\lambda(x) = \frac{\text{tr}(\rho^\#(x))}{d_\rho}$. On the other hand,

$$\text{tr}(\rho^\#(x)) = \int_G \text{tr}(\rho(gxg^{-1}))dg = \chi_\rho(x),$$

so that for all $x \in G$,

$$\int_G \rho(gxg^{-1})dg = \frac{\chi_\rho(x)}{d_\rho}I.$$

Hence

$$\int_G \rho(gxg^{-1}y)dg = \frac{\chi_\rho(x)}{d_\rho} \rho(y).$$

Taking traces yields the functional equation

$$\int_G \chi_\rho(gxg^{-1}y)dg = \frac{\chi_\rho(x)\chi_\rho(y)}{d_\rho}.$$

Conversely, let f be an arbitrary continuous function satisfying the functional equation. From it we see $f(1) \neq 0$ for otherwise $f \equiv 0$. Let $y = 1$ in the equation. Then $f(x)f(1) = f(1) \int_G f(gxg^{-1})dg = f(1)f^\#(x)$. Since $f(1) \neq 0$ $f(x) = f^\#(x)$ so f is central. We will show that for every $\rho \in \mathcal{R}(G)$ and every $x \in G$,

$$\frac{f(x)}{f(1)} \langle f, \chi_\rho \rangle = \frac{\chi_\rho(x)}{\chi_\rho(1)} \langle f, \chi_\rho \rangle. \quad (5.2)$$

Having done so we complete the proof by choosing a $\rho \in \mathcal{R}(G)$ such that $\langle f, \chi_\rho \rangle \neq 0$. For then we can cancel and conclude from (5.2) that

$$\frac{f(x)}{f(1)} = \frac{\chi_\rho(x)}{\chi_\rho(1)}. \quad (5.3)$$

Since f is central such a ρ must exist by Theorem 5.5.9. The ρ satisfying (5.3) is unique because the characters of distinct representations are linearly independent. It remains only to prove (5.2). To do so consider $\int_G \int_G f(gxg^{-1}y) \overline{\chi_\rho(y)} dg dy$. By hypothesis this is

$$\int_G \int_G \overline{\chi_\rho(y)} \frac{f(x)f(y)}{f(1)} dg dy = \frac{f(x)}{f(1)} \int_G f(y) \overline{\chi_\rho(y)} dy = \frac{f(x)}{f(1)} \langle f, \chi_\rho \rangle.$$

On the other hand by Fubini's theorem, left translating

$$\begin{aligned} \int_G \int_G f(gxg^{-1}y) \overline{\chi_\rho(y)} dg dy &= \int_G \left(\int_G f(gxg^{-1}y) \overline{\chi_\rho(y)} dy \right) dg \\ &= \int_G \left(\int_G f(y) \overline{\chi_\rho(gx^{-1}g^{-1}y)} dy \right) dg. \end{aligned}$$

Using $\chi_\rho(t^{-1}) = \overline{\chi_\rho(t)}$, the latter is

$$\begin{aligned} \int_G \int_G f(y) \chi_\rho(y^{-1} g x g^{-1}) dy dg &= \int_G f(y) \left(\int_G \chi_\rho(y^{-1} g x g^{-1}) dg \right) dy \\ &= \int_G f(y) \left(\int_G \chi_\rho(g x g^{-1} y^{-1}) dg \right) dy. \end{aligned}$$

By the part of the theorem already proved this is just

$$\int_G f(y) \frac{\chi_\rho(x) \chi_\rho(y^{-1})}{\chi_\rho(1)} dy = \frac{\chi_\rho(x)}{\chi_\rho(1)} \int_G f(y) \overline{\chi_\rho(y)} dy$$

or

$$\frac{\chi_\rho(x)}{\chi_\rho(1)} \langle f, \chi_\rho \rangle.$$

□

5.6 Induced Representations

We now study induced representations of a compact group, G . Recall (see Theorem 2.3.5) that if H is a closed subgroup of G and dg and dh are the respective normalized Haar measures, then there is a (finite) G -invariant measure μ on the homogeneous space G/H satisfying

$$\int_G f(g) dg = \int_{G/H} \int_H f(gh) dh d(\mu).$$

Now suppose we have a finite dimensional representation σ of H on V_σ . We now define the induced representation of σ to G . This representation, written $\text{Ind}(H \uparrow G, \sigma)$, will be infinite dimensional unless H has finite index in G . We consider only finite dimensional, σ , to avoid technical difficulties and because most of the applications we are interested in are in this situation.

