

CHAPTER 1

INTRODUCTION TO THE NON-PERTURBATIVE RENORMALIZATION GROUP

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An elementary introduction to the non-perturbative renormalization group is presented mainly in the context of statistical mechanics. No prior knowledge of field theory is necessary. The aim of this article is not to give an extensive overview of the subject but rather to focus on conceptual aspects and to explain in detail the main technical steps. It should be taken as an introduction to more advanced readings.

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1. Wilson's Renormalization Group

1.1. Introduction

We give in these notes a short presentation of both the main ideas underlying Wilson's renormalization group (RG) and their concrete implementation under the form of what is now called the non-perturbative renormalization group (NPRG). Prior knowledge of perturbative field theory is not required for the understanding of the core of the article. However, some basic knowledge about phase transitions in Ising and $O(N)$ models is supposed.¹⁻³ We shall mainly work in the framework of statistical field theory but when it will turn out to be illuminating we shall also use the language of particle physics.

The beginning of this article will be rather elementary and known to most physicists working in the field of critical phenomena. Nevertheless, both for completeness and to set up a language (actually a way of thinking at renormalization group) it has appeared necessary to include it. The first part of this article deals with a comparison between perturbative and Wilson's RG. Then, presented in the next part is the implementation of Kadanoff-Wilson's RG on the very peculiar case of the two-dimensional Ising model on the triangular lattice. This allows us to introduce the idea of decimation and block spins and also those of RG flow of coupling con-

stants and fixed point. The last part, which is also the core of this article, deals with the “modern” implementation of Wilson’s ideas. Both the general framework and detailed calculations for the $O(N)$ models will be given.

1.2. The Perturbative Method in Field Theory

The idea behind perturbation theory is to consider an exactly solvable model, either the Gaussian or the mean-field “model”, and to add, in a perturbation expansion, the term(s) present in the model under study and which are not taken into account in the exactly solvable model taken as a reference.^{1,4} For instance, in the “ ϕ^4 ” model (which belongs to the same universality class as the Ising model):

$$Z = \int \mathcal{D}\phi e^{-H(\phi)+\int B\phi} \tag{1}$$

with

$$H(\phi) = \int d^d x \left\{ \frac{1}{2} (\nabla\phi)^2 + \frac{1}{2} r_0 \phi^2 + \frac{1}{4!} u_0 \phi^4 \right\} \tag{2}$$

it is possible to take as a reference model the Gaussian model:

$$Z_0 = \int \mathcal{D}\phi e^{-H_0(\phi)+\int B\phi} \tag{3}$$

where

$$H_0(\phi) = \int d^d x \left(\frac{1}{2} (\nabla\phi)^2 + \frac{1}{2} r_0 \phi^2 \right) . \tag{4}$$

Z is then developed as a series in u_0 around Z_0 :

$$\begin{aligned} Z = \int \mathcal{D}\phi & \left(1 - \frac{u_0}{4!} \int_{x_1} \phi^4(x_1) + \frac{1}{2} \left(\frac{u_0}{4!} \right)^2 \int_{x_1, x_2} \phi^4(x_1) \phi^4(x_2) \right. \\ & \left. + \dots \right) e^{-H_0(\phi)+\int B\phi} . \end{aligned} \tag{5}$$

This expansion leads to the series of Feynman diagrams for the Green functions.

The problem with this approach is that the “fluctuations” induced by the ϕ^4 term around the Gaussian model are large. In the perturbation expansion they lead to integrals – corresponding to the loops in the Feynman diagrams – of the form:

$$\int^\Lambda d^d q_1 \dots d^d q_L \prod_i (\text{propagator}(q_i)) \tag{6}$$

where

$$\text{propagator}(q_i) \sim \frac{1}{(q_i + Q)^2 + r_0}. \quad (7)$$

These integrals are supposed to be cut-off at the upper bound by Λ which is an ultra-violet regulator. In the following, it will be convenient to think Λ as the (analog of the) inverse of a lattice spacing, lattice that would be used to regularize the field theory. In statistical mechanics, it is actually the other way around: the microscopic model is very often a lattice model whereas the field theory is only an effective model useful to describe the long-distance physics.

If Λ were sent to infinity, the integrals in Eq. (6) would be generically divergent for d sufficiently large. This means that for Λ finite but large, the integrals are large and depend crucially on the value of Λ . This is very unpleasant for at least two reasons:

(i) This invalidates the perturbation expansion even if u_0 is small. For instance, in the ϕ^4 model and for the four-point connected correlation function $G_c^{(4)}(x_1, \dots, x_4) = \langle \phi(x_1) \dots \phi(x_4) \rangle_c$, the one-loop approximation writes in Fourier space at zero momentum:

$$G_c^{(4)} \sim u_0 + (\text{constant}) \cdot u_0^2 \cdot \int^\Lambda \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + r_0)^2} + \dots \quad (8)$$

This integral is divergent for $d \geq 4$ in the limit $\Lambda \rightarrow \infty$.

(ii) The universal quantities (critical exponents, etc.) are expected to be independent of the underlying lattice and thus, at least for these quantities, it is paradoxical that the lattice spacing ($\sim \Lambda^{-1}$) plays such a crucial role.

Perturbative renormalization is the method that allows to reparametrize the perturbation expansion in such a way that the sensitive dependence on Λ has been eliminated.^a Then, the renormalization group allows to partially resum the perturbation expansions and thus to compute universal behaviours.^{1,3,4}

Let us now make a list of questions that are not very often addressed in the literature. Some answers are explicitly given in this text. Some others are worth thinking over. They are mainly there to nourish the reader's imagination....

^aLet us emphasize that apart from the field renormalization, the whole renormalization process is nothing but a reparametrization.

Q1: The occurrence of ultra-violet divergences in field theory is often considered as a fundamental property of the theory. Thus, why do they play no role in the few known exact solutions of field theories or statistical models ? For instance, in Onsager's solution of the two dimensional Ising model no divergence occurs. This is also the case when Wilson's RG is implemented in field theoretical models.

Q2: Ultra-violet divergences are often said to be related to the infinite number of degrees of freedom of a field theory (the value of the field at each point). But then, why does a classical field theory that also involves an infinite number of degrees of freedom show no divergence ?

Q3: The answer to the last question often relies in the literature on the fact that a statistical (or quantum) field theory involves fluctuations contrary to a classical field theory. Fluctuations are thus supposed to be responsible for the divergences. The computation of the contributions of the fluctuations — the Feynman diagrams — is thus often considered to be the reason why field theoretical techniques are relevant in statistical mechanics. But there always exist thermal fluctuations in a statistical system whereas field theoretical techniques are most of the time useless in statistical mechanics. Then, which types of fluctuations require field theory and which ones do not?

Q4: Ultra-violet divergences are also often said to be related to the fact that we multiply fields at the same point (in a lagrangian) while fields are distributions the product of which is ill-defined. But what are the distributions in the case of the Ising model ? And since the interaction takes place between spins that are not on the same site but on two neighboring sites why should we take care about this difficulty ?

Q5: In the ϕ^4 theory for instance, the renormalization group flow of the coupling constant — given by the β -function — is determined (in $d = 4$) by the UV divergences. But then why is the (IR stable) zero of the β -function, that is the non-gaussian fixed point, useful to describe the infrared behavior of a field theory and in particular the critical behavior ?

Q6: Why should we bother about the continuum limit in a statistical system — its ultra-violet behavior — for which on one hand there always exist a natural ultra-violet cut-off (such as a lattice spacing or a typical range of interaction) and for which on the other hand we are interested only in its long-distance physics ?

1.3. Wilson's Approach to the Renormalization Group

There are two crucial remarks behind Wilson's method:⁵⁻⁷

(i) in general, we cannot compute exactly the contributions of the fluctuations (otherwise we could solve exactly the model);

(ii) the way fluctuations are summed over in perturbation theory is not appropriate since *all wavelengths are treated on the same footing* in Feynman diagrams. This is what produces integrals: at a given order of the perturbation expansion *all* fluctuations are summed over.^b

Wilson's idea is to organize the summation over fluctuations in a better way. Note that because of remark (i) above, "better way" means "with respect to an approximation scheme".^c What is the idea behind Wilson's method of summation over the fluctuations? Before answering, let us notice that

- in strongly correlated systems (e.g. close to a second order phase transition) the two relevant scales are (i) the microscopic scale a – a lattice spacing, an intermolecular distance, the Planck length, etc. – , (ii) the correlation length ξ . These two scales are very different for $T \simeq T_c$ so that fluctuations exist on *all* wavelengths between a and ξ . In particle physics, ξ corresponds to the Compton wavelength of the particle $(mc/\hbar)^{-1}$ and a to the typical (inverse) energy scale of the "fundamental theory": 10^{16} GeV for a Grand Unified Theory or 10^{19} GeV for quantum gravity.
- for the long distance physics and for universal quantities (magnetization, susceptibility, etc.) the short distance "details" of the model have been completely washed out. This means that these "details" (existence and shape of the lattice for instance) do matter for the short distance physics but that they are averaged out at large distances: there must exist *average processes* that eliminate the microscopic details as the scale at which we "observe" the system is enlarged.^d

^bIn quantum field theory, Feynman diagrams represent the summation over probability amplitudes corresponding to all possible exchanges of virtual particles compatible with a given process at a given order. Note that these integrals are cut-off in the ultraviolet by Λ and in the infrared by the "mass" r_0 (see Eq. (6)). In statistical mechanics, the mass is related to the correlation length ξ by $r_0 \sim \xi^{-2}$ (at the mean-field approximation).

^cIt is extremely rare that renormalization group enables to solve exactly a model that was not already solved by another and simpler method.

^dNote that this is true only for universal quantities. The critical temperatures for in-

Wilson's idea is therefore to build an *effective theory for the long-distance degrees of freedom* we are interested in.⁶⁻⁸ This is achieved by integrating out the short distance ones. Since, at least for universal quantities, these short distance quantities do not matter crucially, it should be possible to devise approximations that preserve the physics at long distance.

Actually, Wilson's idea is more general: it consists in saying that the "best" (approximate) way to study a subset of degrees of freedom of a system is to build an effective theory for them by integrating out the others. For instance, in molecular physics, one should build an effective Hamiltonian for the valence electrons obtained by "integrating out" the core electrons (corresponding to high energy degrees of freedom).

For the Ising model, this consists in integrating out in the partition function the "high energy modes" of the field $\phi(p)$ – those for which $p \in [\Lambda - d\Lambda, \Lambda]$ – and in computing the effective Hamiltonian for the remaining modes. By iterating this procedure down to a scale k , one should obtain an effective Hamiltonian for the "low energy modes", those corresponding to $p < k$. The long distance physics, obtained for $p \rightarrow 0$, should then be readable on the effective Hamiltonian corresponding to $k \rightarrow 0$ since no fluctuation would remain in this limit.

Once again, let us emphasize that if we could perform exactly the integration on these "rapid" modes, we could iterate this integration and obtain the exact solution of the model. In most cases, this is impossible and the interest of this method, beyond its conceptual aspect, lies in the possibility to implement new approximation schemes better than the usual perturbation expansion.^e

Schematically, to implement Wilson's method, we divide $\phi(p)$ into two pieces: $\phi_>(p)$ that involves the rapid modes $p \in [\Lambda/s, \Lambda]$ of $\phi(p)$ and $\phi_<(p)$ that involves the slow modes $p \in [0, \Lambda/s]$:

$$Z = \int \mathcal{D}\phi e^{-H[\phi, \vec{K}, \Lambda]} = \int \mathcal{D}\phi_< \mathcal{D}\phi_> e^{-H[\phi_<, \phi_>, \vec{K}, \Lambda]}, \quad (9)$$

where $\vec{K} = (K_1, K_2, \dots)$ represents *all possible coupling constants* compatible with the symmetries of the system. Here, we have supposed that H involves all these couplings although the initial Hamiltonian (that is, at scale Λ) involves in general only a finite number of them. For instance, in

stance, which are non-universal, depend on microscopic details such as the shape of the lattice.

^eLet us already mention that if Wilson's RG equations are truncated in a perturbation expansion, all the usual perturbative results are recovered as expected.

the ϕ^4 model all (initial) couplings K_i are vanishing, but those corresponding to the terms $(\nabla\phi)^2$, ϕ^2 and ϕ^4 . The integration of the rapid modes consists in integrating out the $\phi_>$ field. In this integration, *all the couplings K_i that were initially vanishing start to grow* (this is why we have considered them from the beginning) but since we have considered the most general Hamiltonian H (compatible with the symmetries of the problem), its functional form remains unchanged. Let us call \vec{K}' the new coupling constants obtained after integrating out $\phi_>$. By definition of \vec{K}' :

$$Z = \int \mathcal{D}\phi_< e^{-H[\phi_<, \vec{K}', \Lambda/s]} \quad (10)$$

with

$$e^{-H[\phi_<, \vec{K}', \Lambda/s]} = \int \mathcal{D}\phi_> e^{-H[\phi_<, \phi_>, \vec{K}, \Lambda]} . \quad (11)$$

We thus build a series of coupling constants, each associated with a given scale:

$$\begin{aligned} \Lambda &\rightarrow \vec{K}, \\ \frac{\Lambda}{s} &\rightarrow \vec{K}', \\ \frac{\Lambda}{s^2} &\rightarrow \vec{K}'', \quad \text{etc.} \end{aligned} \quad (12)$$

This method has several advantages compared with the usual, à la Feynman, approach:

- There is no longer any summation over all length scales since the integration is performed on a momentum shell, $|q| \in [\Lambda/s, \Lambda]$. Thus, there can be no divergence and there is no need for any renormalization in the usual sense (subtraction of divergences).
- The coupling constants K_i are naturally associated with a scale whereas this comes out in the perturbative scheme as a complicated by-product of regularization and renormalization. Wilson's method by-passes completely renormalization to directly deals with renormalization group.
- The method is not linked with a particular expansion and there is therefore a hope to go beyond perturbation expansion.
- The "flow" of coupling constants $\vec{K} \rightarrow \vec{K}' \rightarrow \vec{K}'' \rightarrow \dots$ is sufficient to obtain much information on the physics of the system under

study. In particular, the notion of “fixed point” of this flow will play a particularly important role in statistical mechanics.

2. Renormalization Group Transformations

2.1. Blocks of Spins

As a pedagogical introduction, let us start by a simple and illuminating example of Wilson’s method implemented in x -space instead of momentum space and without having recourse to field theory.

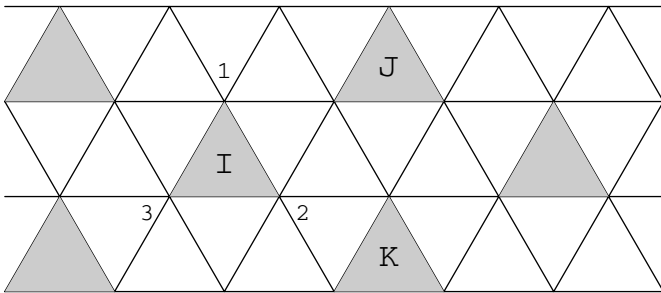


Fig. 1. Partition of the triangular lattice into plaquettes. The plaquettes are labelled by capital letters I, J, K, \dots and the spins inside the plaquettes are denoted, in obvious notations, S_i^I , $i = 1, 2, 3$. The lattice of plaquettes is again triangular with a lattice spacing $a\sqrt{3}$.

We consider a triangular lattice with Ising spins to exemplify block spin transformations:

$$H = -J \sum_{\langle ij \rangle} S_i S_j - B \sum_i S_i \tag{13}$$

where S_i are Ising spins: $S_i = \pm 1$, the summation $\langle \dots \rangle$ runs only on nearest neighbours and B is a uniform magnetic field. The lattice is partitioned into triangular plaquettes labelled by capital letters I, J, \dots . We call S_i^I , $i = 1, 2, 3$ the spin number i of the I -th plaquette. As a first step, we separate the 8 configurations of the three spins S_i^I of plaquette I into 24 configurations of (i) the block spin $\mathcal{S}_I = 1$ – which is chosen here to be an Ising spin – and (ii) the four configurations, called σ_I^α , corresponding to a given value of \mathcal{S}_I either $+1$ or -1 . We choose (and this will be modified in the following) to define \mathcal{S}_I by a majority rule:

$$\mathcal{S}_I = \text{sign}(S_1^I + S_2^I + S_3^I) = 1. \tag{14}$$

Thus, for the four configurations of the S_i^I compatible with $\mathcal{S}_I = +1$, we define the four variables $\sigma_I^{\alpha+}$ by:

$$\begin{aligned} \sigma_I^{1+} &\text{ corresponds to } \uparrow\uparrow\uparrow \\ \sigma_I^{2+} &\text{ corresponds to } \uparrow\uparrow\downarrow \\ \sigma_I^{3+} &\text{ corresponds to } \uparrow\downarrow\uparrow \\ \sigma_I^{4+} &\text{ corresponds to } \downarrow\uparrow\uparrow \quad . \end{aligned} \tag{15}$$

For $\mathcal{S}_I = -1$, the $\sigma_I^{\alpha-}$ correspond to the opposite configurations of the spins. Note that it will not be necessary in the following to compute the S_i^I in terms of the σ_I^α . The sum over all spin configurations in the partition function can be written as:

$$\sum_{\{S_i\}} = \sum_{\{S_I^I\}} = \sum_{\{S_I\}} \sum_{\{\sigma_I^\alpha\}} . \tag{16}$$

The partition function can thus be rewritten as

$$Z[B, T, n] = \sum_{\{S_I\}} \sum_{\{\sigma_I^\alpha\}} e^{-H[\mathcal{S}_I, \sigma_I^\alpha, B, T, n, a]} , \tag{17}$$

where n is the number of lattice sites ($n \rightarrow \infty$ corresponds to the thermodynamical limit) and a is the lattice spacing. We have also chosen to redefine the coupling constant J and the magnetic field B so that the prefactor $1/k_B T$ of H in the Boltzmann weight is absorbed into the normalization of these quantities. In this example, the lattice spacing of the blocked lattice is $a\sqrt{3}$. The summation over the short distance degrees of freedom σ_I^α can be formally performed

$$Z[B, T, n] = \sum_{\{S_I\}} e^{-H'[\mathcal{S}_I, B, T, n/3, a\sqrt{3}]} \tag{18}$$

with, by definition of H' :

$$e^{-H'[\mathcal{S}_I, B, T, n/3, a\sqrt{3}]} = \sum_{\{\sigma_I^\alpha\}} e^{-H[\mathcal{S}_I, \sigma_I^\alpha, B, T, n, a]} . \tag{19}$$

Let us make some remarks here.

