A COMBINATORIAL PROPERTY OF BURNSIDE VARIETIES OF GROUPS

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Let G be an infinite group and $n \in \{2,3\}$. It is proved that $x^n = 1$ for all $x \in G$ if and only if $1_G \in X^n$ for all infinite subsets X of G, where $X^n := \{x_1x_2\cdots x_n \mid x_i \in X\}$.

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1. Introduction and results

Let $w = x_{i_1}^{\epsilon_1}\cdots x_{i_t}^{\epsilon_t}$ be a (reduced) word in the free group of rank $n \in \mathbb{N}$, on the letters $x_1,\dots,x_n$, where $\epsilon_1,\dots,\epsilon_t \in \{-1,1\}$. Suppose that G is a group and $X_1,\dots,X_n$ are n non-empty subsets of G. Define

$$w(X_1,\dots,X_n) := \{a_1^{\epsilon_1}\cdots a_t^{\epsilon_t} \mid a_j \in X_{i_j},\ 1 \le j \le t\}.$$

For example, if $w_1 = x_1^{-1}x_2^{-1}x_1x_2$, $w_2 = x_1^{-1}x_2^{-2}x_1x_2^2$ and $w_3 = x_1^3$, then

$$w_1(X_1,X_2) = \{a^{-1}b^{-1}cd \mid a,c \in X_1,\ b,d \in X_2\},$$

$$w_2(X_1,X_2) = \{a^{-1}b_1^{-1}b_2^{-1}cd_1d_2 \mid a,c \in X_1,\ b_1,b_2,d_1,d_2 \in X_2\},$$

and $w_3(X_1) = \{abc \mid a,b,c \in X_1\}$.

Let W and V be two non-empty subsets of the free group of rank $n \in \mathbb{N}$. Let $P(V,W)$ be the class of all groups G such that for every n-tuple $(g_1,\dots,g_n)$ of elements of G there exist $v \in V$ and $w \in W$ such that $v \neq w$ and $v(g_1,\dots,g_n) = w(g_1,\dots,g_n)$ (cf. [9]). We denote by $P^*(V,W)$ (respectively $P^{\#}(V,W)$) the class of all groups G satisfying the following condition:
$G \in P^*(V,W)$ (respectively $G \in P^{\#}(V,W)$) if and only if for all infinite

subsets $X_1, \ldots, X_n$ of G there exist $v \in V$ and $w \in W$ (respectively, also there exist elements $a_1 \in X_1, \ldots, a_n \in X_n$) such that $v \neq w$ and $1_G \in vw^{-1}(X_1, \ldots, X_n)$ (respectively $v(a_1, \ldots, a_n) = w(a_1, \ldots, a_n)$).
Clearly, we have $\mathcal{F} \cup P(V, W) \subseteq P^*(V, W)$ and

$$\mathcal{F} \cup P(V, W) \subseteq P^{\#}(V, W) \subseteq P^*(V, W),$$

where $\mathcal{F}$ is the class of finite groups. The following questions arise naturally.

Question 1.1. For which non-empty subsets V and W of a free group of finite rank, the equality

$$P(V, W) \cup \mathcal{F} = P^{\#}(V, W),$$

holds?

Question 1.2. For which non-empty subsets V and W of a free group of finite rank, the equality

$$P^{\#}(V, W) = P^*(V, W),$$

holds?

Question 1.3. For which non-empty subsets V and W of a free group of finite rank, the equality

$$P(V, W) \cup \mathcal{F} = P^*(V, W), \qquad (*)$$

holds?

Question 1.1 considered by many people, where the pair (V, W) is of the form $(\{1\}, \{w\})$ with w is a non-trivial word in a free group (see e.g., [1,8,10]). Note that $P(\{v\}, \{w\}) = P(\{vw^{-1}\}, \{1\})$ and $P^{\#}(\{v\}, \{w\}) = P^{\#}(\{vw^{-1}\}, \{1\})$. Question 1.3 first appeared in [3], where this question has been answered positively for $V = \{x_1x_2\}$ and $W = \{x_2x_1\}$, and in this case $P(V, W)$ is the class of abelian groups. If $n > 1$ is an integer and

$$V = \{x_1 \cdots x_n\} \text{ and } W = \{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$$

then $P(V, W)$ is exactly the class of n-permutable groups (see [7]). Also if

$$V = W = \{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$$

then $P(V, W)$ is precisely the class of n-rewritable groups (see [6]). Therefore the main result of [4] says that Question 1.3 has positive answer for

$$V = W = \{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\},$$

and the result of [5] says that the equality (*) in Question 1.3 is true, whenever

$$V = \{x_1 \cdots x_n\} \text{ and } W = \{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}.$$