Consider the vector space \mathcal{W} consisting of functions $F : G \rightarrow V_\sigma$ satisfying

- (1) F is measurable,
- (2) $F(gh) = \sigma(h)^{-1} F(g)$, for $h \in H$ and $g \in G$,

$$(3) \int_{G/H} \|F(g)\|_{V_\sigma}^2 d(\mu)(\bar{g}) < \infty.$$

Such functions clearly form a complex vector space under pointwise operations. If $\langle \cdot, \cdot \rangle_{V_\sigma}$ denotes the Hermitian inner product on V_σ we can use this to define an inner product on this space as follows. For F_1 and F_2 here, the function $g \mapsto \langle F_1(g), F_2(g) \rangle_{V_\sigma}$ is a continuous function on G , which by condition 2 descends to a function on G/H . In particular, $g \mapsto \|F(g)\|_{V_\sigma}^2$ is a non-negative measurable function on G/H .

Now \mathcal{W} is actually a Hilbert space whose inner product is given by $\langle F_1, F_2 \rangle = \int_{G/H} \langle F_1(g), F_2(g) \rangle_{V_\sigma} d(\mu)(\bar{g})$. This inner product converges by the Schwarz inequality which comes built in.

$$\begin{aligned} & \int_{G/H} \langle F_1(g), F_2(g) \rangle_{V_\sigma} d(\mu)(\bar{g}) \\ & \leq \int_{G/H} \|F_1(g)\|_{V_\sigma}^2 d(\mu)(\bar{g}) \int_{G/H} \|F_2(g)\|_{V_\sigma}^2 d(\mu)(\bar{g}). \end{aligned}$$

Now let G act on \mathcal{W} by left translation $(x \cdot F)(g) = F(x^{-1}g)$, where $F \in \mathcal{W}$ and $x, g \in G$.

Proposition 5.6.1. *Ind($H \uparrow G, \sigma$) is a unitary representation of G on \mathcal{W} .*

Exercise 5.6.2. The proof of this is routine and is left to the reader. We also leave to the reader to check that the left regular representation, L acting on L_2 , is an induced representation. Here σ is the trivial 1-dimensional representation of $H = \{1\}$. (This is a good example of an induced representation to keep in mind.)

We now show \mathcal{W} contains a certain dense set of functions to be described below. Let $f \in C(G, V_\sigma)$, the continuous vector valued functions on G and define for $x \in G$, $F_f(x) = \int_H \sigma(h)f(xh)dh$. Since the integrand is a V_σ valued continuous function on H which is compact the integral exists and is a V_σ valued function $F_f : G \rightarrow V_\sigma$ on G .

Lemma 5.6.3. *The F_f are continuous and in \mathcal{W} .*

Proof. We prove 2). After we show F_f is continuous, 1) and 3) follow automatically.

$$\begin{aligned} F_f(gh_1) &= \int_H \sigma(h)f(gH_1h)dh = \int_H \sigma(h_1^{-1}h)f(gh)dh \\ &= \int_H \sigma(h_1^{-1})\sigma(h)f(gh)dh = \sigma(h_1^{-1}) \int_H \sigma(h)f(gh)dh. \end{aligned}$$

Thus $F_f(gh_1) = \sigma(h_1^{-1})F_f(g)$ proving 2).