- We have chosen the majority rule to define the block spin \mathcal{S}_I so that it is again an Ising spin. The price to pay is that relation (14) between \mathcal{S}_I and the S_i^I 's is non-linear. This is a source of many difficulties in more complicated systems than the Ising model on

the triangular lattice. Moreover, it is difficult to generalize this rule to continuous, N -component, spins.

- The explicit computation of H' , Eq. (18), shows that it involves infinitely many interaction terms, even if H involves only a nearest neighbour interaction. Thus, as it stands, the form of H is not stable under block-spin transformations.

There is a solution to both problems.

- We define \mathcal{S}_I through a linear transformation instead of the majority rule:

$$\mathcal{S}_I \propto \sum_{i \in I} S_i^I \tag{20}$$

The Ising character is lost but the relation (20) is simpler than (14) and can be generalized to other models.

- We take for H a Hamiltonian involving all possible couplings among the S_i compatible with the \mathbb{Z}_2 symmetry ($S_i \rightarrow -S_i$ for all i) of the Ising model (they will all be generated) :

$$\begin{aligned}
 H = & -K_1 \sum_{\langle ij \rangle} S_i S_j + K_2 \sum_{\ll ij \gg} S_i S_j \\
 & + K_3 \sum_{\langle ijkl \rangle} S_i S_j S_k S_l + \dots
 \end{aligned} \tag{21}$$

where $\ll ij \gg$ means summation over the next nearest neighbours.

Now $H = H(\vec{K}, S_i, n)$ where $\vec{K} = (K_1, K_2, \dots)$ represents the set of all \mathbb{Z}_2 -symmetric coupling constants as well as the magnetic field if necessary. For the initial Hamiltonian: $K_{i \neq 1} = 0$. Of course, this seems extremely complicated but it is the only possibility to have a form-invariant Hamiltonian. The fact that all couplings are generated when fluctuations are integrated out simply means that even if a Hamiltonian involves a finite number of couplings, all correlation functions, involving an arbitrary number of spins, are non-trivial. We shall come back on this point later.

Let us finally remark that for $n = \infty$, $n/3 = \infty$, $n/3^2 = \infty$, etc., and the number of “spins” remains identical. However, the lattice spacing varies: $a \rightarrow \sqrt{3}a \rightarrow 3a \rightarrow \dots$. Since, we shall look for *fixed point Hamiltonians*^f

^fFixed point Hamiltonian is meant in the usual sense. If $H(\vec{K}^*, S_i)$ is a fixed point Hamiltonian this means that $\vec{K}^* \rightarrow \vec{K}^*$ by summation over the σ_i^α 's, Eq. (19). The rescaling of

in the following, it will be necessary to rescale the lattice spacing by a factor $1/\sqrt{3}$ after each summation of the rapid modes in such a way that we obtain, after summation over the σ_I^α 's, the same Hamiltonian for the same system. We shall come back on this point later.

We can now rewrite Eq. (19) for $n = \infty$ as

$$e^{-H[\vec{K}', S_I, a\sqrt{3}]} = \sum_{\{\sigma_I^\alpha\}} e^{-H[\vec{K}, S_I, \sigma_I^\alpha, a]}. \quad (22)$$

This transformation, together with the rescaling of the lattice spacing, is called a renormalization group transformation. Such a RG transformation

- preserves the partition function Z and thus its singularities and thus the critical behaviour; more generally, all thermodynamical quantities are preserved;[§]
- maps a Hamiltonian onto another Hamiltonian (a system onto another system) in such a way that they have the *same long distance physics*;
- consists in integrating out (averaging over) short distance degrees of freedom to obtain an *effective Hamiltonian* for the long distance degrees of freedom;
- can be summarized in a change of (infinitely many) coupling constants: $\vec{K} \rightarrow \vec{K}'$.

And now, two questions:

Question 1: “Why is it interesting to integrate out the σ_I^α 's? Isn't it as complicated to integrate them out as would be the full integration over all degrees of freedom?”

It is true that integrating out *exactly* the σ_I^α 's is of the same difficulty as calculating Z completely. However,

- the complete calculation of Z contains much more information than what we want to obtain to get a satisfactory description of the critical physics. Moreover, for universal quantities, we guess that we shall be able to make rather drastic approximations as for the microscopic details of the model, that is the integration of the short

the lattice spacing, which is equivalent to measuring all dimensionful quantities in terms of the *running* lattice spacing and not in terms of the (fixed) initial lattice spacing, is a necessary step to obtain fixed point Hamiltonians.

[§]If we wanted to compute correlation functions of the original spins, we would have first to couple the system to an arbitrary magnetic field (in order to be able to compute derivatives of Z with respect to the magnetic field B_i). This is a complicated task.

distance degrees of freedom, since they probably play a minor role; this opens the possibility of new *approximation schemes*;

- the qualitative (or semi-quantitative) behaviour of the RG flow of coupling constants $\vec{K} \rightarrow \vec{K}' \rightarrow \vec{K}'' \rightarrow \dots$ is enough to predict many non-trivial behaviours occurring around a second order phase transition.

Question 2: “Why should we make a series of small block-spins (coarse-graining) instead of directly a large one?”

This question, which is not independent of the first one, is somewhat subtle and requires some developments. Once again, if we were able to perform exactly the integration over the σ_I^α 's, small or large blocks would make no difference. Thus, the problem comes from the approximations and is therefore not fully under control before precise calculations are performed. However, the general idea is not difficult to grasp.

Let us call $\vec{T}(\cdot, p)$ the function that maps $\vec{K} = \vec{K}^{(0)}$ onto $\vec{K}^{(p)}$ after p iterations of the RG transformations:

$$\vec{K}^{(p)} = \vec{T}(\vec{K}^{(0)}, p) \tag{23}$$

We, of course, have the property

$$\vec{K}^{(p)} = \vec{T}(\vec{K}^{(r)}, p - r) = \vec{T}(\vec{T}(\vec{K}^{(0)}, r), p - r) \tag{24}$$

and thus

$$\vec{T}(\cdot, p) = \vec{T}(\vec{T}(\cdot, r), p - r) . \tag{25}$$

This is called a *self-similarity*⁹ property.^h If \vec{T} were exactly known, this property would be trivially verified. However, once approximations are performed, it is generically violated as is the case for instance in perturbative renormalization.

Let us illustrate the concept of self-similarity on the simple example of differential equations.¹⁰ We consider the trivial differential equation:

$$\dot{y} = -\epsilon y \tag{26}$$

^hSomething is said to be self-similar if it is everywhere the same. In our case, the self-similar character comes from the fact that the functional form of the RG flow does not depend on the initial couplings $\vec{K}^{(0)}$ since the same function \vec{T} is used to transform $\vec{K}^{(0)}$ into $\vec{K}^{(p)}$ or $\vec{K}^{(r)}$ into $\vec{K}^{(p)}$. This results in the fact that the right hand side of Eq. (25) is independent of r since the left hand side is. This independence is completely similar to the independence of the renormalized theory on the renormalization scale in perturbative renormalization. This is what allows to derive the Callan-Symanzik RG equations in the perturbative context.

with $y(t_0) = y_0$. The solution is

$$y = f(t - t_0, y_0) = y_0 e^{-\epsilon(t-t_0)}. \quad (27)$$

Of course, f obeys a self-similarity property which means that we can either (i) first integrate (26) between t_0 and τ to obtain $y(\tau) = y_\tau$ and then integrate again (26) between τ and t with y_τ as new initial condition or (ii) directly integrate (26) between t_0 and t :

$$y(t) = f(t - t_0, y_0) = f(t - \tau, f(\tau - t_0, y_0)). \quad (28)$$

This is trivially verified by the exact solution (27) since

$$f(u, v) = v e^{-\epsilon u} = v e^{-\epsilon a} e^{-\epsilon(u-a)} = f(u - a, f(a, v)). \quad (29)$$

However, this property is violated *at any finite order* of the perturbation expansion in ϵ of $y(t)$. Let us show this at first order in ϵ for which we, of course, obtain:

$$y(t) = y_0(1 - \epsilon(t - t_0)) + O(\epsilon^2). \quad (30)$$

This defines the approximation of order one of f :

$$f^{(1)}(t - t_0, y_0) = y_0(1 - \epsilon(t - t_0)). \quad (31)$$

We obtain at this order:

$$\begin{aligned} f^{(1)}(u - a, f^{(1)}(a, v)) &= v(1 - \epsilon a)(1 - \epsilon(u - a)) \\ &= f^{(1)}(u, v) + \epsilon^2 v a(u - a) \end{aligned} \quad (32)$$

By comparing this result with Eq. (29), we find that self-similarity is obeyed at order ϵ , as expected, but is violated at order ϵ^2 . The problem is that this violation can be arbitrarily large if u (which represents $t - t_0$) is large. Thus, even if ϵ is small, the self-similarity property is violated for large time intervals. This is true at *any finite order* of perturbation theory. This large violation comes ultimately from the fact that the perturbation expansion is *not an expansion in ϵ but in $\epsilon(t - t_0)$* . This is completely reminiscent of the perturbation expansion in field theory where the expansion is not performed in terms of u_0 but in terms of $u_0 \log \Lambda$ where u_0 is the bare coupling constant and Λ the cut-off (see Eq. (8) for the ϕ^4 theory in $d = 4$). Reciprocally, it is clear that if $u = t - t_0$ is small, so is the violation in Eq. (32) since, in this case, both a and $u - a$ are small. Thus, using perturbation expansions on small or, even better, on infinitesimal time intervals preserves self-similarity at each step. In geometrical terms, this means that we can safely use perturbation theory to compute the envelope of the curve $f(u, v)$

– the field of tangent vectors $\beta(u)$ – but not the curve itself.¹⁰ The curve $f(u, v)$ can only be obtained in a second step by integration of its envelope.

The analogue for the RG is that small blocks will be under control. Coarse graining in this case respects self-similarity *even when approximations are used* while large ones lead inevitably to large errors.

Before studying the structure of the RG flow, let us make two remarks about the RG transformations and their physical meaning.

2.2. Two Remarks Concerning RG Transformations

The first remark is that it is still widely believed that the correlation length $\xi(T)$ is a measure of the typical size of clusters of spins having the same orientation, that is of ordered domains (in the Ising case). As a consequence, it is believed that the divergence of ξ at T_c is a consequence of the divergence of the size of these (so-called) naive clusters. The traditional metaphor is that at T_c there would exist oceans of up spins with continents of down spins that would contain themselves lakes of up spins with islands of down spins, etc, with some kind of fractal geometry. This is wrong. It has been shown long ago that the distribution of cluster boundaries *does not scale at criticality*. Rather, at a temperature T_p well below T_c the clusters of spins having the opposite sign of the spontaneous magnetization merge into a large percolating cluster. An important point is that, strictly speaking, no phase transition occurs at T_p since no local order parameter of the Ising model can be built out the spins in order to describe this transition: there is no singularity of the partition function at T_p . In fact, it is possible to construct clusters of spins that are critical at T_c . These are the famous Fortuin and Kasteleyn clusters.¹¹ They are used in the Swendsen-Wang algorithm of Monte Carlo simulations of the Ising model since they partially defeat critical slowing down.¹²

The second remark is that the “microscope analogy” is often used to give an intuition of the physical meaning of the RG transformations. In this analogy, the coarse-graining implemented in the RG transformations would be similar to what occurs when the magnification of a microscope is decreased. Let us imagine that we look at an image made of small pixels of definite colours (say blue, green or red). At a mesoscopic scale, the pixels are no longer seen and only a smearing of the colors of blocks of pixels can be observed. As a result, the “physics” observed would depend on the scale as in the RG transformations. This analogy has several virtues but also several drawbacks. Let us mention some. First, our brain plays a crucial

role for the color vision. From the three colors blue, green and red the cones in the retina are sensitive to, our brain is smart enough to reconstruct the impression of a continuous spectrum of colors. Although the analogy leads us to believe that our perception of the colors at a mesoscopic scale is a linear combination at this scale of the elementary colors of the pixels, this is not so. Second, in a RG transformation, there are two main steps (not to mention the final change of scale to go back to the original lattice spacing). The first one is to build a stochastic variable for the block.ⁱ The second is to build an effective hamiltonian for this block variable by integration over short distance fluctuations. We can imagine that the first step is analogous to the superimposition of the electromagnetic field produced by the different pixels. But then what is the analog of the second step? The laws of classical electrodynamics for the propagation of light do not change from one scale to the other. Let us repeat here that the effective hamiltonians for the block variables in the Ising model are extremely complicated: they involve all powers of the fields and not only interactions among nearest neighbors.^j There is no analog for this step in the microscope analogy although it is the crucial one from the RG point of view. In fact, things go almost the other way around. Whereas the electromagnetic field emitted by several pixels is the linear superposition of the field produced by each of them, the β -function in quantum field theory that gives the evolution of the coupling constant with the scale is a measure of the deviation to the trivial rescaling invariance (in the case of quantum electrodynamics). Thus, although the microscope analogy can be useful it should be employed with some care (and a grain of salt).

Let us now show how linear RG transformations can be implemented. This will allow us to prove a simple relation about the behavior under RG transformations of the two-point correlation function $\langle S_i S_j \rangle$ and thus on the correlation length.^{1,7}

ⁱThis can be performed either by a majority rule as in Eq.(14) or by a linear relation as in Eq.(34).

^jOnce the continuum limit has been taken and continuous RG transformations are implemented this means that the effective hamiltonians involve all powers of the field and of its derivatives.

2.3. Linear RG Transformations and Behaviour of the Correlation Length

Instead of the majority rule, we consider a linear transformation between the spins of a plaquette and the block-spin. The simplest idea is to take a *spatial* average (not a thermodynamic one)

$$S'_I = \frac{1}{s^d} \sum_{i \in I} S_i^I \tag{33}$$

where s is the “linear size” of the block, that is s^d is the number of spins per block. In our example of the triangular lattice, $d = 2$ and $s = \sqrt{3}$. As we already said, we shall also need to perform a rescaling of all dimensionful quantities (in order to find fixed points). Thus we take:

$$S_I = \frac{\lambda(s)}{s^d} \sum_{i \in I} S_i^I \tag{34}$$

where $\lambda(s)$ is a function that will be determined in such a way that we find a fixed point. This relation among the stochastic variables S_i^I and S_I leads to relations among their thermodynamic averages. The most important one is the two-point correlation function:

$$\begin{aligned} \langle S_I S_J \rangle &= \frac{1}{Z} \sum_{\{S_I\}} S_I S_J e^{-H[\vec{K}', S_L]} \\ &= \frac{1}{Z} \sum_{\{S_I\}} S_I S_J \sum_{\{\sigma_L^\alpha\}} e^{-H[\vec{K}, S_L, \sigma_L^\alpha]} \\ &= \frac{\lambda^2(s)}{s^{2d}} \frac{1}{Z} \sum_{\{S_I\}} \sum_{\{\sigma_I^\alpha\}} \sum_{i \in I, j \in J} S_i^I S_j^J e^{-H[\vec{K}, S_i]} \\ &= \frac{\lambda^2(s)}{s^{2d}} \sum_{i \in I, j \in J} G^{(2)}(\vec{x}_i, \vec{x}_j) \end{aligned} \tag{35}$$

where, by definition

$$G^{(2)}(\vec{x}_i, \vec{x}_j) = G^{(2)}(r_{ij}, \vec{K}) = \langle S_i S_j \rangle \tag{36}$$

is the two-point correlation function of the spins S_i . If the correlation length is large compared to the size of the plaquettes and if we consider two plaquettes I and J such that their distance is very large compared to a : $|\vec{x}_{i \in I} - \vec{x}_{j \in J}| \gg a$, then $G^{(2)}(\vec{x}_i, \vec{x}_j)$ does not vary much for $i \in I$ and $j \in J$. Thus, in this case:

$$\sum_{i \in I, j \in J} G^{(2)}(\vec{x}_i, \vec{x}_j) \simeq s^{2d} G^{(2)}(\vec{x}_i, \vec{x}_j). \tag{37}$$

Therefore, close to the critical temperature and for distant plaquettes:

$$\langle \mathcal{S}_I \mathcal{S}_J \rangle \simeq \lambda^2(s) \langle S_i^I S_j^J \rangle. \quad (38)$$

The important point is that $\langle \mathcal{S}_I \mathcal{S}_J \rangle$ is also the two-point correlation function of a \mathbb{Z}_2 -invariant magnetic system. The only difference is that it is computed with the set of couplings \vec{K}' instead of \vec{K} . We thus obtain:

$$\langle \mathcal{S}_I \mathcal{S}_J \rangle = G^{(2)}(r_{IJ}, \vec{K}') \quad (39)$$

where $G^{(2)}$ is the same function as in Eq. (36).^k We thus deduce:

$$G^{(2)}(r_{IJ}, \vec{K}') \simeq \lambda^2(s) G^{(2)}(r_{ij}, \vec{K}) \quad (40)$$

for sufficiently distant plaquettes.

Let us now explain the precise meaning of this relation. Let us suppose that we are given a new Ising model on a triangular lattice with a set of couplings \vec{K}' . In principle, we can compute the correlation function $G^{(2)}$ of the spins of this new system. Our claim is that this correlation function is identical to the correlation function of the block-spins of the original lattice. Of course, to compare the two functions we have to say how to compare the distances r_{ij} between spins in the two lattices. Our calculation shows that what we have to do is to measure all distances in the length unit intrinsic to the system, that is in units of the lattice spacing of each system. This means that the quantities r_{ij} and r_{IJ} that appear in Eq. (40) must be *numerically different*

$$r_{IJ} = r_{ij}/s \quad (41)$$

since they correspond to the same “distance” but measured in two different units: a for the original system and $a' = a\sqrt{3}$ for the coarse-grained system. It is important to understand that measured in an extrinsic length unit, like metres, these distances are indeed the same since $r_{IJ}a' = (r_{ij}/s).(sa) = r_{ij}a$ whereas they are different when they are measured in the length unit intrinsic to each system. Put it differently, the dimensionful distances, measured in a common length unit, are equal whereas the dimensionless distances, measured in terms of the lattice spacing of each system, are different. We shall put a bar or a tilde on dimensionless quantities to distinguish them.

^kLet us point out here a subtlety. This statement is not fully rigorous since the original spins are Ising spins whereas the block spins \mathcal{S}_I are not. The correlation functions $\langle S_i^I S_j^J \rangle$ and $\langle \mathcal{S}_I \mathcal{S}_J \rangle$ are therefore not computed exactly in the same way since the summation over the configurations of S_i and of \mathcal{S}_I do not run on the same values. In fact, after several blocking iterations, the spins that are summed over become almost continuous variables and the aforementioned difficulty disappears.