Let $n > 0$ be an integer. It seems that the 'simplest' case of Question 1.3 is when $V = \{x^n\}$ and $W = \{1\}$. Note that in this case $P(V, W)$ is the variety of groups satisfying the law $x^n = 1$, i.e. the Burnside variety of finite exponent n of groups. For this latter choice for (V, W) Question 1.1 has positive answer (see [10]). Here we positively answer Question 1.3 when $V = \{x^2\}$ or $V = \{x^3\}$ and $W = \{1\}$.

Theorem 1.1. *If $n \in \{2, 3\}$, then $P(\{x^n\}, \{1\}) \cup \mathcal{F} = P^*(\{x^n\}, \{1\})$.*

2. Proofs

We begin with a result which shows that the equality (*) of Question 1.3 is true on the class of residually finite groups, whenever V and W are finite.

Proposition 2.1. *Let V and W be any non-empty finite subsets of the free group of rank $n \in \mathbb{N}$. Then every infinite residually finite $P^*(V, W)$-group belongs to the class $P(V, W)$.*

Proof. Let $g_1, \ldots, g_n$ be arbitrary elements of G and

$$S = \{v(g_1, \ldots, g_n)w^{-1}(g_1, \ldots, g_n) \mid v \neq w, \quad v \in V \text{ and } w \in W\}.$$

Suppose, for a contradiction, that $1 \notin S$. Since G is residually finite and S is finite, there exists a normal subgroup N of finite index in G such that $S \cap N = \varnothing$. Now by considering infinite subsets $Ng_1, \ldots, Ng_n$, there exist $v \in V$ and $w \in W$ such that $v \neq w$ and $1_G \in vw^{-1}(Ng_1, \ldots, Ng_n)$ and so $v(g_1, \ldots, g_n)w^{-1}(g_1, \ldots, g_n) \in N$, which is a contradiction. □

For proving the main result, we first show some general results on infinite subsets of an arbitrary group similar to the main result of [2].

Theorem 2.1. *Let $w(x_1, \ldots, x_n)$ be a word in the free group of rank $n > 1$ such that all the letters $x_1, \ldots, x_n$ occur in w. Suppose further that $w = v_1 x_i^{\epsilon} v_2$ where ϵ is a non-zero integer, v_1, v_2 are two (possibly empty) words such that only the letters $x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_n$ occur in them. Then every infinite subset Y of a group with the property that*

$$a, b \in Y, \quad a^{\epsilon} = b^{\epsilon} \Rightarrow a = b$$

contains an infinite subset X such that for every n distinct elements $y_1, \ldots, y_n \in X$, we have $w(y_1, \ldots, y_n) \neq 1$.

Proof. List the elements of Y under some well order $\leq$ as $y_1, y_2, \ldots$. Let $s \in Y^{(n)}$ the set of all n-element subsets of Y, and write the elements of s in the ascending order $y_{j_1}, \ldots, y_{j_n}$, $j_1 < \cdots < j_n$. Create $n! + 1$ sets as follows: One Y_σ for any $\sigma \in S_n$ and Z. Put $s \in Y_\sigma$ if $w(y_{j_{\sigma(1)}}, \ldots, y_{j_{\sigma(n)}}) = 1$ and put $s \in Z$ if $s \notin Y_\sigma$ for all $\sigma \in S_n$. Then by Ramsey's Theorem [11], there exists an infinite subset X of Y such that $X^{(n)} \subseteq Y_\sigma$ for some σ or $X^{(n)} \subseteq Z$. If $X^{(n)} \subseteq Y_\sigma$ for some σ then for any sequence $j_1 < j_2 < \cdots < j_n$ (after restricting the order $\leq$ to X) we have $w(y_{j_{\sigma(1)}}, \ldots, y_{j_{\sigma(n)}}) = 1$. Then