Now since f is uniformly continuous given $\epsilon > 0$ there is a neighborhood U of 1 in G so that $\|f(xh) - f(yh)\|_{V_\sigma} < \epsilon$, whenever $h \in H$ and $xy^{-1} \in U$. Therefore

$$\|F_f(x) - F_f(y)\|_{V_\sigma} \leq \int_H \|\sigma(h)\| \|f(xh) - f(yh)\|_{V_\sigma} dh.$$

Since σ is unitary we see if $xy^{-1} \in U$, then $\|F_f(x) - F_f(y)\|_{V_\sigma} < \epsilon$. \square

Lemma 5.6.4. *The F_f are dense in \mathcal{W} .*

Proof. Clearly the continuous functions in \mathcal{W} form a dense subspace of \mathcal{W} . We will actually show that the F_f are not only dense, but actually comprise all continuous functions in \mathcal{W} . Let F_1 be any continuous function satisfying 2). We want to find an $f \in C(G, V_\sigma)$ so that

$$\|F_1 - F_f\|^2 = \int_{G/H} \langle F_f - F_1, F_f - F_1 \rangle d\mu(\bar{g})$$

is small. Now $F_1(g) = \sigma(h)F_1(gh)$, so

$$F_1(g) = \int_H F_1(g)dh = \int_H \sigma(h)F_1(gh)dh.$$

On the other hand, $F_f(g) = \int_H \sigma(h)f(gh)dh$. Therefore since σ is unitary $\|F_1 - F_f\|^2 = \int_{G/H} \int_H \|f(gh) - F_1(gh)\|_{V_\sigma}^2 dh d\mu(\bar{g})$. Taking $f = F_1$, then this last integral is zero so $F_f = F_1$. Since they are both continuous they are identically equal on G . Therefore the F_f consist of all continuous functions in \mathcal{W} , and hence they are dense in \mathcal{W} . \square

Corollary 5.6.5. $\text{Ind}(H \uparrow G, \sigma)$ is a strongly continuous unitary representation of G on \mathcal{W} .

Proof. We will show that if $g_\nu \rightarrow g$, and $F \in \mathcal{W}$ is fixed, then $\|L_{g_\nu}(F) - L_g(F)\| \rightarrow 0$. Since $\text{Ind}(H \uparrow G, \sigma)$ is a unitary representation we have $\|L_{g_\nu}(F) - L_g(F)\| = \|L_{g^{-1}g_\nu}(F) - F\|$ so we may as well assume $g = 1$. Also if we were to prove this for all F_f , then by density it would hold for all $F \in \mathcal{W}$. We leave this to be checked by the reader. (It is essentially the same argument as in the one for the regular representation in the third paragraph of section 4.) Thus we may assume F is continuous. Hence F is uniformly continuous and $\|F(x^{-1}g) - F(g)\|_{V_\sigma}^2 < \epsilon^2$, if $x \in U$ a neighborhood of 1 and $g \in G$. Then if $x \in U$,

$$\|L_x(F) - F\|^2 = \int_{G/H} \|L_x F(\bar{g}) - F(\bar{g})\|_{V_\sigma}^2 < \epsilon^2 \mu(G/H).$$

So $\|L_x(F) - F\| < \epsilon$. □

Since by Theorem 5.4.19 any continuous unitary representation of a compact group on a Hilbert space is the direct sum of finite dimensional continuous irreducible unitary representations. In particular,

Corollary 5.6.6. $\text{Ind}(H \uparrow G, \sigma)$ is a direct sum of finite dimensional, continuous, irreducible unitary representations of G on \mathcal{W} .

Exercise 5.6.7. The following is a consequence of the fact that a continuous function $F : G \rightarrow V_\sigma$ which satisfies condition 2) is determined by its values on coset representatives of H in G and its proof is left to the reader.

Corollary 5.6.8. (1) $\dim_{\mathbb{C}} \mathcal{W}$ is infinite unless $[G : H]$ is finite.

(2) If $[G : H]$ is finite, then $\dim_{\mathbb{C}} \mathcal{W} = [G : H] \dim_{\mathbb{C}} V_\sigma$.

The next result includes the possibility that γ could be induced, or finite dimensional.

Proposition 5.6.9. Let G be a compact group and γ a strongly continuous representation of G on a Hilbert space, V and $\rho \in \mathcal{R}(G)$. Then $[\gamma : \rho] = \dim_{\mathbb{C}} \text{Hom}_G(V_\rho, V)$.