Let us emphasize that the value in an extrinsic unit like metres of, say, the correlation is almost meaningless. From a physical point of view, the only relevant measure of the correlation length is in units of the lattice spacing. The difficulty in our case is two-fold. First, to write down a field theory, it is necessary to perform the continuum limit $a \rightarrow 0$. It is therefore convenient to rescale the position vectors by a factor a before performing this limit (thus, as usual, $[\vec{x}] = \text{length}$).¹ Second, since we have to consider several systems with different lattice spacings, it will be convenient to work in the continuum with lengths measured in units of the running lattice spacings.

Let us therefore define dimensionless quantities as

$$\bar{r} = \frac{r}{a} \quad , \quad \bar{\xi} = \frac{\xi}{a} \quad , \quad \bar{r}' = \frac{r}{sa} \quad . \quad (43)$$

where r and ξ are the dimensionful quantities (measured in meters) that will be convenient once the continuum limit will be taken, that is in the field theory formalism. Eq. (40) that involves only dimensionless quantities can then be rewritten:

$$G^{(2)}\left(\frac{\bar{r}}{s}, \bar{K}'\right) \simeq \lambda^2(s) G^{(2)}(\bar{r}, \bar{K}) \quad . \quad (44)$$

Let us give a concrete example of the significance of this relation. In three dimensions and at large distances: $r_{ij} \gg 1$, a typical form of the two-point correlation function is:

$$\langle S_i S_j \rangle = G^{(2)}(\bar{r}, \bar{\xi}) \sim \frac{e^{-\bar{r}/\bar{\xi}}}{\bar{r}^\theta} \quad (45)$$

with $\bar{\xi}$ the correlation length in units of the lattice spacing a and $\bar{r} = \bar{r}_{ij}$. We can use the same formula as in Eq. (45) for the correlation function of

¹It will also be convenient to rescale the spin-field by the appropriate power of a : $S_i \rightarrow \phi(\vec{x})$ so that the gradient term $(\nabla\phi)^2$ comes in the Hamiltonian of the field theory with a dimensionless prefactor:

$$H = \int d^d x \left(\frac{1}{2} (\nabla\phi)^2 + U(\phi) \right) \quad . \quad (42)$$

We find from this equation that $[\phi(x)] = [x^{-\frac{d-2}{2}}]$ so that the rescaling involves a factor $a^{-\frac{d-2}{2}}$. Note that the original variables S_i are dimensionless since $S_i = 1$. The function $G^{(2)}$ in Eq. (40) is therefore also dimensionless. This is consistent with the fact that the vectors \vec{x}_i have integer components that label the position of the sites of the lattice and are therefore also dimensionless.

the block-spin system:

$$\langle \mathcal{S}_I \mathcal{S}_J \rangle \sim \frac{e^{-\bar{r}'/\bar{\xi}'}}{\bar{r}'^\theta} . \quad (46)$$

We thus obtain

$$G^{(2)}(\bar{r}', \bar{\xi}') \sim \frac{e^{-\bar{r}'/s}}{\bar{r}'^\theta} s^\theta \quad (47)$$

and, by comparing with Eq. (40), we find that

$$\bar{\xi}' = \frac{\bar{\xi}}{s} \quad \text{and} \quad \lambda(s) = s^{\theta/2} . \quad (48)$$

By comparing all these relations we find that

- the dimensionful correlation lengths of the original and of the block-spin systems are identical: it is a RG-invariant. Thus, the dimensionless correlation lengths decrease as the coarse graining scale s increases. This means that the coarse-grained systems are less correlated than the initial one and that the correlation length varies linearly with the scale. Since the correlation length behaves as a power law close to the critical temperature

$$\xi \sim (T - T_c)^{-\nu} \quad (49)$$

parameterizing $G^{(2)}$ in terms of the (reduced) temperature

$$t = \frac{T - T_c}{T_c} \quad (50)$$

or in terms of the correlation length is equivalent. Thus, saying that $\bar{\xi}$ decreases with the block size s is equivalent to saying that the running reduced temperature $t(s)$ increases with s : the coarse-grained system is “less critical” than the original one. We call relevant a parameter that increases with the scale s .

- the reduced temperature t is one particular coupling among all the couplings \vec{K} . We shall explain in the following why the form of $G^{(2)}$ given above and in which only the correlation length appears is valid at large distance.
- if we combine two RG transformations of scale s_1 and s_2 we must obtain the same result as a unique transformation of scale $s_1 s_2$ (this is self-similarity). This clearly implies that

$$\lambda(s_1)\lambda(s_2) = \lambda(s_1 s_2) . \quad (51)$$

It is straightforward to show that the only solution of this equation is a power law. The example above shows that the exponent of this power law is directly related to the power law behaviour of $G^{(2)}(\bar{r})$ at $T = T_c$, Eq. (48).

3. Properties of the RG Flow: Fixed Points, Critical Surface, Relevant Directions

The RG flow takes place in the space of Hamiltonians, that is in the space of coupling constants \vec{K} (see Fig. 2). We now study this flow. One of its nice properties is that, without specifying any particular statistical system, very general information on second order transitions can be obtained from it by only making very natural assumptions.

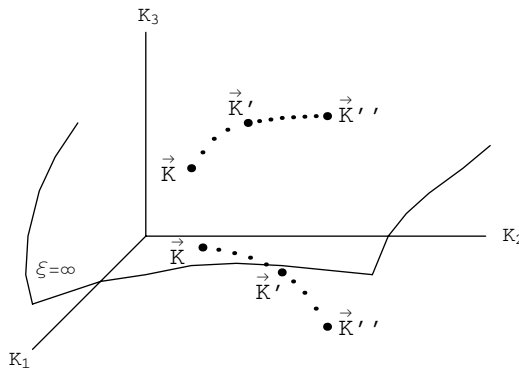


Fig. 2. Schematic representation of the RG flow in the space of couplings, K_1, K_2, \dots . This space is infinite dimensional and the critical surface, defined by the set of points for which the correlation length is infinite is of co-dimension one. Under RG transformations the critical surface is stable whereas a point away from the critical surface is mapped onto another one “further away” from it.

At T_c , $\xi = \infty$ ($\bar{\xi} = \infty$) at a second order phase transition. Thus the point $\vec{K}^{(0)}$ is mapped onto $\vec{K}^{(1)}$ for which $\bar{\xi} = \infty$ again (the block-spin system is also critical). We define the *critical surface* as the set of points \vec{K} in the coupling constant space for which $\bar{\xi} = \infty$. For a second order phase transition only one parameter needs to be fine-tuned to make the system critical (the temperature for instance). Thus, the critical surface is of co-dimension one. It is stable under RG transformations.

If we now consider a system described by a point $\vec{K}^{(0)}$ such that $\bar{\xi} < \infty$

then $\bar{\xi}' = \bar{\xi}/s$ and the block-spin system, being “less critical”, is described by $\vec{K}^{(1)}$ which is “further away” from the critical surface than $\vec{K}^{(0)}$. If we iterate the blocking process, we obtain points $\vec{K}^{(2)}, \vec{K}^{(3)}, \dots$ that will be further and further away from the critical surface.

We shall consider in the following the continuum limit of the Ising model and this will allow us to perform continuous RG transformations. We call in this case the set of points $\vec{K}_s, s \in \mathbb{R}$ a RG trajectory. To different \vec{K}_s on the same RG trajectory correspond systems that are different microscopically (this means at the scale of their own lattice spacing) but that lead to the *same long-distance physics* since they all have the same partition function. Let us now make the fundamental hypothesis (that must be checked on each example):

Hypothesis: For points in a (finite or infinite) domain on the critical surface, the RG flow converges to a fixed point \vec{K}^* : $\vec{K}^* = \vec{T}(\vec{K}^*, s)$.

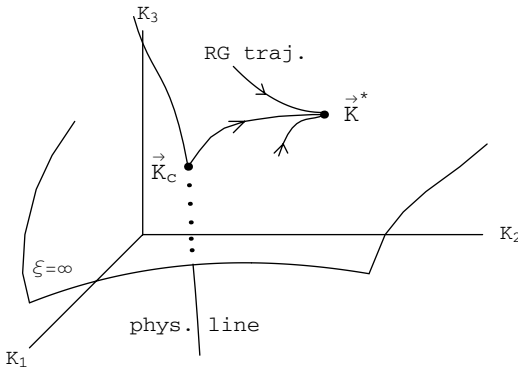


Fig. 3. Schematic representation of continuous RG trajectories on the critical surface. The flow converges to the fixed point \vec{K}^* . For a given model, the “physical line” corresponds to a change of the temperature. It is not a RG trajectory.

This domain is called the basin of attraction of the fixed point \vec{K}^* . Under this hypothesis, the typical topology of the flow on the critical surface is summarized in Fig.3. We have called “physical line” in this figure the line on which the temperature alone is varied. It is *not* a RG trajectory. Reciprocally, a RG trajectory does not, in general, correspond to any transformation doable on a physical system by a human being. It is only a mapping that preserves the partition function without any connection to a

physical transformation.

All the systems that belong to the basin of attraction of \vec{K}^* belong to the same universality class since they all have the same long-distance physics. Note that \vec{K}^* depends on the choice of RG transformations $\vec{T}(\cdot, s)$. Apart from being fixed for this particular choice of RG transformations, this point is nothing else than a particular critical point.

3.1. *Scaling Relations – Linearization of the Flow Around the Fixed Point*

The existence of an attractive (in the critical surface) fixed point is sufficient to explain universality since, independently of the starting point \vec{K}_c on the critical surface, all RG trajectories end at the same point \vec{K}^* . However, universality holds also for systems that are not right at T_c (which is anyway impossible to reach experimentally) but close to T_c . It is thus natural to assume that the flow is continuous in the vicinity of \vec{K}^* . In this case, starting at a point \vec{K} close to the point \vec{K}_c and on the same physical line, the RG trajectory emanating from \vec{K} remains close to the one emanating from \vec{K}_c during many steps before it diverges from the critical surface. It is easy to estimate the typical value of s for which the RG trajectory diverges from the critical surface.

As long as the running (dimensionless) correlation length $\bar{\xi}(s) = \bar{\xi}/s$ remains large, the system behaves as if it were critical and the representative point $\vec{K}(s)$ must be close to the critical surface. When the running correlation length becomes of order 1, the coarse grained system is no longer critical and $\vec{K}(s)$ must be at a distance of order 1 of the critical surface. More precisely, at the beginning of the flow, $\vec{K}(s)$ moves towards \vec{K}^* (by continuity). It remains close to it as long as $\bar{\xi}(s)$ remains large so that the memory of the initial point \vec{K} is largely lost. Finally, it departs from \vec{K}^* when $s \sim \bar{\xi}$. Another and more precise way to state the same result is to say that the running reduced temperature $t(s)$ is of order 1 when $s \sim \bar{\xi}$:

$$t(s) \sim 1 \quad \text{for} \quad s \sim \bar{\xi}. \quad (52)$$

We shall see in the following that the hypothesis of the existence of a fixed point together with this relation are sufficient to predict the existence of power law behaviours for many thermodynamical quantities with critical exponents that are universal and that satisfy “scaling relations” *independently of any specific microscopic model*. Clearly, to obtain these relations, only the vicinity of \vec{K}^* is important since after a few RG steps, all RG

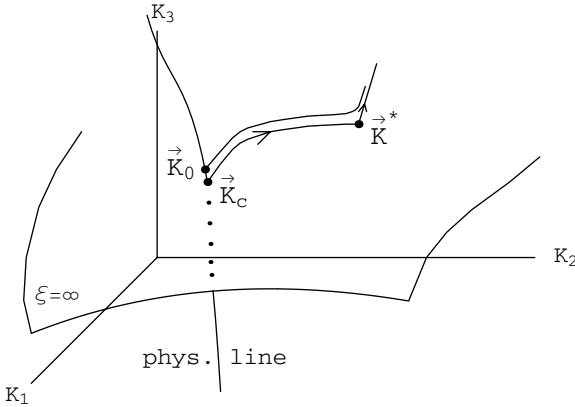


Fig. 4. Schematic representation of two continuous RG trajectories corresponding to the same model for two different temperatures. The trajectory starting at \vec{K}_c is on the critical surface ($T = T_c$) and the other, starting at \vec{K}_0 , is slightly away from it. There exists a RG trajectory emanating from the fixed point \vec{K}^* and which is not on the critical surface. It is an eigendirection of the RG flow corresponding to the relevant direction. Note that we could parametrize the coupling constant space in such a way that the axis denoted K_3 represents the temperature. It is not necessary that this axis coincides with the relevant eigendirection of the RG flow at \vec{K}^* but it is necessary that it has a non vanishing projection onto this axis since the temperature is for sure a relevant parameter. Note that this is non trivial in an infinite dimensional space.

trajectories emanating from points close to criticality are in the vicinity of this point. This will allow us to linearize the RG flow around \vec{K}^* .

For the sake of simplicity, we assume in the following that s can take continuous values (we work in the continuum, that is the continuous limit, $a \rightarrow 0$, has been taken). We also suppose that the RG transformation with $s = 1$ is the identity (no block spin) and that the composition law is such that a RG transformation of parameter s_1 followed by another one of parameter s_2 is equivalent to a transformation of parameter $s_1 s_2$.

As already emphasized, to obtain approximations of the RG flow that are under control, it is preferable to perform a series of infinitesimal RG transformations rather than directly a transformation with s large. It is thus necessary to study the differential form of these transformations. There are two ways to do so:

(i) The first way consists in comparing two RG transformations of parameters s and $s + \epsilon$ performed from an initial set of couplings \vec{K}_{in} :

$$\vec{K}_{s+\epsilon} - \vec{K}_s = \vec{T}(\vec{K}_{in}, s + \epsilon) - \vec{T}(\vec{K}_{in}, s) \tag{53}$$

and thus

$$\frac{\partial \vec{K}_s}{\partial s} = \frac{\partial \vec{T}}{\partial s} \Big|_{(\vec{K}_{in}, s)} \cdot \tag{54}$$

This formula is not convenient since it is expressed in terms of the initial couplings \vec{K}_{in} and, as a consequence, in terms of a parameter s that can be large (and thus for RG transformations that will not be under control once approximations will be implemented). Using approximate expressions of \vec{T} together with this expression would lead to violations of the composition law of RG transformations.

(ii) The second way consists in comparing two sets of running couplings differing by an infinitesimal RG transformation: \vec{K}_s and $\vec{K}_{s(1+\epsilon)}$ and in expressing the result as a function of \vec{K}_s :

$$\vec{K}_{s(1+\epsilon)} - \vec{K}_s = \vec{T} \left(\vec{K}_s, 1 + \epsilon \right) - \vec{T} \left(\vec{K}_s, 1 \right) \tag{55}$$

and thus

$$s \frac{\partial \vec{K}_s}{\partial s} = \frac{\partial \vec{T}}{\partial s} \Big|_{(\vec{K}_s, 1)} \tag{56}$$

It is important to notice here several things.

- The composition law of RG transformations will be automatically satisfied if $\vec{K}(s)$ is obtained by integration of this expression, *even if an approximate expression of \vec{T} is used*. This comes from the fact that, by construction, $\vec{K}(s')$ is computed in terms of $\vec{K}(s)$ by composing the infinitesimal group law between s and s' : this is the meaning of integrating Eq. (56).
- Eq. (56) leads naturally to the logarithmic evolution of the couplings with s , contrary to Eq. (54).
- In Eq. (54), the variations of \vec{T} had to be known for only one set of couplings \vec{K}_{in} but for all s . This was the real problem of this approach since we are interested in large values of s . In Eq. (56) only the variation of \vec{T} for $s = 1$ have to be known. The price to pay is that this must be known for all values of \vec{K}_s . In perturbation theory this is not problematic as long as the running coupling used in the perturbation expansion remains small all along the flow. Of course, if \vec{K}_s converges to \vec{K}^* such that the coupling(s) of the perturbation expansion at the fixed point is large, perturbation theory runs into trouble.

We define

$$\vec{\beta}(\vec{K}_s) = \frac{\partial \vec{T}}{\partial s} \Big|_{(\vec{K}_s, 1)} \tag{57}$$

which gives the evolution of the couplings of the model with the scale. Note that $\vec{\beta}(\vec{K}_s)$ does not depend explicitly on s since the right hand side of Eq. (57) is evaluated at $s = 1$ and thus does not depend on this variable.^m The only dependence on s of this function is implicit and comes through the dependence of the couplings \vec{K}_s on s . By definition of the fixed point \vec{K}^* of the RG

$$\vec{\beta}(\vec{K}^*) = \vec{0} . \tag{58}$$

We thus obtain

$$s \frac{d\vec{K}_s}{ds} - s \frac{d\vec{K}^*}{ds} = \vec{\beta}(\vec{K}_s) - \vec{\beta}(\vec{K}^*) = \frac{d\vec{\beta}}{d\vec{K}_s} \Big|_{\vec{K}^*} \cdot \delta\vec{K}_s + O(\delta\vec{K}_s^2) \tag{59}$$

where, by definition

$$\delta\vec{K}_s = \vec{K}_s - \vec{K}^* \tag{60}$$

and

$$\frac{d\vec{\beta}}{d\vec{K}_s} \Big|_{\vec{K}^*} \quad \text{is a matrix} \quad \mathcal{M}_{ij} = \frac{d\beta_i}{dK_{s,j}} \Big|_{\vec{K}^*} . \tag{61}$$

Thus, in the neighbourhood of \vec{K}^* :

$$s \frac{d\delta\vec{K}_s}{ds} = \mathcal{M} \delta\vec{K}_s . \tag{62}$$

\mathcal{M} is not symmetric in general. We suppose in the following that it can be diagonalized and that its eigenvalues are realⁿ (this must be checked on each example). We moreover suppose that the set of eigenvectors $\{\vec{e}_i\}$ is a complete basis:

$$\mathcal{M}\vec{e}_i = \lambda_i \vec{e}_i \quad \text{with} \quad \lambda_i \in \mathbb{R} \tag{63}$$

and

$$\delta\vec{K}_s = \sum_i v_i(s) \vec{e}_i . \tag{64}$$

^mIn perturbation theory, the fact that the β -function of the marginal coupling is cut-off independent and thus scale independent is a consequence of the renormalizability of the model.

ⁿIt can happen that some eigenvalues are complex. In this case, the RG flow around the fixed point is spiral-like (focus) in the corresponding eigendirections.