$$y^{\epsilon}_{j_{\sigma(i)}} = v_1^{-1} v_2^{-1} \qquad (I)$$

where v_1 and v_2 are evaluated on $(y_{j_{\sigma(1)}}, \ldots, y_{j_{\sigma(i-1)}}, y_{j_{\sigma(i+1)}}, \ldots, y_{j_{\sigma(n)}})$ and theirs values do not depend on $y_{j_{\sigma(1)}}$. Let $\sigma(i) = k$. Now since X is infinite, there exist two sequences as follows:

$$j_1 < \cdots < j_{k-1} < j_k < j_{k+1} < \cdots < j_n$$

and

$$j_1 < \cdots < j_{k-1} < j'_k < j_{k+1} < \cdots < j_n,$$

where $j_k \neq j'_k$. By equation (I), we have

$$y^{\epsilon}_{j_k} = y^{\epsilon}_{j'_k}$$

which, by hypothesis implies that $j_k = j'_k$, a contradiction. Hence $X^{(n)}$ must be a subset of Z. Therefore for every n distinct elements $y_1, \ldots, y_n$ of X, we have that $w(y_1, \ldots, y_n) \neq 1$. This completes the proof. ∎

Corollary 2.1. *Let w be a word with the properties stated in Theorem 2.1. Then the conclusion of Theorem 2.1 is valid if either G is a torsion-free nilpotent group or $\epsilon \in \{-1, 1\}$.*

Proof. It follows easily from Theorem 2.1 and by noting that in a torsion-free nilpotent group G, the property stated for the infinite set Y in Theorem 2.1 holds for G. □

Corollary 2.2. *Let X be an infinite subset of a group. Then there exists an infinite subset X_0 of X such that for all distinct elements $x, y, z, t \in X_0$ and for all $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \{-1, 1\}$ we have $x^{\epsilon_1} y^{\epsilon_2} z^{\epsilon_3} t^{\epsilon_4} \neq 1$, $x^{2\epsilon_1} y^{\epsilon_2} z^{\epsilon_3} \neq 1$, $x^{\epsilon_1} y^{2\epsilon_2} \neq 1$ and $x^{\epsilon_1} y^{\epsilon_2} \neq 1$.*

Proof. By applying Theorem 2.1 on the word x_1x_2 with the subset X, we find an infinite subset X_1 of X such that $xy \neq 1$ for all distinct elements $x, y \in X_1$. Now again apply Theorem 2.1 on the word $x_1^{-1}x_2$ with the subset X_1, then we get an infinite subset X_2 of X_1 with the property that $x^{-1}y \neq 1$ for all distinct $x, y \in X_2$. Continuing in this manner with the words $x_1^{-1}x_2^{-1}$, $x_1^2x_2$, $x_1^{-2}x_2$, $x_1^2x_2^{-1}$, $x_1^{-2}x_2^{-1}, \dots, x_1^{-1}x_2^{-1}x_3^{-1}, \dots, x_1^{-2}x_2^{-1}x_3^{-1}, \dots$, we find an infinite subset X_0 of X with properties stated in the corollary. This completes the proof. □

Proof of Theorem 1.1. Since it is proved in [10] that $P^{\#}(\{x^m\}, \{1\}) = P(\{x^m\}, \{1\}) \cup \mathcal{F}$ for all $m > 1$, it is enough to show that $P^*(\{x^n\}, \{1\}) = P^{\#}(\{x^n\}, \{1\})$. Now assume that $n = 3$ and $G \in P^*(\{x^3\}, \{1\})$. Suppose that X is an infinite subset of X, we prove that there exists an element $x \in X$ such that $x^3 = 1$. Suppose, for a contradiction, that $a^3 \neq 1$ for all $a \in X$. By Corollary 2.2, there is an infinite subset X_0 of X with the property stated therein. Since $G \in P^*(\{x^3\}, \{1\})$, $1 \in X_0^3$. It follows that

$$xyz = 1 \tag{1}$$

for some elements $x, y, z \in X_0$. By assumption $|\{x, y, z\}| \in \{2, 3\}$, thus the equality (1) gives a contradiction to one of the properties of X_0. This completes the proof for the case $n = 3$. The proof of the case $n = 2$ is very easy and it may be done by a similar argument. □

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