Proof. By 5.4.19 we can write V as the orthogonal direct sum of finite dimensional irreducible continuous unitary subrepresentations, $(V_i, \gamma|_{V_i})$, where $i \in I$ and $V = \bigoplus V_i$. Let π_i be the orthogonal projection of V onto V_i . Partition $I = I_1 \cup I_2$, where I_1 contains those representations equivalent to ρ , while I_2 contains those representations which are not equivalent to ρ . For $T \in \text{Hom}_G(V_\rho, V)$, $\pi_i \circ T \in \text{Hom}_G(V_\rho, V_i)$. So if $i \in I_2$, Schur's lemma tells us $\pi_i \circ T = 0$, while if $i \in I_1$, $\pi_i \circ T$ is a scalar multiple of the identity. Thus the former components have dimension 0 while the latter have dimension 1. Let W be the closure of the sum of the V_i for $i \in I_1$. Hence the dimension of $\text{Hom}_G(V_\rho, V)$ is the same as that of $\text{Hom}_G(V_\rho, W)$ which is the cardinality of I_1 . \square

We now come to the Frobenius reciprocity theorem a particular case of which states that each irreducible representation ρ of G is contained in the induced from σ with the same multiplicity that its restriction contains the irreducible σ of H . In particular the multiplicity of ρ in the induced is always finite.

Theorem 5.6.10. *Let G be a compact group, H be a closed subgroup, σ be any finite dimensional continuous unitary representation of H and ρ a finite dimensional continuous unitary representation of G . Then*

$$\text{Hom}_G(\rho, \text{Ind}(H \uparrow G, \sigma)) \simeq \text{Hom}_H(\rho|_H, \sigma). \quad (5.4)$$

In particular, these have the same dimension. Hence by Proposition 5.6.9 if ρ and σ are each irreducible, then $[\text{Ind}(H \uparrow G, \sigma) : \rho] = [\rho|_H : \sigma]$.

Proof. Our proof of this result is functorial, In this way it does not really depend on compactness of G at all. For example it also works for any (not necessarily unitary, but) finite dimensional representations if $[G : H] < \infty$. Nor does it depend on irreducibility!

We will prove (5.4) by constructing a vector space isomorphism between them. Let T be a G -linear map $T : V_\rho \rightarrow \mathcal{W}$. Then for each $v_\rho \in V_\rho$ we know $T(v_\rho) \in \mathcal{W}$ and so $T(v_\rho)(1) \in V_\sigma$. This gives us a linear map T^* from V_ρ to V_σ . So $T^* \in \text{Hom}_{\mathbb{C}}(V_\rho, V_\sigma)$. Moreover, $T \mapsto T^*$ is itself a \mathbb{C} -linear map.

Now let us consider how the action of H fits into this picture. Let $h \in H$. Then $T^*(\rho_h(v_\rho)) = T(\rho_h(v_\rho))(1)$ and since by 2) $F(h^{-1}) = \sigma(h)F(1)$ we get

$$T(\rho_h(v_\rho))(1) = L_h T(v_\rho)(1) = T(v_\rho)(h^{-1}) = \sigma(h)T(v_\rho)(1) = \sigma(h)T^*(v_\rho).$$

This says $T^*\rho_h = \sigma_h T^*$, for all $h \in H$. Thus $T^* \in \text{Hom}_H(\rho|_H, \sigma)$. We now construct the inverse of this map.

Let $S \in \text{Hom}_H(\rho|_H, \sigma)$ and define $S_* : V_\rho \rightarrow \mathcal{W}$ by $S_*(v_\rho)(g) = S(\rho_g^{-1}(v_\rho)) \in V_\sigma$. Since $S_*(v_\rho)$ is a mapping from G to V_σ , it has a chance of being in \mathcal{W} . Because ρ is continuous as is S , we see $S_*(v_\rho)$ is continuous and so measurable.

It also satisfies 2).

$$\begin{aligned} S_*(v_\rho)(gh) &= S(\rho(gh)^{-1}(v_\rho)) = S(\rho(h)^{-1}\rho(g)^{-1}(v_\rho)) \\ &= \sigma(h)^{-1}S(\rho(g)^{-1}(v_\rho)) = \sigma(h)^{-1}S_*(v_\rho)(g). \end{aligned}$$

Since G/H is compact and this function is continuous, it has finite square integrable norm. Thus $S_*(v_\rho) \in \mathcal{W}$ and so we have a linear map $S_* : V_\rho \rightarrow \mathcal{W}$.