Under these hypotheses, we obtain:

$$s \frac{d\delta \vec{K}_s}{ds} = \sum_i v_i(s) \mathcal{M} \vec{e}_i = \sum_i \lambda_i v_i(s) \vec{e}_i \quad (65)$$

and thus

$$s \frac{dv_i(s)}{ds} = \lambda_i v_i(s) \quad \Rightarrow \quad v_i(s) = v_i(1) s^{\lambda_i} . \quad (66)$$

We conclude that around the fixed point, the RG flow behaves as power laws along its eigendirections (but if $\lambda_i = 0$). There are three possibilities:

- $\lambda_i > 0$ and $v_i(s) \nearrow$ when $s \nearrow$ which means that the flow in the direction \vec{e}_i goes away from \vec{K}^* . \vec{e}_i is called a relevant direction and v_i a relevant coupling. As we have already seen, the reduced temperature is relevant since the system is less and less critical along the RG flow which means that $t(s)$ increases with s .
- $\lambda_i < 0$ and $v_i(s) \searrow$ when $s \nearrow$ which means that the flow in the direction \vec{e}_i approaches \vec{K}^* . \vec{e}_i is called an irrelevant direction and v_i an irrelevant coupling.
- $\lambda_i = 0$ and v_i is said to be marginal. It is necessary to go beyond the linear approximation to know whether it is relevant or irrelevant. The flow in this direction is slow: it is logarithmic instead of being a power law. This is important since it is the case of the renormalizable couplings in the upper critical dimension (that is $d = 4$ for the Ising model).

Physically, we expect a small number of relevant directions since, clearly, there are as many such directions as there are parameters to be fine-tuned to be on the critical surface (the co-dimension of this surface). For second order phase transitions there is one coupling to be fine-tuned to make the system critical: the temperature (for instance). There is actually one more parameter which is relevant but which is also \mathbb{Z}_2 non-invariant: the magnetic field. All the other directions of the RG flow are supposed to be irrelevant (this can be checked explicitly once a specific model is given).

In the following, we shall use the language of magnetic systems although our discussion will be completely general. We shall suppose that together with the magnetic field, there is only one other relevant direction. We shall show that this implies scaling laws for all thermodynamical quantities with only two independent critical exponents.

3.2. The Correlation Length and the Spin-Spin Correlation Function

Let us start by studying the critical physics of a model at zero external magnetic field having only one relevant coupling. We call v_1 this coupling and λ_1 the eigenvalue of the flow at \vec{K}^* associated with v_1 . We order the other eigen-couplings v_2, v_3, \dots in such a way that $0 > \lambda_2 > \lambda_3 \dots$. Physically, the reduced temperature is expected to be a relevant parameter. It is not necessary that it corresponds to the relevant eigen-direction of the RG flow but it is that its projection onto this eigen-direction be non vanishing. For the sake of simplicity we ignore this subtlety that does not play any role in the following and we assume that $v_1 = t$. We have seen, Eq. (40), that for large \bar{r} :

$$G^{(2)}(\bar{r}, \vec{K}) \simeq \lambda^{-2}(s) G^{(2)}\left(\frac{\bar{r}}{s}, \vec{K}_s\right) \quad \text{with} \quad \lambda(s) = s^{\theta/2} . \quad (67)$$

From these relations we can deduce two relations, one at T_c and another one away from T_c :

- For $t = 0$ and in the neighbourhood of \vec{K}^* :

$$G^{(2)}(\bar{r}, 0, v_2, v_3, \dots) \simeq s^{-\theta} G^{(2)}\left(\frac{\bar{r}}{s}, 0, s^{\lambda_2} v_2, s^{\lambda_3} v_3, \dots\right) . \quad (68)$$

Let us now suppose that we integrate out all fluctuations between scales a and r . This amounts to taking $s = \bar{r}$:

$$G^{(2)}(\bar{r}, 0, v_2, \dots) \simeq \bar{r}^{-\theta} G^{(2)}(1, 0, \bar{r}^{\lambda_2} v_2, \dots) . \quad (69)$$

Since $\lambda_2 < 0$, we obtain that for \bar{r} sufficiently large: $\bar{r}^{\lambda_2} v_2 \ll 1$ and thus

$$G^{(2)}(\bar{r}, 0, v_2, \dots) \simeq \bar{r}^{-\theta} G^{(2)}(1, 0, 0, \dots) . \quad (70)$$

From the definition of the anomalous dimension

$$G^{(2)}(r) \underset{T=T_c}{\simeq} \frac{1}{r^{d-2+\eta}} . \quad (71)$$

we find that

$$\theta = d - 2 + \eta . \quad (72)$$

Note that the $d - 2$ part of θ is purely dimensional: it corresponds to the engineering dimension of the spin field. The η part corresponds to a dynamical contribution. It can be proven rigorously and for many field theories (as the ϕ^4 theory) that $\eta \geq 0$. This means that the fluctuations contribute to decrease the correlations of the system with the distance.

• For $t \neq 0$, $\xi < \infty$ and we can therefore integrate all fluctuations between scales a and ξ by taking $s = \bar{\xi}$. As explained above, Eq. (52), the running temperature at this scale must be of order 1 since the coarse-grained system at this scale is no longer critical. We thus deduce

$$t(s = \bar{\xi}) = t \bar{\xi}^{\lambda_1} \sim 1 . \tag{73}$$

From the definition of the critical exponent ν :

$$\xi \sim t^{-\nu} \tag{74}$$

we find that

$$\nu = \frac{1}{\lambda_1} . \tag{75}$$

The behaviour of the correlation function close to the critical temperature follows from Eqs. (67), (73) and from $\bar{\xi} \gg 1$:

$$\begin{aligned} G^{(2)}(\bar{r}, t, v_2, \dots) &\simeq \bar{\xi}^{-\theta} G^{(2)}\left(\frac{\bar{r}}{\bar{\xi}}, t \bar{\xi}^{\lambda_1}, v_2 \bar{\xi}^{\lambda_2} \dots\right) \simeq \\ &\simeq \bar{\xi}^{-\theta} G^{(2)}\left(\frac{\bar{r}}{\bar{\xi}}, 1, 0 \dots\right) \simeq \bar{r}^{-\theta} f\left(\frac{\bar{r}}{\bar{\xi}}\right) . \end{aligned} \tag{76}$$

We can see on this relation that close to T_c and at large distance the correlation function is no longer a function of infinitely many coupling constants but of only one parameter, the correlation length. One can also see that the Yukawa-like form of the correlation function that we have considered in Eq. (45) has the right form.

3.3. *Scaling of the Correlation Function in the Presence of a Magnetic Field – Relation Among Exponents*

We now couple the system to a uniform magnetic field. In a RG transformation:

$$B \sum_i S_i = B \sum_I \frac{s^d}{\lambda(s)} S_I \stackrel{\text{def}}{=} B_s \sum_I S_I . \tag{77}$$

Thus

$$B_s = s^{d-\theta/2} B . \tag{78}$$

We call

$$\lambda_B = d - \theta/2 = \frac{d + 2 - \eta}{2} \tag{79}$$

the magnetic eigenvalue. Since η is always smaller than $d + 2$ (even for $d = 1$), $\lambda_B > 0$ so that the magnetic field is a relevant variable.

Relation (67) can be trivially generalized in the presence of a magnetic field and to any correlation function. We now consider the magnetization per spin which is the one-point function: $G^{(1)} = m = \langle S \rangle$. Clearly, it behaves as:

$$m(t, B, \dots) \simeq s^{-\theta/2} m(s^{\lambda_1} t, s^{\lambda_B} B, \dots) . \quad (80)$$

Once again, we can obtain several relations among exponents by considering the scaling of physical quantities at and away from T_c .

- For $t = 0$ and $B \neq 0$

$$m(0, B, \dots) \simeq s^{-\theta} m(0, s^{\lambda_B} B, \dots) . \quad (81)$$

and, by taking s such that $s^{\lambda_B} B \simeq 1$, we obtain

$$m(0, B, \dots) \simeq B^{\theta/2\lambda_B} m(0, 1, \dots) \sim B^{1/\delta} \quad (82)$$

by definition of the exponent δ . We thus find:

$$\delta = \frac{d + 2 - \eta}{d - 2 + \eta} \quad (83)$$

- For $t < 0$ and $B = 0$ we find:

$$m(t, 0, \dots) \simeq s^{-\theta/2} m(s^{\lambda_1} t, 0, \dots) . \quad (84)$$

and by taking $s = \bar{\xi}$:

$$m(t, 0, \dots) \simeq \bar{\xi}^{-\theta/2} m(1, 0, \dots) \simeq (T_c - T)^\beta \quad (85)$$

by definition of the exponent β . Thus we find

$$\beta = \nu \frac{d - 2 + \eta}{2} . \quad (86)$$

- For the susceptibility $\chi = \partial m / \partial B$, at $t \neq 0$, we obtain:

$$\chi(t, B, \dots) \simeq s^{-\theta/2} \frac{\partial}{\partial B} m(s^{\lambda_1} t, s^{\lambda_B} B, \dots) \simeq s^{\lambda_B - \theta/2} \chi(s^{\lambda_1} t, s^{\lambda_B} B, \dots) . \quad (87)$$

By taking $B = 0$ and $s = \bar{\xi}$, we find:

$$\chi(t, 0, \dots) \simeq \bar{\xi}^{\lambda_B - \theta/2} \chi(1, 0, \dots) \sim (T - T_c)^{-\gamma} . \quad (88)$$

Thus, by definition of γ

$$\gamma = \nu(2 - \eta) . \quad (89)$$

Finally, for the exponent α of the specific heat, the calculation is subtler and requires to consider the free energy. One finds:

$$\alpha = 2 - \nu d . \quad (90)$$

To conclude, we have found that the hypothesis of the existence of a fixed point in the RG flow is sufficient to explain:

- universality since the critical exponents depend only on the RG flow around the fixed point and not on the point \vec{K}_c representing the system when it is critical;
- the scaling behaviour of the thermodynamic quantities such as the magnetization, the susceptibility, the correlation function, etc.;
- the relations existing between critical exponents;
- the irrelevance of infinitely many couplings and the fact that the scaling of the correlation length with the temperature drives the scaling of many other thermodynamic quantities.

Two remarks are in order here. First, for second order transitions, only two exponents are independent, ν and η for instance. Second, universality is a much more general concept than what we have seen here on critical exponents. It is possible to show in particular that the RG enables to understand why it is possible to keep track of only a small number of coupling constants in field theory while these theories involves infinitely many degrees of freedom and thus, *a priori*, infinitely many couplings.

3.4. *The Example of the Two-Dimensional Ising Model on the Triangular Lattice*

This model is very famous as an example where the block-spin method à la Kadanoff can be implemented rather easily. We shall not repeat the explicit calculation of the RG flow that can be found in most textbooks on this subject. We shall only give the main ideas that will be relevant for our purpose.

As already explained above, the idea is to partition the lattice in triangular plaquettes, to build an Ising block-spin \mathcal{S}_I for each plaquette by a majority rule, Eq. (14), and to integrate out the fluctuations inside the plaquettes compatible with a given value of \mathcal{S}_I . We have already said that the implementation of this idea imposes to take into account all possible \mathbb{Z}_2 -invariant couplings among the spins. From a practical point of view it is thus necessary to perform truncations. The approximations usually

performed consists in keeping only some couplings and projecting the RG flow onto this restricted space of couplings. We shall see that almost the same idea is used in the field theoretical implementation of non-perturbative renormalization group. The simplest truncation consists in keeping only the nearest neighbour interaction, that is K_1 , in Hamiltonian (21) as well as the magnetic field B . For concreteness, let us give here the result of the RG transformation on the triangular lattice for a small magnetic field:

$$K_{1,s} = 2K_1 \frac{e^{-K_1} + e^{3K_1}}{3e^{-K_1} + e^{3K_1}}, \quad (91)$$

$$B_s = 3B \frac{e^{-K_1} + e^{3K_1}}{3e^{-K_1} + e^{3K_1}}. \quad (92)$$

A fixed point is trivially found

$$K_1^* = 0.336, \quad (93)$$

$$B^* = 0. \quad (94)$$

The exponent ν can be computed as well as δ and we find:

$$\nu = 1.118, \quad \delta = 2.17. \quad (95)$$

All the other exponents can be found using the scaling relations among them. The exact values, given by Onsager's solution are

$$\nu = 1, \quad \delta = 15. \quad (96)$$

Let us notice that these RG results can be systematically improved by keeping more and more couplings. They drastically and rapidly improve with the next orders of approximations.

4. The Non-Perturbative Renormalization Group

4.1. Introduction

All the different implementations of the non-perturbative renormalization group (NPRG) rely on Kadanoff-Wilson's ideas of block spins, coarse graining and *effective long-distance theories*. However, they can substantially differ as for the way they are implemented. In the framework of field theory, there exists two main formulations: the Wilson (also called Wilson-Polchinski) approach^{6-8,13,14} and the "effective average action" approach.¹⁵⁻²⁸ We shall deal with the second one which is not the best known, probably for historical reasons. Since it is nevertheless interesting to have an idea of the Wilson-Polchinski formulation, we start by this approach although we shall not study it in detail.

4.1.1. The Wilson-Polchinski Approach

We shall work in the context of statistical field theory (at equilibrium). This means that we shall not deal with a Minkowski metrics (this brings its own difficulties) and that we suppose a continuum description of the systems under study. The microscopic physics is supposed to correspond to a scale Λ in momentum space which is – up to a factor of unity – the inverse of a microscopic length (a lattice spacing, an intermolecular distance, etc.). The partition function is thus given by a functional integral:

$$Z[B] = \int d\mu_{C_\Lambda}(\phi) \exp\left(-\int V(\phi) + \int B\phi\right) \tag{97}$$

where $d\mu_{C_\Lambda}$ is a (functional) Gaussian measure with a cut-off at scale Λ :

$$d\mu_{C_\Lambda} = \mathcal{D}\phi(x) \exp\left(-\frac{1}{2} \int_{x,y} \phi(x) C_\Lambda^{-1}(x-y) \phi(y)\right) \tag{98}$$

with (in momentum space)

$$C_\Lambda(p) = (1 - \theta_\epsilon(p, \Lambda)) C(p) \tag{99}$$

and C is the usual free propagator:

$$C(p) = \frac{1}{p^2 + r} . \tag{100}$$

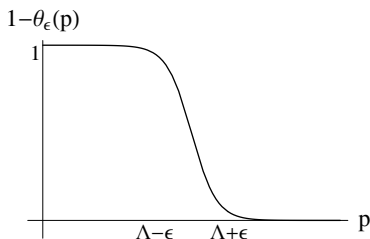


Fig. 5. A typical cut-off function in the Wilson-Polchinski approach.

The “cut-off” function θ_ϵ is a step function in p -space starting at Λ and smoothed around Λ on an interval of typical width ϵ , see Fig.5. If $\epsilon = 0$, the quadratic part of the Hamiltonian becomes the usual gradient and “mass” term, cut-off at scale Λ :

$$\frac{1}{2} \int_0^\Lambda \frac{d^d q}{(2\pi)^d} \phi(-p)(p^2 + r)\phi(p) . \tag{101}$$

The role of ϵ is to smoothen the sharp cut-off at Λ which is conceptually simple but technically unpleasant.

We want to implement the block-spin idea in our field theory framework, that is to separate $\phi_p = \phi(p)$ into “rapid” and “slow” modes (with respect to a scale k). The slow modes will play the role of block-spins while the rapid ones will correspond to fluctuations inside the blocks. It is convenient to work in Fourier space where the derivative operators are diagonalized (and the cut-off is simple).

We define

$$\phi_p = \phi_{p,<} + \phi_{p,>} \quad (102)$$

and associate

$$\phi_p \rightarrow C_\Lambda(p), \quad \phi_{p,<} \rightarrow C_k(p), \quad \phi_{p,>} \rightarrow C_\Lambda(p) - C_k(p). \quad (103)$$

It is important to notice that ϕ is the sum of $\phi_{p,<}$ and $\phi_{p,>}$ for all $p \in \Lambda$: it does not coincide with $\phi_{p,<}$ on $[0, k]$ and with $\phi_{p,>}$ on $[k, \Lambda]$. The meaning of $\phi_{<}$ and $\phi_{>}$ comes from their propagator $C_k(p)$ and $C_\Lambda(p) - C_k(p)$ respectively. A beautiful identity allows us to rewrite the original partition function in terms of $\phi_{<}$ and $\phi_{>}$, $C_k(p)$ and $C_\Lambda(p) - C_k(p)$ and to perform (at least formally) the integration on the slow modes:

$$d\mu_{C_\Lambda}(\phi) = d\mu_{C_k}(\phi_{<}) d\mu_{C_\Lambda - C_k}(\phi_{>}). \quad (104)$$

Let us show it on a one-dimensional integral:

$$I = \int_{-\infty}^{+\infty} dx e^{-x^2/2\gamma}. \quad (105)$$

where x is the analogue of ϕ and γ of C_Λ . We now define

$$\begin{cases} x = y + z \\ \gamma = \alpha + \beta \end{cases} \quad (106)$$

y and z are the analogues of $\phi_{>}$ and $\phi_{<}$ respectively and α and β are the analogues of $C_\Lambda - C_k$ and of C_k . Then

$$I \propto \int_{-\infty}^{+\infty} dy dz e^{-y^2/2\alpha} e^{-z^2/2\beta}. \quad (107)$$

Demonstration.