For $g \in G$, one checks easily that $S \mapsto S_*$ is linear and $L_g S_* = S_* \rho_g$. Hence $S_* \in \text{Hom}_G(V_\rho, \mathcal{W})$. It remains only to see that these maps invert one another.

Now

$$\begin{aligned} T^*(v_\rho)_*(g) &= T^*(\rho(g)^{-1}(v_\rho)) = T\rho(g)^{-1}(v_\rho)(1) \\ &= L_{g^{-1}}T(v_\rho)(1) = T(v_\rho)(g). \end{aligned}$$

Since this holds for all $g \in G$ and $v_\rho \in V_\rho$ we conclude $T^*_* = T$. Also, $S_*(v_\rho)(1) = S(v_\rho)$. Hence $(S_*)^*(v_\rho) = S(v_\rho)$ so $(S_*)^* = S$. \square

5.7 Some Consequences of Frobenius Reciprocity

Let $\text{SO}(3, \mathbb{R})$ act on S^2 with isotropy group $\text{SO}(2, \mathbb{R})$. Then $\text{SO}(3, \mathbb{R})$ also acts on $C(S^2)$ and therefore on $L_2(S^2)$. Hence (see 5.1.3) we can de-

compose this representation into irreducible components, called *spherical harmonics*. An interesting question is then which spherical harmonics occur and with what multiplicities?

It is easy to verify directly that $SU(2, \mathbb{C})$ is a compact real form of $SL(2, \mathbb{C})$, that is the complexification of $\mathfrak{su}(2, \mathbb{C})$ is $\mathfrak{sl}(2, \mathbb{C})$. The general fact follows from Corollary 7.4.10. Since $SU(2, \mathbb{C})$ is simply connected, its finite dimensional irreducible representations are the same as those of $\mathfrak{su}(2, \mathbb{C})$ by Corollary 1.4.15. Hence the finite dimensional irreducible representations of $SU(2, \mathbb{C})$ are in bijective correspondence with those of $\mathfrak{sl}(2, \mathbb{C})$, that is to say the positive integers by the degree of the representation (see Section 3.1.5). Since $SU(2, \mathbb{C})$ is two-sheeted covering of $SO(3, \mathbb{R})$, its irreducibles are those of odd degree.

Exercise 5.7.1. Show that the irreducibles of $SU(2, \mathbb{C})$ which are trivial on $\pm id$ are those of odd degree.

Theorem 5.7.2. *In the action of $SO(3, \mathbb{R})$ on $L_2(S^2)$ each irreducible representation of $SO(3, \mathbb{R})$ occurs and with multiplicity 1.*

Proof. We know $S^2 = SO(3, \mathbb{R})/SO(2, \mathbb{R})$. Consider the trivial irreducible representation σ of $SO(2, \mathbb{R})$. Then the representation of $SO(3, \mathbb{R})$ on $L_2(S^2)$ is $\text{Ind}(SO(2, \mathbb{R}) \uparrow SO(3), \sigma)$. If ρ is an irreducible representation of $SO(3)$, then $[\text{Ind}(SO(2) \uparrow SO(3), \sigma) : \rho]$ is the same as $[\rho|_{SO(2, \mathbb{R})} : 1_{SO(2, \mathbb{R})}]$. But we know what the irreducibles of $SO(3)$ are; these are just the irreducibles of $SU(2, \mathbb{C})$ of odd degree, or what is the same thing, the complex Lie algebra irreducibles of $\mathfrak{sl}(2, \mathbb{C})$ of odd degree. So the question is: given an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ of odd degree, how many times does its restriction to \mathfrak{h} (the line through H) contain the 0 representation of \mathfrak{h} ? Our study of these representations tells us the answer is 1. \square

We now study the relationship between the representations of G and those of a proper subgroup H .

Proposition 5.7.3. *Let G be a compact group and H a proper closed subgroup. Then there exists $\rho \in \mathcal{R}(G) \setminus \{1\}$ whose restriction to H contains the 1-dimensional trivial representation. That is, there exists*

$v_0 \in V_\rho$ with $\rho_h(v_0) = v_0$ for all $h \in H$.

Proof. If $\underline{\rho} \in \mathcal{R}(G) \setminus \{1\}$ the orthogonality relations show $\int_G \rho_{ij}(x) \overline{1(x)} dx = 0$ for all $i, j = 1, \dots, d_\rho$. Hence $\int_G r(x) dx = 0$ for all $r \in \mathcal{R}(\rho)$ and all such ρ . For each such ρ , $\rho|_H$ is a direct sum of irreducibles, $\sigma^1, \dots, \sigma^m$ of H . If the statement of the proposition were false none of the σ^i would be 1_H . If $r \in \mathcal{R}(\rho)$, then $r|_H$ is a linear combination of the coefficients of the σ^i . By the orthogonality relations on H , $\int_H \sigma_{kl}^i(h) \overline{1(h)} dh = 0$ for all i . Hence $\int_H r(h) dh = 0$. Thus

$$\int_G r(x) dx = \int_H r(h) dh \text{ for all } r \in \mathcal{R}(\rho), \rho \in \mathcal{R}(G) \setminus \{1\}. \quad (5.5)$$

On the other hand consider $\rho = 1_G \in \mathcal{R}(G)$. Here $\rho|_H = 1$ and the representative functions associated with this are $r(x) = \lambda \cdot 1 = \lambda$ so $\int_G r(x) dx = \lambda = \int_H r(h) dh$. Hence (5.5) holds for all $\rho \in \mathcal{R}(G)$. Now let $f \in C(G)$ and $\epsilon > 0$. Choose $r \in \mathcal{R}(G)$ so that $\|f - r\|_G < \epsilon$. Then

$$\begin{aligned} \left| \int_H f(h) dh - \int_G f(g) dg \right| &\leq \left| \int_H f(h) dh - \int_H r(h) dh \right| + \\ &\quad \left| \int_H r(h) dh - \int_G r(g) dg \right| + \\ &\quad \left| \int_G r(g) dg - \int_G f(g) dg \right| \\ &\leq 2\epsilon \end{aligned}$$

and since ϵ is arbitrary, $\int_G f(g) dg = \int_H f(h) dh$ for all $f \in C(G)$. Now $H \neq G$ so there must be another coset, x_0H . Choose a neighborhood U of x_0H which is disjoint from H and a continuous non-negative real valued function, f , which is $\equiv 1$ on x_0H and $\equiv 0$ on H . Then $\int_G f(g) dg > 0$ and $\int_H f(h) dh = 0$, a contradiction. \square

Because of Frobenius reciprocity and the fact that $[\rho|_H : 1] \geq 1$ this proposition has the following corollary.

Corollary 5.7.4. *Let G be a compact group and H a proper closed subgroup. Then there exists $\rho \in \mathcal{R}(G) \setminus \{1\}$ for which $[\text{Ind}(H \uparrow G, 1) : \rho] \geq 1$.*

We will prove the following

Theorem 5.7.5. *Let H be a closed subgroup of the compact group G . For each $\sigma \in \mathcal{R}(H)$ there is some $\rho \in \mathcal{R}(G)$ with $[\rho|_H : \sigma] \geq 1$.*

Hence by Frobenius reciprocity we would get

Corollary 5.7.6. *Let H be a closed subgroup of the compact Lie group G . For each $\sigma \in \mathcal{R}(H)$ there is some $\rho \in \mathcal{R}(G)$ with $[\text{Ind}(H \uparrow G, \sigma) : \rho] \geq 1$.*

Proof of Theorem 5.7.5. Let ρ_0 be faithful representation of G (and also H). Earlier 5.4.18 we proved that σ is a subrepresentation of $\rho_0^{(n)} \otimes \rho_0^{-m}|_H$ for some choice of n and m . Therefore σ is a subrepresentation of some irreducible component of ρ since ρ is an irreducible component of some $\rho_0^{(n)} \otimes \rho_0^{-m}$. Therefore some irreducible component ρ of $\rho_0^{(n)} \otimes \rho_0^{-m}$ must be restricted to σ .

Exercise 5.7.7. Let G be a compact group and H be a closed subgroup.