We define:

$$J = \int_{y,z} e^{-y^2/2\alpha - z^2/2\beta} \quad (108)$$

and we rewrite the exponent

$$-\frac{y^2}{2\alpha} - \frac{z^2}{2\beta} = -\frac{1}{2} \left(\frac{1}{\alpha} + \frac{1}{\beta} \right) y^2 + \frac{xy}{\beta} - \frac{x^2}{2\beta} \tag{109}$$

$$= -\frac{1}{2} \frac{\gamma}{\alpha\beta} \left(y - \frac{\alpha}{\gamma} x \right)^2 + \frac{\alpha}{2\beta\gamma} x^2 - \frac{x^2}{2\beta}. \tag{110}$$

We now define

$$u = y - \frac{\alpha}{\gamma} x \tag{111}$$

and change variables: $(y, z) \rightarrow (u, x)$. The Jacobian is 1 and thus:

$$J = \int_{u,x} e^{-\gamma u^2/2\alpha\beta - x^2/2\gamma} = \sqrt{\frac{2\pi\alpha\beta}{\gamma}} I. \tag{112}$$

Actually, we are not only interested in Gaussian integrals and our result can be trivially generalized :

$$\int_{-\infty}^{+\infty} dx e^{-x^2/2\gamma - V(x)} \propto \int_{-\infty}^{+\infty} dy dz e^{-y^2/2\alpha - z^2/2\beta - V(y+z)}. \tag{113}$$

This result can be generalized straightforwardly to functional integrals since it is a property of the Gaussian integrals.^o It becomes

$$\int d\mu_{C_\Lambda}(\phi) e^{-\int V(\phi)} \propto \int d\mu_{C_k}(\phi_<) d\mu_{C_\Lambda - C_k}(\phi_>) e^{-\int V(\phi_< + \phi_>)}. \tag{114}$$

Thus, by performing formally the integration on $\phi_>$, we define a running “potential” V_k at scale k :

$$e^{-\int V_k(\phi_<)} = \int d\mu_{C_\Lambda - C_k}(\phi_>) e^{-\int V(\phi_< + \phi_>)} \tag{115}$$

with, by definition, $V_\Lambda = V$, the initial potential. Let us emphasize that it is called a potential because we do not have included in it the quadratic derivative term. However, generically, as soon as $k < \Lambda$, V_k involves derivative terms with, moreover, any power of the derivatives.

It is possible to write down a differential equation for the evolution of V_k with k : this is the Wilson-Polchinski equation derived in B.1.⁶⁻⁸ It is a possible starting point for a non-perturbative formulation of the RG. The one we shall use is mathematically equivalent to this one although it is more convenient in many respects when approximations are used. Before going to this second formulation, let us make some remarks here.

^oWe first consider a N -dimensional Gaussian integral and then take the limit $N \rightarrow \infty$.

- $\phi_{<}$ at scale k represents approximately a spatial average of ϕ over a volume of order k^{-d} . It is not a thermal average. For $k = 0$, $\phi_{p=0,<}$ would represent only what is (improperly) called the magnetization mode: $\int d^d x \phi(x)$ which is *not* the magnetization. It has a highly non-trivial probability distribution and the true magnetization is the thermal average of this mode computed with this distribution. Thus $\phi_{<}$ at scale $k \neq 0$ is *not* a precursor of the order parameter: it is still a stochastic variable whose physical interpretation is not so trivial.
- The flow of “potentials” $V_k(\phi_{<})$ does not contain all the information on the initial theory: the correlation functions of the rapid modes cannot be computed from this flow. It is necessary to first couple the system to an arbitrary “source” $B(x)$ and to follow the flow of this term to reconstruct the correlations of the initial fields in the whole momentum range $p \in [0, \Lambda]$. Thus the information is split into two different kinds of terms: the k -dependent Hamiltonians which give rise to a flow for all the couplings involved in these Hamiltonians and the source term. Fortunately, much information about the theory (e.g. critical behaviour) can be obtained from the flow of Hamiltonians alone. It is nevertheless an open question to know if the difficulties encountered with this method are related to the fact that the information on the Green functions is not contained in the flow of Hamiltonians.
- The effective Hamiltonians V_k (and the corresponding Boltzmann weights) are highly abstract objects! One should remember in particular that the RG transformations do not correspond to any transformations that a human being can perform on the system. This is a purely theoretical idea that moreover will be useful only when approximations are used.
- The flow equation on V_k was written more than thirty years ago but was not much used in actual physical problems before the mid 90’s (for exceptions see, Golner, Newman, Riedel, Bagnuls, Bervillier, Zumbach, etc.).^{13,29–35} There are three main reasons for this strange fact. First, perturbation theory was extremely successful for $O(N)$ models (as well as in particle physics) and the need for NPRG was not obvious in many situations. Second, perturbation theory was believed to correspond to a controlled approximation whereas approximations performed in the NPRG framework seemed uncontrolled. However, this is *completely wrong*. Perturbation series are

not convergent.⁴ They are asymptotic series, at best.^P For the $O(N)$ model at l loops, the coefficient in front of u^l of a correlation function behaves at large l as $l!(-a)^l$ with a a real number. Thus, even in cases where many orders of the perturbation expansion have been computed, resummation methods of the renormalized series are required (Padé-Borel, conformal-Borel, etc.). In many cases, these resummation techniques fail to produce convergent results. As for the NPRG, there is no general theorem about the convergence of the series of approximations that are used. However, from the few results yet obtained, it seems that this method has good convergence properties.^{28,36-40} It is however too early to draw any firm conclusion on this question. Third, it seemed that the anomalous dimension was crucially depending on the choice of cut-off function θ_ϵ that separates the rapid and the slow modes whereas it should be independent on it. This was especially important when studying the $O(N)$ models in two dimensions where it seemed impossible to reproduce the perturbative results obtained from the non-linear sigma model. This difficulty is very simply overcome in the “effective average action” approach.^{28,41,42}

Let us finally mention two other “psychological” difficulties related to NPRG.

- The NPRG equation on V_k can be truncated in perturbation theory. This enables, of course, to recover the usual perturbation expansion (what else could it lead to?). However, the way it was implemented most of the time in the 70’s did not allow to retrieve the two-loop results.^Q It is still widely believed for this reason, even by “specialists”, that Wilson’s method does not work at two-loop order and beyond!
- $V_k(\phi)$ involves infinitely many couplings contrary to perturbation theory that involves only the renormalizable ones. For this rea-

^PThe $O(N)$ models are completely exceptional in this respect since they are the only ones for which it has been proven that the series of the β -function is Borel-summable in $d = 3$ (in the massive scheme). In all other cases, either this is not known or it is known that the series are not Borel-summable. For QED, this is not yet a problem because the smallness of the fine structure constant ensures up to now an apparent convergence of the perturbative results.

^QThese calculations did not correspond to a series expansion of the exact NPRG equation on V_k . They enabled to retrieve the one-loop results easily but became very cumbersome beyond one-loop.

son, it is widely believed that the recourse to numerical methods is unavoidable in the NPRG approach whereas it is not in the perturbative one. This is not fully correct for two reasons. First, even in perturbation theory the RG flow cannot be integrated analytically in general. Second, even in the NPRG approach, very crude approximations, involving only very few couplings, often lead both to analytically tractable computations and to highly non-trivial non-perturbative results.²⁸

Let us now turn to the other implementation of the NPRG formalism.

4.2. *The Effective Average Action Method*

Many formal results about the NPRG method as well as some “physical” results have been obtained within the Wilson-Polchinski approach.^r However, the revival of Wilson’s ideas as well as their concrete implementation in the last fifteen years is largely linked with the development of an alternative, although formally equivalent, formulation.^{19–28} In practice, this has allowed to compute in a reasonable way the anomalous dimension η and, more importantly, to study the physics of the $O(N)$ models and of many others, in all dimensions, including two.^{28,42} Moreover, the whole scheme is more intuitive, allows to retrieve very easily the one-loop results in both $4 - \epsilon$ and $2 + \epsilon$ dimensions and in the large N limit. This has convinced many specialists of the subject to work with this formalism.

4.2.1. *Block-Spins, Coarse Graining, Legendre Transform, etc.*

The original Kadanoff-Wilson’s idea is to perform coarse graining and to map Hamiltonians onto other Hamiltonians at larger scales. The Hamiltonians thus obtained are the Hamiltonians of the modes *that have not yet been integrated out* in the partition function, that is $\phi_{<}$. As already emphasized, these Hamiltonians are very abstract objects.^s Instead of computing this sequence of Hamiltonians, we can compute the Gibbs free energy $\Gamma[M]$ of the rapid modes (that is $\phi_{>}$) that have *already been integrated out*.^t

^rSee the impressive and inspiring works of Bagnuls and Bervillier about the formal aspects of Wilson’s RG, as well as their criticisms of the perturbative approach.^{43–45}

^sIt is impossible to get easily any physical information from it apart at “mean field-like” level: a functional integral has still to be performed.

^tThe Helmholtz free energy is $F = -k_B T \log Z[B]$. It is a functional of the source $B(x)$. The Gibbs free energy is obtained by a Legendre transform from F and is a functional

Thus, the idea is to build a one-parameter family of models, indexed by a scale k such that²⁸

- when $k = \Lambda$, that is when no fluctuation has been integrated out, the Gibbs free energy $\Gamma_k[M]$ is equal to the microscopic Hamiltonian:¹⁴

$$\Gamma_{k=\Lambda}[M] = H[\phi = M] . \quad (116)$$

- when $k = 0$, that is when all fluctuations are integrated out, $\Gamma_{k=0}$ is nothing but the Gibbs free energy of the original model:

$$\Gamma_{k=0}[M] = \Gamma[M] . \quad (117)$$

Thus, as k decreases more and more fluctuations are integrated out. The magnetization at scale k is therefore a precursor of the true magnetization (obtained at $k = 0$) and the free energy Γ_k , also called the effective average action, a precursor of the true free energy Γ (also called the effective action).

Let us notice two points. First, k plays the role of an *ultra-violet* cut-off for the slow modes $\phi_{<}$ in the Wilson-Polchinski formulation (analogous to Λ in the original model). It plays the role of an *infrared* cut-off in the effective average action method since Γ_k is the free energy of the rapid modes. Second, the slow modes play a fundamental role in the Wilson-Polchinski approach whereas they are absent of the effective average action method which involves only the (free energy of the) rapid modes. As a by-product, we shall see that all the information on the model (RG flow, existence of a fixed point, computations of correlation functions, etc.) are contained in a single object: $\Gamma_k[M]$. This is a rather important advantage of this method compared to the Wilson-Polchinski one.

Now the question is to build explicitly this one-parameter family of $\Gamma_{k\neq}$. The idea is to decouple the slow modes of the model in the partition function. A very convenient implementation of this idea is to give them a large mass.²⁸ In the language of particle physics, a large mass corresponds to a small Compton wavelength (\hbar/mc) and thus to a small range of distances where quantum fluctuations are important. A very heavy particle decouples from the low energy (compared to its mass) physics since it can play a role at energies below its mass threshold only through virtual processes.

of the magnetization $M(x)$. It is the generating functional of the one-particle irreducible (1PI) correlation functions.

¹⁴Let us emphasize that at the mean-field approximation, the Gibbs free energy of the system is identical to the Hamiltonian. Eq. (116) is an exact version of this statement (remember that no fluctuation is taken into account at the mean-field level).

These processes are themselves suppressed by inverse powers of the mass of the heavy particle coming from its propagator. In the language of critical phenomena, the “mass” term $r\phi^2/2$ in the Hamiltonian corresponds to the deviation to the critical temperature: $r \propto T - T_c$ (at the mean-field level, at least). Thus a large “mass” r corresponds to a theory which is far from criticality ($\xi \sim a$), that is where thermal fluctuations are small.

Therefore, the idea is to build a one-parameter family of models for which a “momentum-dependent mass term” has been added to the original Hamiltonian:²⁸

$$Z_k[B] = \int \mathcal{D}\phi(x) \exp \left(-H[\phi] - \Delta H_k[\phi] + \int B\phi \right) \quad (118)$$

with

$$\Delta H_k[\phi] = \frac{1}{2} \int_q R_k(q) \phi_q \phi_{-q} . \quad (119)$$

The function $R_k(q)$ – called the cut-off function from now on – must be chosen in such a way that

- when $k = 0$, $R_{k=0}(q) = 0$ identically ($\forall q$) so that:

$$Z_{k=0}[B] = Z[B] . \quad (120)$$

This will ensure that the original model is recovered when all fluctuations are integrated out, see Eq. (117).

- when $k = \Lambda$, all fluctuations are frozen. This will ensure that relation (116) is obeyed. Giving an infinite mass to all modes $q \in [0, \Lambda]$ freezes their propagation completely and we must therefore take

$$R_{k=\Lambda}(q) = \infty, \quad \forall q . \quad (121)$$

An approximate, but convenient way to achieve this goal is to choose a function $R_{k=\Lambda}$ not infinite but of the order of Λ^2 for all momenta.

- when $0 < k < \Lambda$, the rapid modes (those for which $|q| > k$) must be almost unaffected by $R_k(q)$ which must therefore almost vanish for these modes:

$$R_k(|q| > k) \simeq 0 . \quad (122)$$

On the contrary, the slow modes must have a mass that almost decouple them from the long distance physics.

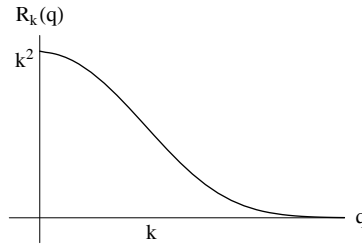


Fig. 6. A typical cut-off function in the effective average action approach.

Remembering that R_k is homogeneous to a mass square, it is not difficult to guess its generic shape, at least if we require that it is an analytic function of q^2 , see Fig.6.

We shall discuss in the following some convenient choices for R_k but let us first define precisely what Γ_k is. In principle, having defined Z_k by Eq. (118), the Legendre transform of $\log Z_k$ should be unambiguous and should lead to Γ_k . Let us follow this program and see that there is a subtlety. We thus define

$$W_k[B] = \log Z_k[B] \tag{123}$$

which is the Helmholtz free energy, up to the $-k_B T$ term that plays no role in what follows. The Legendre transform of W_k is defined by

$$\Gamma'_k[M] + W_k[B] = \int BM \tag{124}$$

where the magnetization $M(x)$ is, by definition the average of $\phi(x)$ and is therefore:

$$M(x) = \frac{\delta W_k}{\delta B(x)}. \tag{125}$$

Of course, for $k \rightarrow 0$, $R_k \rightarrow 0$, $W_k \rightarrow W$ and thus $\Gamma'_k \rightarrow \Gamma =$ Gibbs free energy of the system. However, it is easy to show that $\Gamma'_\Lambda[M] \neq H[M]$ contrary to what is expected, see Eq. (116). This comes from the $\Delta H_{k=\Lambda}$ term, which is large. Thus, we prefer to work with a modified free energy where the R_k term has been subtracted in Γ_k .²⁸

$$\Gamma_k[M] + W_k[B] = \int BM - \frac{1}{2} \int_q R_k(q) M_q M_{-q}. \tag{126}$$

The R_k term in Eq. (126) does not spoil the limit $k \rightarrow 0$ since in this limit it vanishes for $\forall q$. Let us now show that Eq. (126) is the correct definition of Γ_k leading to the limit $\Gamma_{k=\Lambda}[M] = H(M)$, Eq. (117).

4.3. An Integral Representation of Γ_k and the Limit $k \rightarrow \Lambda$

We start from the definition of Z_k , Eq. (118) and from the definition of Γ_k , Eq. (126). We deduce by differentiation (see, B.2):

$$B_x = \frac{\delta \Gamma_k}{\delta M_x} + \int_y R_k(x-y) M_y . \quad (127)$$

Thus, by substituting Eq. (126) and Eq. (127) into the definition of W_k we obtain:

$$e^{-\Gamma_k[M]} = \int \mathcal{D}\phi \exp \left(-H[\phi] + \int_x \frac{\delta \Gamma_k}{\delta M_x} (\phi_x - M_x) \right) \quad (128)$$

$$\exp \left(-\frac{1}{2} \int_{x,y} (\phi_x - M_x) R_k(x-y) (\phi_y - M_y) \right) . \quad (129)$$

If we choose a function $R_k(q)$ that diverges for all q as $k \rightarrow \Lambda$ then, in this limit:

$$\exp \left(-\frac{1}{2} \int_{x,y} (\phi_x - M_x) R_k(x-y) (\phi_y - M_y) \right) \sim \delta(\phi - M) \quad (130)$$

that is, it behaves as a functional Dirac delta. Therefore,

$$\Gamma_k[M] \rightarrow H[\phi = M] \quad \text{as } k \rightarrow \Lambda \quad (131)$$

if the cut-off R_k is such that it diverges in this limit. If R_k does not diverge and is only very large,

$$\Gamma_{k=\Lambda} \sim H . \quad (132)$$

5. The Exact RG Equation and Its Properties

The RG equation on Γ_k , that is the differential equation $\partial_k \Gamma_k = f(\Gamma_k)$ is derived in detail in B.2. The strategy is to obtain first an evolution equation for Z_k , then to deduce the equation on Γ_k . It writes

$$\partial_k \Gamma_k = \frac{1}{2} \int_q \partial_k R_k(q) \left(\Gamma_k^{(2)}[M] + \mathcal{R}_k \right)_{q,-q}^{-1} \quad (133)$$

where $\mathcal{R}_k(x, y) = R_k(x-y)$. The inverse $\left(\Gamma_k^{(2)}[M] + \mathcal{R}_k \right)_{q,-q}^{-1}$ has to be understood in the operator sense. It is convenient in practice to rewrite Eq. (133) as²⁸

$$\partial_k \Gamma_k = \frac{1}{2} \tilde{\partial}_k \text{Tr} \log \left(\Gamma_k^{(2)} + R_k \right) \quad (134)$$

where $\tilde{\partial}_k$ acts only on the k -dependence of R_k and not on $\Gamma_k^{(2)} \mathfrak{x}$:

$$\tilde{\partial}_k = \frac{\partial R_k}{\partial k} \frac{\partial}{\partial R_k} \tag{135}$$

and the trace means integral over q .

5.1. Some General Properties of the Effective Average Action Method

Let us mention some important properties of Γ_k .

- If the microscopic Hamiltonian H and the functional measure are symmetric under a group G and if there exists a cut-off function R_k such that the mass term ΔH_k respects this symmetry, then Γ_k is symmetric under G for all k and thus so is $\Gamma = \Gamma_{k=0}$. It can happen that there is no mass-like term that respects the symmetry whereas the theory, that is Γ is invariant under G . This means that the symmetry is broken for all finite k and that the symmetry is recovered only for $k \rightarrow 0$. This is the case of gauge symmetry. This symmetry breaking term can be controlled by modified Ward identities that become, in the limit $k \rightarrow 0$, the true Ward identities. It remains nevertheless difficult up to now to compute RG flows in a completely controlled way in gauge theories.
- An exact RG equation for theories involving fermions can also be derived along the same line.
- Eq. (134) looks very much like a one-loop result. At one-loop:

$$\Gamma_k = H + \frac{1}{2} \text{Tr} \log \left(H^{(2)} + R_k \right) . \tag{136}$$

Thus, substituting $H^{(2)}$ by the full $\Gamma_k^{(2)}[M]$ function changes the one-loop result into an exact one! There exists a diagrammatic representation of the RG equation written as in Eq. (133) and that emphasizes its one-loop structure, see Figs.7 and 8.