- (1) Each $\sigma \in \mathcal{R}(H)$ is an irreducible component of the restriction $\rho|_H$ of some $\rho \in \mathcal{R}(G)$.
- (2) If H happens to be a Lie group, then there is a finite dimensional continuous representation ρ of G whose restriction to H is faithful.
- (3) The restriction map $R(G) \rightarrow R(H)$ is surjective.

We conclude this chapter with the following result connected with equivariant imbeddings of compact G -spaces.

Theorem 5.7.8. *Let H be a closed subgroup of the compact Lie group G . Then there exists a finite dimensional continuous unitary representation ρ of G on V_ρ and a nonzero vector $v_0 \in V_\rho$ so that $H = \text{Stab}_G(v_0)$.*

We first need a lemma which tells us that the dimension together with the number of components determines the size of a compact Lie group.

Lemma 5.7.9. *Let G be a compact Lie group and $G \supseteq G_1 \supseteq G_2 \dots$ be a chain closed subgroups. Then this chain must eventually stabilize.*

Proof. Since $\dim G_i \geq \dim G_{i+1} \dots$ and all these dimensions are finite, then for $i \geq n_0$ all the dimensions must be constant. Hence for $i \geq n_0$ each G_{i+1} is open in G_i which is itself open in G_{n_0} , the number of components of G_{i+1} is \leq the number of components of G_i which is \leq the number of components of G_{n_0} . Since G_{n_0} is closed in G , it is compact and therefore has a finite number of components. It follows that eventually these must also stabilize and hence the conclusion. \square

Proof of Theorem 5.7.8. We may assume $H < G$ since if $H = G$ we may take ρ to be the trivial 1-dimensional representation and v_0 any nonzero vector. We will now prove

(**) If $g_0 \in G - H$, then there exists a representation ρ of G on V and $v_0 \neq 0 \in V_\rho$ such that $\rho_{g_0}(v_0) \neq 0$ and $\rho_h(v_0) = 0$ for all $h \in H$.

Suppose we can do this. Then $G \supseteq \text{Stab}_G(v_0) \supseteq H$ and g_0 is not in $\text{Stab}_G(v_0)$. Replacing G by the closed and therefore compact group $\text{Stab}_G(v_0)$ we can apply (**) again to this subgroup. In this way we get a descending chain of closed subgroups terminating in H which must terminate by Lemma 5.7.9. Therefore they must terminate in H . This would prove the Theorem.

*Proof of (**).* Since H and Hg_0^{-1} are disjoint compact sets we can find a continuous function f on G for which $f|_H = \alpha$ and $f|_{Hg_0^{-1}} = \beta$, where $\alpha < \beta$. Approximate f by $r \in R(G)$ to within $\epsilon = \frac{\beta - \alpha}{2}$. Let $F(g) = \int_H r(hg)dh$. Then F is continuous and therefore in $L_2(G)$. Since $r \in R(G)$, $F \in R(G)$ also. For $h_1 \in H$,

$$F(h_1) = \int_H r(hh_1)dh = \int_H r(h)dh \leq \epsilon + \alpha.$$

So $F|_H \leq \epsilon + \alpha$. On the other hand,

$$F(h_1 g_0^{-1}) = \int_H r(h h_1 g_0^{-1}) dh = \int_H r(h g_0^{-1}) dh \geq \beta - \epsilon.$$

So $F|_{H g_0^{-1}} \geq \beta - \epsilon$. In particular, $F(1) \leq \epsilon + \alpha$ and $F(g_0^{-1}) \geq \beta - \epsilon$ so $F(1) \neq F(g_0^{-1})$. Now apply L , the left regular representation of G on L_2 . Hence because $L_{g_0} F(1) = F(g_0^{-1})$ we see $L_{g_0} F \neq F$. On the other hand,

$$L_{h_1}(F)(g) = F(h_1^{-1} g) = \int_H r(h h_1^{-1} g) dh = \int_H r(h g) dh = F(g).$$

Thus $L_{h_1}(F) = F$ for all $h_1 \in H$. Since $F \in R(G)$ it lies in a finite dimensional L -invariant subspace V_ρ of $C(G)$. So there is a finite dimensional continuous unitary representation ρ of G and a nonzero vector F in it with $\rho_h(F) = F$ for all $h \in H$ and $\rho_{g_0}(F) \neq F$, proving (**).