We define

$$G_k[M] = \left(\Gamma_k^{(2)}[M] + \mathcal{R}_k \right)^{-1} , \tag{137}$$

which is the “full”, that is M -dependent, propagator, see Fig.7.

- This one-loop structure has a very important practical consequence: only one integral has to be computed and thanks to rotational invariance, it is one-dimensional. This is very different from

$$G_{k, q, -q'} = q \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} -q'$$

Fig. 7. Diagrammatic representation of the “full” propagator that is the M -dependent function $G_k[M] = \left(\Gamma_k^{(2)}[M] + \mathcal{R}_k \right)_{q, -q}^{-1}$.

$$\partial_t \bar{\Gamma}_k = \frac{1}{2} \begin{array}{c} \circlearrowright \\ \times \\ \circlearrowleft \end{array}$$

Fig. 8. Diagrammatic representation of the RG equation on the effective average action. The line represents the “full” propagator, Fig.7 and the cross $\partial_k R_k(q)$ and the loop, the integral over q .

perturbation theory where l loop-diagrams require l d -dimensional integrals. This is a tremendous simplification of the NPRG method compared to perturbation theory.

- The perturbation expansion can be retrieved from the NPRG equation (and the all-order proof of renormalizability can be simpler in this formalism).
- Because of the term $\partial_k R_k$ in the NPRG equation, only momenta q^2 of the order of k^2 or less contribute to the flow at scale k (we come back in detail to this point in the following). Thus the RG flow is regular both in the ultra-violet and in the infra-red. All the divergences of perturbation theory are avoided: we compute directly the RG flow and not first the relationship between bare and renormalized quantities from which is computed, in a second step, the RG flow.
- k acts as an infrared regulator (for $k \neq 0$) somewhat similar to a box of finite size $\sim k^{-1}$. Thus, for $k > 0$, there is no phase transition and thus no singularity in the free energy $\bar{\Gamma}_k$. At finite k , everything is regular and can be power-expanded safely. We can therefore conclude that

- (i) the singularities of Γ_k build up as k is lowered and are thus smoothed by k ,
- (ii) the precursor of the critical behaviour should already show up at finite k for $|q| \gg k$.

- An important consequence of the regularity of Γ_k at $k > 0$ is that it can be expanded in a power series of $\nabla M(x)$. For slowly varying fields $M(x)$ this expansion is expected to be well-behaved. This is the basis of the *derivative expansion* that consists in proposing an *ansatz* for Γ_k involving only a finite number of derivatives of the field.²⁸ Two of the most used approximations, based on the derivative expansion, are

$$\Gamma_k = \int d^d x \left(U_k(M(x)) + \frac{1}{2} (\nabla M)^2 \right) \quad (138)$$

called the local potential approximation (LPA) since no field renormalization in front of the derivative term is included and

$$\Gamma_k = \int d^d x \left(U_k(M(x)) + \frac{1}{2} Z_k(M) (\nabla M)^2 \right) \quad (139)$$

often called the $O(\partial^2)$ approximation or the leading approximation.^v Of course, in principle, Γ_k involves all powers of ∇M compatible with rotational and \mathbb{Z}_2 invariance. We shall come back in detail to these approximations in the following.

- We shall see in the following that by working with dimensionless (and renormalized) quantities, the NPRG equation can be rewritten in a way that makes no explicit reference to the scales k and Λ . This will allow us to find fixed points from which quantities like critical exponents can be computed. The fact that Λ disappears from the flow equation also ensures that the “group law” of composition of RG transformations is obeyed. This self-similarity property of the flow will be automatic *even if Γ_k is truncated* (as in Eqs. (138), (139) for instance). This is a major advantage of this method compared to many others where renormalizability, that is self-similarity, is spoiled by approximations (Schwinger-Dyson approximations for instance).
- A property that follows from the last point is the decoupling of massive modes. Let us consider a theory with one very massive

^vThe function $Z_k(M)$ has, of course, nothing to do with the partition function $Z_k[B]$ introduced in Eq. (118) although it is customary to use the same symbol.

mode (m_1) and another one with a lower mass (m_2). Since only the modes $|q| \leq k$ contribute to the flow of Γ_k , once $k \ll m_1$, the massive mode does (almost) no longer contribute to the flow: it decouples. This means that it contributes to the flow when k is between Λ and m_1 (or, in real space, when the running lattice spacing is between a and its correlation length m_1^{-1}) and almost not below m_1 . This also means that if we are given a model at a scale $k_0 \ll m_1$, we cannot know whether the underlying “fundamental theory” (at scales larger than m_1) involves or not a massive particle since there is no longer any signal of the presence of this mode in the theory (that is, in the RG flow).

- Self-similarity and decoupling of massive modes will have universality (in the vicinity of a second order transition) as a consequence. We shall see that universality is not just the consequence of the existence of a fixed point. It is the consequence of a very peculiar geometry of the RG flow.^{44,46}

6. Approximation Procedures

The NPRG equation is an extremely complicated equation. It is a functional partial differential equation since it involves the functionals $\Gamma_k[M]$ and $\Gamma_k^{(2)}[M]$. Needless to say that we do not know how to solve it in general. Some approximations are thus required. Two main kinds of approximations are usually considered: the Green function approach and the derivative expansion. In both cases, the strategy consists in solving the RG equation in a *restricted functional space* and not as a series expansion in a small parameter. This is why we can hope to obtain non-perturbative results. Both methods need to project consistently the exact RG equation in the functional space that has been chosen. Of course, the quality of the result will depend crucially on the choice of space in which we search a solution. Depending on the problem, one choice can be drastically better than another. In all cases, it is impossible to know whether we have missed some physically crucial ingredient by making one choice rather than another one. But this problem is not specific to NPRG. It is generic in physics...

Let us review briefly the Green function approach and then explain in some details the derivative expansion which is the most employed method in statistical mechanics.

6.1. The Green Function Approach

From Eq. (133) we can deduce the infinite hierarchy of RG equations for the correlations functions defined by

$$\bar{\Gamma}_{k,p_1\dots p_n}^{(n)} = \frac{\delta^n \Gamma_k}{\delta M_{p_1} \dots \delta M_{p_n}} \tag{140}$$

taken in a particular field configuration (the zero and the uniform field configurations being the most studied). We define $t = \log k/\Lambda$ which is often called the RG “time”. From Eq. (133), we obtain

$$\partial_t \frac{\delta \Gamma_k}{\delta M_p} = -\frac{1}{2} \int_{q_1, q_2, q_3} \partial_t R_k(q_1) G_k{}_{q_1, -q_2} \frac{\delta \Gamma_k^{(2)}{}_{q_2, -q_3}}{\delta M_p} G_k{}_{q_3, -q_1} . \tag{141}$$

Therefore, setting $\dot{R}_k = \partial_t R_k = k \partial_k R_k$ we obtain

$$\begin{aligned} \partial_t \frac{\delta^2 \Gamma_k}{\delta M_p \delta M_{p'}} &= \int_{\{q_i\}} \dot{R}_{k, q_1} G_k{}_{q_1, -q_2} \Gamma_k^{(3)}{}_{p, q_2, -q_3} G_k{}_{q_3, -q_4} \Gamma_k^{(3)}{}_{q_4, -q_5, p'} \\ &G_k{}_{q_5, -q_1} - \frac{1}{2} \int_{\{q_i\}} \dot{R}_{k, q_1} G_k{}_{q_1, -q_2} \Gamma_k^{(4)}{}_{q_2, -q_3, p, p'} G_k{}_{q_3, -q_1} \end{aligned} \tag{142}$$

where both the left and the right hand sides are functions of M_q since they have not yet been evaluated in a particular configuration. These equations look terrible but in fact they are not since there exists a diagrammatic way to obtain them automatically. We represent $\Gamma_k^{(3)}{}_{p_1, p_2, p_3}$ as in Fig.9. We

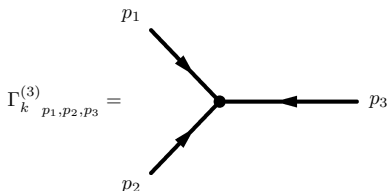


Fig. 9. Diagrammatic representation of $\Gamma_k^{(3)}{}_{p_1, p_2, p_3}[M]$.

obtain for $\partial_t \Gamma_k^{(1)}$ the graph in Fig.10. Of course, if we evaluate $\partial_t \Gamma_k^{(1)}$ in a uniform field configuration, the momentum is conserved at each vertex and for each “propagator”. The function is thus non-vanishing only at zero momentum and we obtain the equation in Fig.11. It is clear on the diagrammatic representation that only one momentum integral remains. The RG equation for $\partial_t \Gamma_k^{(2)}$ evaluated in a uniform field configuration is given

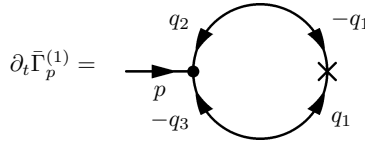


Fig. 10. Diagrammatic representation of $\partial_t \Gamma_k^{(1)}$.

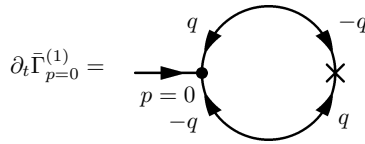


Fig. 11. Diagrammatic representation of $\partial_t \Gamma_k^{(1)}$ evaluated in a uniform field configuration.

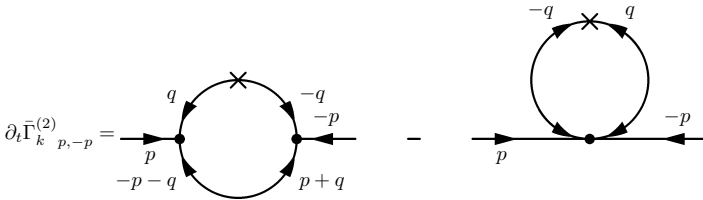


Fig. 12. Diagrammatic representation of $\partial_t \Gamma_k^{(2)}$ evaluated in a uniform field configuration. Note that the factor $\frac{1}{2}$ has not been represented on the figure for the tadpole (contrary to what we did for the other figures) because it can be retrieved from the topology of the graph itself (in fact the minus sign can also be retrieved from the graph).

in Fig.12. It is clear from these diagrammatic representations that $\partial_t \Gamma_k^{(n)}$ involves $\Gamma_k^{(n+1)}$ and $\Gamma_k^{(n+2)}$. If we want to solve this infinite tower of equations, we have to truncate it. A possible truncation consists, for instance, in keeping $\Gamma_k^{(2)}$ and $\Gamma_k^{(4)}$ evaluated at $M = 0$ and to neglect the contribution of $\Gamma_k^{(6)}$ in the equation on $\partial_t \Gamma_k^{(4)}$ ($\Gamma_k^{(3)}[M = 0] = 0$ in a \mathbb{Z}_2 -invariant model). A better method is to find an ansatz for $\Gamma_k^{(6)}$ in terms of $\Gamma_k^{(2)}$ and $\Gamma_k^{(4)}$. In both cases, the system of equations becomes closed and can, at least in principle, be solved. Let us finally notice that this method consists in truncating the field-dependence of $\Gamma_k[M]$ while keeping the momentum dependence of $\Gamma_k^{(2)}$

and $\Gamma_k^{(4)}$. Tremendous improvements of this type of approximation has been performed these last years.⁴⁷⁻⁴⁹

We now study another truncation method which is somewhat the reverse.

6.2. The Derivative Expansion

The principle of this approximation has already been introduced previously, see Eqs. (138), (139). The underlying idea is that we are mostly interested (for the study of critical phenomena) in the long distance physics, that is the $|q| \rightarrow 0$ region of the correlation functions.^w Thus, we keep only the lowest orders of the expansion of Γ_k in ∇M while we keep all orders in the fields M

$$\Gamma_k = \int d^d x \left(U_k(M(x)) + \frac{1}{2} Z_k(M) (\nabla M)^2 \right) + O(\nabla^4). \quad (143)$$

This approximation is based on a somewhat opposite philosophy as the one that prevailed in the Green function approach. However, it should be clear that for statistical mechanics, the most important information – e.g. the equation of state – is hidden in the effective potential $U_{k=0}$ of the theory that, therefore, needs to be computed as accurately as possible (see however Ref. [48]).

It is in fact remarkable that we can combine both methods by making a field expansion of U_k, Z_k, \dots on top of the derivative expansion while preserving many non-trivial results. The simplest such truncation consists in using the LPA and in keeping only the first two terms of the expansion of U_k in powers of M :^x

$$\Gamma_k[M] = \int d^d x (g_{2,k} M^2 + g_{4,k} M^4 + \frac{1}{2} (\nabla M)^2) \quad (144)$$

With this kind of ansatz, the RG equation on Γ_k becomes a set of ordinary differential equations for the couplings retained in the *ansatz*:

$$\partial_t g_{n,k} = \beta_n (\{g_{p,k}\}) . \quad (145)$$

^wThe computation of quantities like the total magnetization or the susceptibilities require only the knowledge of the spin-spin correlation function at zero momentum. The same thing holds for the correlation length.

^xLet us notice that if the k -dependence of the couplings were neglected, this ansatz would exactly coincide with the *ansatz* chosen by Landau to study second order phase transitions. We know that it would lead to the mean field approximation. It is remarkable that keeping the scale dependence of the couplings and substituting precisely this *ansatz* into the RG equation of Γ_k is sufficient to capture almost all the qualitative features of the critical physics of the Ising and $O(N)$ models in all dimensions (see the following).

For instance, if the truncation above is considered we find (see section 9.2 for the derivation of these equations)

$$\partial_t \kappa_k = -(d-2)\kappa_k + 6v_d l_1^d (2\kappa_k \lambda_k) \quad (146)$$

$$\partial_t \lambda_k = (d-4)\lambda_k + 6v_d \lambda_k^2 l_2^d (2\kappa_k \lambda_k) \quad (147)$$

where l_1^d, l_2^d are defined in the Appendix, Eq.(A.13). These equations are already non perturbative since the functions l_1^d, l_2^d are non polynomial. If U_k, Z_k, \dots are not truncated in a field expansion, the RG equation on Γ_k becomes a set of coupled partial differential equations for these functions. The initial condition at scale Λ is given by the Hamiltonian of the model. Before studying in detail this approximation method let us make an important remark.

Contrary to perturbation theory where only the renormalizable couplings are retained in the renormalized action, all powers of the fields appear in the ansatz of Eq. (143). There is no longer any distinction – at this level at least – between the two kinds of couplings, renormalizable and non-renormalizable. Although we shall not enter into any detail, let us comment briefly on this difference between the perturbative and non-perturbative approaches.^{8,44,46}

To understand the origin of this difference, it is necessary to remember that in perturbative field theory, the action (bare or renormalized) is *not* a physical quantity. It is only the “object” that generates the vertices of the theory from which Feynman diagrams are computed. As for the Feynman diagrams they are physical objects in the sense they are the building blocks of the Green functions that are physical (in particle physics, the S -matrix is physical). The effective action $\Gamma[M]$ is also physical. The action of perturbative field theory can well be polynomial and involve only a few couplings, $\Gamma[M]$ always involve all powers of the field in a non-trivial manner. In the particle physics language, this means that $\Gamma^{(n)}$ with $n = 6, 8, \dots$, corresponding to the scattering of n particles, cannot, in general, be factorized trivially into products of $\Gamma^{(p)}$ with $p < n$ corresponding to the scattering of fewer particles. Thus all couplings are non-trivial, even those corresponding to an arbitrarily large number of fields. They all have a non-trivial RG evolution that, *a priori*, needs to be taken into account into the flow of Γ_k . This is the reason why we must keep a complete function of M for U_k and not just the first terms of its expansion in M .

This is paradoxical because it seems in conflict with what we know from perturbation theory where only the marginal couplings drive the whole RG flow. In fact, it can be shown that at sufficiently long distance, that is

for sufficiently small k , all RG trajectories – that belong to an infinite dimensional space – are attracted towards a submanifold of dimension the number of renormalizable couplings (including the masses). This is particularly clear, and has been studied in detail by Bagnuls and Bervillier, for the critical theories of the \mathbb{Z}_2 -invariant theories in $d = 3$.^{44,46} In this case, the trajectories belong to the critical surface. For all of them, after a transient regime, they (almost) collapse on a line joining the Gaussian fixed point to the non-trivial fixed point describing the phase transition of the Ising model (called the Wilson-Fisher fixed point). Thus, if one is only interested in the long-distance physics compared to Λ^{-1} , the RG flow behaves as if it were driven by a unique coupling. As long as the projection of this line on the axis corresponding to the ϕ^4 -coupling is non-singular, it is possible to describe the flow along this line by the flow of the ϕ^4 -coupling alone. This is what perturbation theory does. Of course, this flow is universal: it depends only on the flow between the Gaussian and the Wilson-Fisher fixed point. It is therefore impossible to get any non-universal information on a given system from this RG trajectory alone because once we focus on this particular RG trajectory, it is impossible to reverse the flow (in the ultra-violet direction) to go back to the microscopic system we started with. Bagnuls and Bervillier have used the following metaphor:^{44,46} the RG trajectories on the critical surface are like rivers in the mountains. In the valley, there is a large river along which the flow is slow. Many small rivers, coming from the mountains, flow very rapidly into the large one. Once in the large river, it is almost impossible to know where the water came from. The large river has its source at the Gaussian fixed point and stops at the Wilson-Fisher fixed point. Perturbation theory can only focus on the large river whereas NPRG, because it is functional in essence, can follow any RG trajectory. Of course, the “continuum limit”, $\Lambda \rightarrow \infty$, (at fixed large distance physics) can only be taken on a trajectory where this limit is defined. There is only one such trajectory: the large river joining the Gaussian to the Wilson-Fisher fixed point. On this trajectory, the RG flow does not blow up at infinity in the continuum limit since the trajectory has a fixed point – the Gaussian – at its source. This theory is asymptotically free in the ultra-violet. Let us emphasize that the continuum limit is probably irrelevant from a physical viewpoint: who cares about the physics at asymptotically small distances or, in particle physics, at asymptotically large energies?

7. The Local Potential Approximation for the Ising Model

7.1. The Flow Equation of the Potential

We now consider the *ansatz* Eq. (138) for a \mathbb{Z}_2 symmetric theory. As already mentioned, the problem is to project the RG equation (133) on the potential U_k . This is naturally performed by *defining* the potential as Γ_k computed for *uniform* field configurations:

$$U_k(M_{\text{unif.}}) = \frac{1}{\Omega} \Gamma_k[M_{\text{unif.}}] \quad (148)$$

where Ω is the volume of the system. To compute the RG flow of U_k we act on both sides of this equation with ∂_t and we evaluate the right hand side thanks to Eq. (133). The only “difficulty” of this calculation is to invert $\Gamma_k^{(2)}[M] + R_k$ for uniform field configurations using the LPA *ansatz* to compute $\Gamma_k^{(2)}[M]$. This is performed in detail in B.3. The final result reads:

$$\partial_k U_k(\rho) = \frac{1}{2} \int_q \frac{\partial_k R_k(q)}{q^2 + R_k(q) + U'_k(\rho) + 2\rho U''_k(\rho)}, \quad (149)$$

where

$$\rho = \frac{1}{2} M^2 \quad (150)$$

is the \mathbb{Z}_2 -invariant and $U'_k(\rho)$ and $U''_k(\rho)$ are derivatives of U_k with respect to ρ . An important remark is in order here. Once the angular integral has been performed, the integrand of Eq. (149) is (up to a constant factor)

$$f(q^2, w) = |q|^{d-1} \frac{\partial_k R_k(q^2)}{q^2 + R_k(q^2) + w}. \quad (151)$$

Let us consider a typical cut-off function:

$$R_k(q^2) = \frac{q^2}{e^{q^2/k^2} - 1}. \quad (152)$$

Then, for generic values of w , the typical shape of f at fixed k is given in Fig.13 (for $d > 1$). As expected, only a window of momenta around k contributes to the flow at scale k . We see in particular that the rapid modes are efficiently integrated out by this kind of cut-off function. This explains the decoupling of massive modes already discussed section 5.1. The cut-off function (152) has been used many times in the literature but it turns out that another one is convenient because it allows to perform analytically the integral in Eq. (149). It writes⁵⁰

$$R_k(q^2) = (k^2 - q^2) \theta(k^2 - q^2). \quad (153)$$

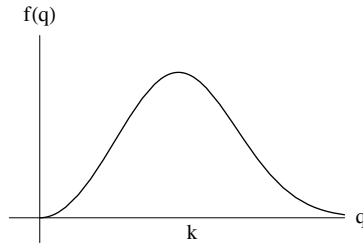


Fig. 13. The typical shape of the integrand $f(q)$.

With this choice of R_k we easily find for the RG flow of U_k

$$\partial_t U_k(\rho) = \frac{4v_d}{d} \frac{k^{d+2}}{k^2 + U'_k(\rho) + 2\rho U''_k(\rho)} \tag{154}$$

where v_d is defined in Eq. (A.3). All we can learn about the model at this approximation is contained in the solution of this equation. We can already see that, as expected, this equation does not admit a fixed point U^* . We have already seen that it is necessary to go first to the dimensionless variables to find a fixed point. This is what we now do.

7.2. The Scaling Form of the RG Equation of the Dimensionless Potential

We have already emphasized when studying block-spins that it is necessary to measure all lengths in units of the running lattice spacing to find a fixed point of the RG flow. Looking for a fixed point is a convenient method to study the critical behaviour of a model and we shall now spend some time deriving the RG equation for the dimensionless potential.

In our formalism, k is the analogue of the running lattice spacing and we must therefore “de-dimension” all dimensionful quantities thanks to k to find a fixed point. We have

$$[\Gamma_k] = k^0 \implies [M] = k^{\frac{d-2}{2}} \quad \text{and} \quad [U_k] = k^d . \tag{155}$$

We define the dimensionless variables by

$$y = \frac{q^2}{k^2}, \tag{156}$$

$$R_k(q^2) = q^2 r(y) = k^2 y r(y), \tag{157}$$

$$\tilde{x} = kx, \tag{158}$$

$$\tilde{M}(\tilde{x}) = k^{\frac{2-d}{2}} M(x), \quad (159)$$

$$\tilde{U}_t(\tilde{M}(\tilde{x})) = k^{-d} U_k(M(x)) . \quad (160)$$

To derive the RG equation for \tilde{U}_t we must keep in mind that the ∂_t in Eq. (154) is taken at fixed ρ whereas we want now to take it at fixed $\tilde{\rho}$. The detailed calculation is performed in the Appendix and the result is:

$$\partial_t \tilde{U}_t = -d \tilde{U}_t + (d-2) \tilde{\rho} \tilde{U}'_t - 2v_d \int_0^\infty dy y^{d/2+1} \frac{r'(y)}{y(1+r(y)) + \tilde{U}'_t + 2\tilde{\rho} \tilde{U}''_t} . \quad (161)$$

Once again, with the cut-off (153) the integral can be performed analytically and we find

$$\partial_t \tilde{U}_t = -d \tilde{U}_t + (d-2) \tilde{\rho} \tilde{U}'_t + \frac{4v_d}{d} \frac{1}{1 + \tilde{U}'_t + 2\tilde{\rho} \tilde{U}''_t} . \quad (162)$$

We clearly see on this equation that the flow of \tilde{U}_t has two parts, one that comes from the dimensions of U_k and ρ and one that comes from the dynamics of the model. This RG equation on \tilde{U}_t is a rather simple partial differential equation that can be easily integrated numerically. We are thus in a position to discuss the critical behaviour of the Ising model and to look for fixed points.²⁸

8. The Critical and Non-Critical Behaviour of the Ising Model within the LPA

Having derived the RG flow of the potential, we can relate a “microscopic” model defined at scale Λ by an Hamiltonian (or directly by Γ_Λ) with the free energy $\Gamma = \Gamma_{k=0}$. Let us emphasize that there is no reason why the Hamiltonian H should involve only ϕ^2 and ϕ^4 terms and not ϕ^6 , ϕ^8 , ... terms. In fact, the Hubbard-Stratonovich transformation enables us to obtain the exact potential at the scale of the lattice spacing of a magnetic system. In the Ising case, the potential thus obtained is

$$U_\Lambda(\phi) \propto \log \cosh \phi \quad (163)$$

and is thus non-polynomial. In the NPRG framework this is not a problem since, even if it were a polynomial at scale Λ , it would become non-polynomial at any other scale.

Let us anyway, for the sake of simplicity, consider a dimensionless potential at scale Λ of the form:

$$\tilde{U}_\Lambda(\tilde{\rho}) = \frac{\lambda_\Lambda}{2}(\tilde{\rho} - \kappa_\Lambda)^2 \tag{164}$$

with $\kappa_\Lambda > 0$. At the mean-field level, $\Gamma^{\text{MF}} = \Omega U_\Lambda$ and we would deduce at this approximation that the system is in its broken phase with a spontaneous magnetization per unit volume $M_{\text{sp}} = \Lambda^{\frac{d-2}{2}} \sqrt{2\kappa_\Lambda}$.^y However, the integration of the fluctuations can drastically change this picture: the minimum of the potential has a non-trivial RG flow that can drive it to 0. If this happens, the system is in fact in its symmetric (high temperature) phase.

Let us call $\kappa(k)$ the running minimum of the dimensionless potential at scale k (more appropriately at “time” $t = \log k/\Lambda$):

$$\partial_{\tilde{\rho}} \tilde{U}_k |_{\kappa_k} = 0. \tag{165}$$

It is physically clear, and this can be checked on the flow of U_k , that the spontaneous magnetization decreases because of the fluctuations. This means that the true spontaneous magnetization is always less than the mean-field spontaneous magnetization $\Lambda^{\frac{d-2}{2}} \sqrt{2\kappa_\Lambda}$. There are thus three possibilities.

(i) The system is in its broken phase (low temperature), the spontaneous magnetization is given by

$$M_{\text{sp}} = \sqrt{2\rho_0(k=0)} \tag{166}$$

where $\rho_0(k)$ is the location of the minimum of the (dimensionful) potential $U_k(\rho)$.^z The relation between $\kappa(k)$ and $\rho_0(k)$ is

$$\rho_0(k) = k^{d-2} \kappa(k) \tag{167}$$

from Eq. (159). Thus, if $\rho_0(k) \rightarrow M_{\text{sp}}^2/2$ when $k \rightarrow 0$, $\kappa(k)$ flows to infinity (for $d > 2$) as k^{2-d} in the same limit.

(ii) The system is in the high temperature phase, the spontaneous magnetization is vanishing. Thus, as k decreases the minimum $\kappa(k)$ must decrease and, at a finite scale k_s , hit the origin. It is easy to guess that k_s must be of the order of the inverse correlation length since as long as $k^{-1} \ll \xi$ the coarse-grained system remains strongly correlated and still behaves as if it were critical. It is only when $k^{-1} \sim \xi$ that “the system can realize” that

^yAt vanishing external magnetic field, the magnetization is given by $B = 0 = \partial U/\partial M$ and corresponds therefore to the location of the minimum of the potential.

^zWe shall see in the following that there is a subtlety here because of the convexity of the potential in the limit $k \rightarrow 0$.

its correlation length is finite and that its magnetization is vanishing. Thus $k_s \simeq \xi^{-1}$.

(iii) The system is critical. The spontaneous magnetization is vanishing which means that $\rho_0(k) = k^{d-2}\kappa(k) \rightarrow 0$ as $k \rightarrow 0$. Note that this does not require that $\kappa(0) = 0$ since $\kappa(k)$ is multiplied by a positive power of k , at least for $d > 2$. In fact, $\kappa(k)$ reaches a finite fixed point value κ^* . Note that for $d < 2$ it is necessary to take into account the field renormalization, that is the anomalous dimension, to obtain a coherent picture of the physics. This requires to go beyond the LPA (see section 9).

We can summarize the three cases in Fig.14. κ_c is the critical initial value

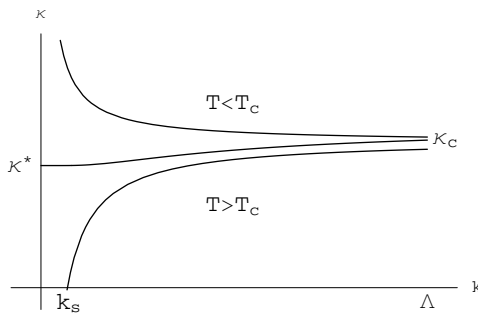


Fig. 14. Behavior of the running minimum $\kappa(k)$ of the dimensionless potential. From top to bottom: $T < T_c$, $T = T_c$ and $T > T_c$. κ_c is the critical initial value of κ_Λ for which the system is critical. This value should not be confused with κ^* which is the fixed point value of $\kappa(k)$. In the low temperature phase the dimensionless running minimum diverges whereas the dimensionful minimum converges to the value of the spontaneous magnetization since it is multiplied by a positive power of the scale k that compensates exactly the divergence of $\kappa(k)$. For the high temperature phase $k_s \simeq \xi^{-1}$.

of κ_Λ for which the system is critical. This value should not be confused with κ^* . For a generic initial potential

$$\tilde{U}_\Lambda(M) = \frac{\lambda_\Lambda}{2}(\tilde{\rho} - \kappa_\Lambda)^2 + \frac{u_{3,\Lambda}}{3!}(\tilde{\rho} - \kappa_\Lambda)^3 + \dots \tag{168}$$

the critical value κ_c of κ_Λ is a function of all the other couplings $\lambda_\Lambda, u_{3,\Lambda}, \dots$. This means that in the space of dimensionless coupling constants there exists a “critical hypersurface” of co-dimension 1 that corresponds to systems that are critical. Generically, κ_Λ is a regular function of the temperature around T_c and at first order we can assume that

$$\kappa_\Lambda - \kappa_c \propto T_c - T \tag{169}$$

This allows us to relate the coefficients of the microscopic Hamiltonian to the temperature. Note that contrary to the mean-field analysis for which criticality is reached when the coefficient r_0 of the quadratic term of the potential (that is $r_0\phi^2/2$) is vanishing: $r_0 \propto T - T_c$, this is not true here since criticality does not correspond to the vanishing of the bare mass term. The mean-field analysis is wrong in this respect.

Let us now show what we expect for $U = U_{k=0}$ and $\tilde{U}_{k=0}$.

8.1. The Low and the High Temperature Phases

In the low temperature phase, we expect to have a spontaneous magnetization, either up or down. More precisely, the equation

$$B = \frac{\partial U}{\partial M} \tag{170}$$

is expected to have a solution $+M_{sp}$ for $B \rightarrow 0^+$ and $-M_{sp}$ for $B \rightarrow 0^-$. Moreover, U must be a convex function of M since it is obtained from a Legendre transform. Thus, U must have a very peculiar shape since it must have two minima at M_{sp} and no maximum in between (otherwise it should not be convex). The only possibility is that it is flat in between $-M_{sp}$ and $+M_{sp}$.

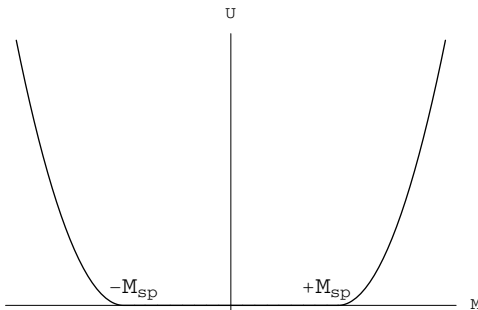


Fig. 15. Shape of the effective potential $U = U_{k=0}$ for $T < T_c$.

The convexity of the effective potential is not preserved by perturbation theory: the convex envelop has to be taken by hand. In fact, it is notoriously difficult to compute “safely” a convex effective potential and this is a property that is reproduced by NPRG already at the LPA. Note that this is no longer the case if a field expansion of the potential is performed (a polynomial can never be flat on a whole interval). In fact, starting with a potential

U_Λ showing a double-well structure and with parameters $\kappa_\Lambda, \lambda_\Lambda, \dots$ such that the system is in its low temperature phase, the RG flow of U_k is such that as k is lowered²⁸

- the minimum $\rho_0(k)$ moves towards the origin and eventually reaches a limit equal to the spontaneous magnetization M_{sp} ;
- the maximum of the potential located between the two minima decreases and goes to 0 as $k \rightarrow 0$ (the “inner part” of the potential flattens). At $k = 0$, $U_{k=0}$ looks like the potential of Fig.15.

As $T \rightarrow T_c$, that is as $\kappa_\Lambda \rightarrow \kappa_c$, M_{sp} moves towards the origin and at $T = T_c$ coincides exactly (by definition of T_c) with the origin. In the high temperature phase, that is $\kappa_\Lambda < \kappa_c$, the spontaneous magnetization vanishes. Thus U has a single minimum at the origin in this case.

Let us finally notice that when $k \neq 0$, U_k is not convex in general. This is normal since Γ_k is not the Legendre transform of W_k (W_k is convex) because of the term $1/2 \int R_k MM$ that has been subtracted.

8.2. The Critical Point

For $\kappa_\Lambda = \kappa_c$, the minimum of the potential U_k (or \tilde{U}_k) for $k > 0$ is non vanishing. It is only for $k \rightarrow 0$ that the minimum $\rho_0(k)$ reaches the origin.

To characterize the critical point it is interesting to work with \tilde{U}_k instead of U_k since, at this point, the long distance physics (compared with Λ^{-1}) is scaleless. This means that the potential, properly rescaled thanks to k , should be k -independent at $T = T_c$ and for sufficiently small k . It must be a fixed potential $\tilde{U}^*(\tilde{\rho})$

$$\partial_t \tilde{U}^*(\tilde{\rho}) = 0 . \tag{171}$$

We deduce from this equation and from Eq. (154) that $\tilde{U}^*(\tilde{\rho})$ is a solution of

$$0 = -d \tilde{U}^* + (d - 2) \tilde{\rho} \tilde{U}^{*'} + \frac{4vd}{d} \frac{1}{1 + \tilde{U}^{*'} + 2\tilde{\rho} \tilde{U}^{*''}} . \tag{172}$$

At first sight, the situation looks paradoxical since this is a second order differential equation that should admit infinitely many solutions indexed by two numbers whereas we expect only one fixed point in $d = 3$ corresponding to the universality class of the Ising model. In fact, it can be shown that among all these solutions, *only one* is well defined for all $\tilde{\rho} \in [0, \infty]$.¹⁴

An easy way to show this is the “shooting method”. We first write down the fixed point equation for $\tilde{U}^{*'}(\tilde{M})$ analogous to Eq. (172) and which is

more convenient. Then, one initial condition is given by the \mathbb{Z}_2 symmetry: $\tilde{U}^{*'}(\tilde{M} = 0) = 0$. The other initial condition: $\tilde{U}^{*''}(\tilde{M} = 0)$ is then adjusted so that $\tilde{U}^{*'}(\tilde{M})$ exists for all \tilde{M} . For a generic $\tilde{U}^{*''}(\tilde{M} = 0)$ this is not the case: at finite \tilde{M} , $\tilde{U}^{*'}$ either blows up at $+\infty$ or at $-\infty$. By dichotomy, the value $\tilde{U}^{*''}(\tilde{M} = 0)$ can be fine-tuned so that this occurs for larger and larger \tilde{M} . The fixed point potential thus obtained is shown in Fig.16.

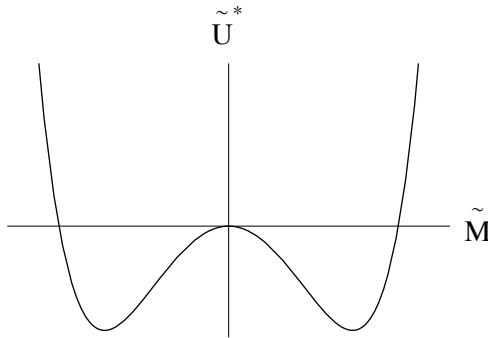


Fig. 16. Dimensionless fixed point potential of the Ising model at $d = 3$. The minima of this potential are located at $\sqrt{2\kappa^*}$. The spontaneous magnetization is vanishing although $\kappa^* \neq 0$ (see the text).

The same fixed point potential can be found “dynamically” by integrating the RG flow and by fine-tuning the value of κ_Λ to get closer and closer to κ_c . We find that for an initial κ_Λ very close to κ_c , the potential \tilde{U}_t spends a very long RG time close to \tilde{U}^* before either departing in the high or low temperature phase. Thus, all couplings reach a plateau before blowing up. On this plateau, we obtain a very good approximation of $\tilde{U}^*(\tilde{M})$ very close to the one found by the shooting method. For $\kappa(k)$ very close to κ_c this is represented in Fig.17.

One can observe that during a transient regime, $\kappa(k)$ for $k \simeq \Lambda$ is not stationary even at the critical temperature. This regime simply corresponds to the RG “time” necessary to approach the fixed point. It is non-universal since it depends on the starting point κ_c on the critical surface.

Let us finally point out a subtlety. When $k > 0$, it is possible to reconstruct $U_k(M)$ from $\tilde{U}_k(\tilde{M})$ by a somewhat trivial rescaling:

$$U_k(M) = k^d \tilde{U}_k \left(k^{-\frac{d-2}{2}} M \right) . \tag{173}$$

The k factor acts as a magnification scale when we go from M to \tilde{M} (at

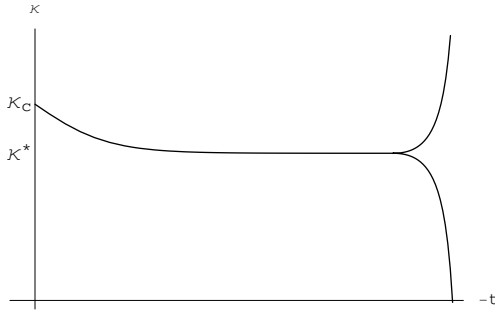


Fig. 17. Behavior of $\kappa(k)$ very close to κ_c . During the transient regime at small $-t$ the representative point of the system approaches the fixed point. Then, this point remains close to the fixed point and κ remains almost constant and very close to its fixed point value κ^* . Then, if the initial condition of the RG flow was not right on the critical surface, it departs either in the low or in the high temperature phase where κ either blows up or goes to 0.

least for $d > 2$). For $k \rightarrow 0$ an infinitesimal range of M around the origin is mapped onto a finite range of \tilde{M} . It is therefore possible – and this is what indeed occurs at $T = T_c$ – that $U_{k=0}$ has a trivial shape around $M = 0$ showing a minimum only at 0, see Fig.15 for $M_{\text{sp}} \rightarrow 0$, while $\tilde{U}_{k=0}$ shows a double-well structure!

8.3. The Critical Exponents

Within the LPA, there is no possibility to modify the mean-field like q^2 -dependence of $\Gamma_k^{(2)}(q)$ (evaluated at $M = 0$) since there is no renormalization of the derivative term. The exponent η is therefore vanishing at this order of the derivative expansion (see Eq. (71) and $\Gamma^{(2)}(q) = 1/G^{(2)}(q)$):

$$\eta^{\text{LPA}} = 0. \quad (174)$$

The exponent β defined in Eq. (85) can be calculated directly by fitting the behaviour of M_{sp} defined in Fig.15 as a function of $\kappa_\Lambda - \kappa_c$ which is itself proportional to $T_c - T$. Of course, this is not very convenient although feasible.

As for the exponent ν , we have seen that it is related to the behaviour of the RG flow around the fixed point, Eq. (75) and, more precisely, is the inverse of the positive eigenvalue of the flow at the fixed point. This means that very close to the fixed point and away from the critical surface, the

potential is such that

$$\tilde{U}(\tilde{M}, t) = \tilde{U}^*(\tilde{M}) + \epsilon e^{-t/\nu} u(\tilde{M}) . \quad (175)$$

It can be shown that $u(\tilde{M})$ must behave as a power law at large \tilde{M} and that this ensures that there is a unique value of ν such that Eq. (175) holds. Note that Eq.(175) is not general since if we choose a point on the critical surface, the corresponding potential is attracted towards $\tilde{U}^*(\tilde{M})$. This approach is governed by the so-called critical exponent ω corresponding to correction to scaling. Thus, in general, infinitesimally close to the fixed point, the evolution of the potential is given by the sum of two terms, one describing the approach to \tilde{U}^* on the critical surface and one describing the way the RG flow escapes the fixed point if one starts close but away from the critical surface. With the θ -cut-off function, Eq. (153), one finds at $d = 3$:^{28,38,39}

$$\nu^{\text{LPA}} = 0.65 \quad (176)$$

to be compared with $\nu = 0.6297(5)$ obtained by Monte Carlo simulations.

Several remarks are in order here.

- In principle, from ν and η all the other critical exponents can be calculated thanks to the scaling relations derived in section 3.3. It is interesting to verify that these relations are not spoiled by the LPA. Indeed, by computing separately all the exponents, it is found that the scaling relations among them are very well satisfied.
- In the exact theory, no physical result should depend on the choice of cut-off function R_k . However, once approximations are performed a spurious dependence on the choice of R_k is observed. As a consequence, the whole scheme makes sense only if this dependence is smooth. It is of course very difficult to obtain general results on this point since the space of cut-off functions is infinite dimensional and that we cannot sample it efficiently. In practice, we should choose a space of “reasonable” cut-off functions and study the variations of physical quantities like critical exponents in this space. This has been done in some details and it is observed that the dependence of ν on R_k is rather mild.^{38–40} Let us emphasize that this problem is not specific to NPRG. Even in the perturbative schemes the critical exponents acquire a spurious dependence on several choices made during their calculations.
- The LPA is certainly not appropriate for the computation of critical exponents in $d = 2$. We have seen several times that this dimension plays a special role in the formalism although nothing spectacular

is expected in $d = 2$ for the critical behaviour, at least for the Ising model. This comes from the fact that $\eta = 0$ starts to be a bad approximation at and below $d = 2$ for this model (it is worse for the $O(N)$ models with $N \geq 2$). The exact value of η is known in $d = 2$ from Onsager's solution of the Ising model: $\eta = 0.25$. Going to the next order of the derivative expansion cures most of the problems encountered at the level of the LPA at low dimensions.^{28,51}

- As explained in the following for the $O(N)$ models, it is possible to go beyond the LPA and to compute with greater accuracy the critical exponents, η in particular. This kind of calculations has been performed by several authors⁵² at order $O(\partial^2)$ of the derivative expansion for the Ising and $O(N)$ models in $d = 3$ and $d = 2$ ⁵³ and also at $O(\partial^4)$ for the Ising model in $d = 3$. Let us quote the best results obtained for the Ising model in $d = 3$:

order	ν	η
∂^0	0.6506	0
∂^2	0.6281	0.044
∂^4	0.632	0.033
7-loops	0.6304(13)	0.0335(25)

where the $O(\partial^4)$ results come from⁴⁰ and the 7-loops results from.⁴

Let us now study the $O(N)$ models at order $O(\partial^2)$ of the derivative expansion.

9. The $O(N)$ Models at $O(\partial^2)$ of the Derivative Expansion

Although the Ising and $O(N)$ models have much in common, there are several non-trivial points specific to the $O(N)$ models that are worth studying. Among them is the presence of Goldstone modes in the low temperature phase, the Mermin-Wagner theorem in $d = 2$ and the Kosterlitz-Thouless transition for $N = 2$ in $d = 2$. We shall use again the derivative expansion that writes at $O(\partial^2)$:

$$\Gamma_k = \int d^d x \left(U_k(\vec{M}^2(x)) + \frac{1}{2} Z_k(\vec{M}^2) (\nabla M)^2 + \frac{1}{4} Y_k(\vec{M}^2) \left(\vec{M} \cdot \nabla \vec{M} \right)^2 \right) \quad (177)$$

where \vec{M} is a N -component vector. In fact, we shall mainly study the LPA' that consists in neglecting $Y_k(\vec{M}^2)$ and keeping only the first term of the field-expansion of $Z_k(\vec{M}^2)$ that we call Z_k :

$$\Gamma_k = \int d^d x \left(U_k(\vec{M}^2(x)) + \frac{1}{2} Z_k (\nabla M)^2 \right). \tag{178}$$

The RG equation on Γ_k is obtained along the same line as in the Ising case. We start by constructing the partition function $Z_k[\vec{B}]$

$$Z_k[\vec{B}] = \int \mathcal{D}\vec{\phi}(x) \exp \left(-H[\vec{\phi}] - \Delta H_k[\vec{\phi}] + \int \vec{B} \cdot \vec{\phi} \right) \tag{179}$$

with

$$\Delta H_k[\vec{\phi}] = \frac{1}{2} \int_q R_k(q) \vec{\phi}_q \cdot \vec{\phi}_{-q}. \tag{180}$$

Since $\Gamma_k[\vec{M}]$ is a $O(N)$ -scalar, the RG equation involves now a trace on the $O(N)$ indices

$$\partial_k \Gamma_k = \frac{1}{2} \text{Tr} \int_{x,y} \partial_k R_k(x-y) W_k^{(2)}(x,y) \tag{181}$$

where $W_k^{(2)}(x,y)$ is now a NN matrix:

$$W_{k,ij}^{(2)}(x,y) = \frac{\delta^2 W_k}{\delta B_i(x) \delta B_j(y)}. \tag{182}$$

$\Gamma_k^{(2)} + \mathcal{R}_k$ is again the inverse of $W_k^{(2)}$

$$\delta(x-z) \delta_{ik} = \int_y W_{k,ij}^{(2)}(x,y) \left(\Gamma_k^{(2)} + \mathcal{R}_k \right)_{jk}(y,z). \tag{183}$$

where the summation over repeated indices is understood (Einstein's convention). Thus the RG equation writes:

$$\partial_k \Gamma_k = \frac{1}{2} \text{Tr} \int_{x,y} \partial_k R_k(x-y) \left(\Gamma_k^{(2)} + \mathcal{R}_k \right)_{x,y}^{-1}. \tag{184}$$

9.1. The RG Equation for the Potential

Once again we define the potential as $\Gamma_k[\vec{M}]$ evaluated in a uniform field configuration \vec{M} . By symmetry, we can choose any direction we want for \vec{M} . We take

$$\vec{M} = \begin{pmatrix} M \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{185}$$

The RG equation for the potential writes

$$\partial_k U_k = \frac{1}{2} \text{Tr} \int_q \left(\partial_k R_k(q) \left(\frac{\partial^2 U_k}{\partial M_i \partial M_j} + (Z_k q^2 + R_k) \delta_{ij} \right)^{-1} \right). \quad (186)$$

Since

$$\frac{\partial^2 U_k}{\partial M_i \partial M_j} = \frac{\partial U_k}{\partial \rho} \delta_{ij} + \frac{\partial^2 U_k}{\partial \rho^2} M_i M_j \quad (187)$$

where $\rho = 1/2\bar{M}^2$, we obtain

$$\frac{\partial^2 U_k}{\partial M_i \partial M_j} + (Z_k q^2 + R_k) \delta_{ij} = \begin{pmatrix} Z_k q^2 + R_k + U'_k + 2\rho U''_k & & & \\ & Z_k q^2 + R_k + U'_k & & \\ & & \ddots & \\ & & & Z_k q^2 + R_k + U'_k \end{pmatrix}. \quad (188)$$

It is trivial to invert this matrix and to compute the trace. We find:

$$\partial_k U_k = \frac{1}{2} \int_q \partial_k R_k(q) \left(\frac{1}{Z_k q^2 + R_k + U'_k + 2\rho U''_k} + \frac{N-1}{Z_k q^2 + R_k + U'_k} \right). \quad (189)$$

We can see two differences with what we have done in the Ising case within the LPA:

- Since we were working within the LPA we did not have previously a “field renormalization” Z_k in front of the q^2 term. Its presence will have important consequences both from the technical and physical point of view.
- There is a new term proportional to $N - 1$ in Eq. (189). It will take care of the physics of the Goldstone bosons in the low temperature phase.

9.2. The RG Equation for the Dimensionless Potential \tilde{U}_k

Once again, Eq. (189) is not well suited for the search of a fixed point since k appears explicitly in the right hand side. We have to go to dimensionless quantities. But now, even the change of variables to dimensionless quantities, Eq. (159) is not sufficient to get rid of the k -dependence since Z_k depends on k . It can be shown that in the scaling regime⁴⁷

$$Z_{k \rightarrow 0} \sim \left(\frac{k}{\Lambda} \right)^{-\eta}. \quad (190)$$

Z_k never reach a fixed point value and it is therefore necessary to get rid of it to obtain the fixed point. We introduce therefore dimensionless and

“renormalized” quantities defined by

$$y = \frac{q^2}{k^2}, \tag{191}$$

$$R_k(q^2) = Z_k q^2 r(y) = Z_k k^2 y r(y), \tag{192}$$

$$\tilde{x} = k x, \tag{193}$$

$$\tilde{M}(\tilde{x}) = \sqrt{Z_k} k^{\frac{2-d}{2}} M(x), \tag{194}$$

$$\tilde{U}_t(\tilde{M}(\tilde{x})) = k^{-d} U_k(M(x)) . \tag{195}$$

Note that a Z_k has been included in R_k . We can now repeat all the different steps leading to the RG equation on \tilde{U}_k . It is useful to define first

$$k \partial_k Z_k = -\eta_k Z_k \tag{196}$$

which we could call a “running” anomalous dimension. Because of the behaviour of Z_k , Eq. (190), η_k will reach a fixed point value (the anomalous dimension) whereas Z_k does not. From Eqs. (191)-(195) we deduce that $Z_k q^2 + R_k + U'_k + 2\rho U''_k$ becomes proportional to $Z_k k^2$ as $\partial_t R_k(q^2)$ which is at the numerator. Thus, with this rescaling, the explicit k - and Z_k -dependencies will disappear in the equation on \tilde{U}_k (in this sense, working with the dimensionless and renormalized quantities consists in going to the “co-moving frame”). The RG equation on \tilde{U}_k writes:²⁸

$$\begin{aligned} \partial_t \tilde{U}_t &= -d \tilde{U}_t + (d - 2 + \eta_k) \tilde{\rho} \tilde{U}'_t - v_d \int_0^\infty dy y^{d/2} (\eta_k r(y) + 2 y r'(y)) \\ &\left(\frac{1}{y(1 + r(y)) + \tilde{U}'_t + 2\tilde{\rho} \tilde{U}''_t} + \frac{N - 1}{y(1 + r(y)) + \tilde{U}'_t} \right). \end{aligned} \tag{197}$$

We shall see in the following that it is sometimes convenient to consider the field-expansion of \tilde{U}_t and the RG flow of the couplings appearing in this expansion. It is a non-trivial question to know around which field configuration one should perform the expansion. Of course, if we do not truncate this expansion and if the expansion has an infinite radius of convergence, the point around which the expansion is performed does not matter. However, the radius of convergence is not infinite and we shall be of course interested in truncating the series expansion at orders as low as possible. The rule of thumb is that each time a field-expansion has to be performed, the best

choice is to do it around the minimum of the (dimensionless) potential κ_k , Eq. (165):

$$\vec{M} \Big|_{\text{Min}} = \begin{pmatrix} \sqrt{2\kappa_k} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{198}$$

with

$$\tilde{U}_k = \frac{\lambda_k}{2}(\tilde{\rho} - \kappa_k)^2 + \frac{u_{3,k}}{3!}(\tilde{\rho} - \kappa_k)^3 + \dots \tag{199}$$

Let us notice that this equation is not sufficient to define completely $\kappa_k, \lambda_k, u_{3,k}$, etc. It is necessary to define them as

$$\frac{\partial \tilde{U}_k}{\partial \tilde{\rho}} \Big|_{\tilde{\rho}=\kappa_k} = 0, \tag{200}$$

$$\frac{\partial^2 \tilde{U}_k}{\partial \tilde{\rho}^2} \Big|_{\tilde{\rho}=\kappa_k} = \lambda_k, \tag{201}$$

$$\begin{aligned} &\vdots \\ \frac{\partial^n \tilde{U}_k}{\partial \tilde{\rho}^n} \Big|_{\tilde{\rho}=\kappa_k} &= u_{n,k}. \end{aligned} \tag{202}$$

Note that Eq. (200) makes sense only if $\kappa_k \neq 0$ (for the search of the fixed point, this is not a problem since $\kappa_k^* \neq 0$).

The flow of all these coupling constants can be obtained trivially by acting on both sides of these equations with ∂_t and using Eq. (197). We find for instance

$$\begin{aligned} \partial_t \kappa_k &= -(d - 2 + \eta_k)\kappa_k + 2v_d \left(3 + 2 \frac{\kappa_k u_{3,k}}{\lambda_k} \right) l_1^d(2\kappa_k \lambda_k) \\ &\quad + 2v_d(N - 1)l_1^d(0), \end{aligned} \tag{203}$$

$$\begin{aligned} \partial_t \lambda_k &= (d - 4 + 2\eta_k)\lambda_k + 2v_d(N - 1)\lambda_k^2 l_2^d(0) \\ &\quad + 2v_d(3\lambda_k + 2\kappa_k u_{3,k})^2 l_2^d(2\kappa_k \lambda_k) \\ &\quad - 2v_d \left(2u_{3,k} + 2\kappa_k u_{4,k} - 2 \frac{\kappa_k u_{3,k}^2}{\lambda_k} \right) l_1^d(2\kappa_k \lambda_k) \end{aligned} \tag{204}$$

where the threshold functions l_n^d have been defined in A.13. One remarks that the flow of λ_k involves $u_{3,k}$ and $u_{4,k}$. This is a general rule: the flow of $u_{n,k}$ involves $u_{n+1,k}$ and $u_{n+2,k}$. The non-perturbative character of these

flows comes from the non-polynomial character of the threshold functions l_n^d . This, in turn, implies that the right hand side of Eqs.(203,204) are not series expansions in the coupling constant λ_k .

The computation of the anomalous dimension η_k requires the computation of the flow of Z_k . As we did for the potential this is possible after a definition of Z_k in terms of Γ_k has been found. It is clear that Z_k corresponds to the term in Γ_k which is quadratic in M and in q . In fact, this definition is not sufficient to completely characterize Z_k since it is the first term in the expansion of the function $Z_k(\tilde{\rho})$ and it is necessary to specify around which value of $\tilde{\rho}$ the expansion is performed. Here again, we choose the minimum κ_k of the potential, Eqs. (198)-(200).

A precise calculation shows that

$$Z_k = \frac{(2\pi)^d}{\delta(p=0)} \lim_{p^2 \rightarrow 0} \frac{d}{dp^2} \left(\bar{\Gamma}_{(2,p),(2,-p)}^{(2)} \Big|_{M_{\min}} \right) \quad (205)$$

where $\bar{\Gamma}_{(2,p),(2,-p)}^{(2)}$ is the second derivative of Γ_k with respect to $M_2(p)$ and $M_2(-p)$. The flow of Z_k is now obtained by acting on both sides of Eq. (205) with ∂_t . After a straightforward although somewhat tedious calculation we obtain:

$$\eta_k = \frac{16v_d}{d} \kappa_k \lambda_k^2 m_{2,2}^d (2\kappa_k \lambda_k) \quad (206)$$

where $m_{2,2}^d$ is a threshold function defined in (A.16). We thus find that η_k is *not an independent quantity*. It is entirely determined by the other couplings.

9.3. The Limits $d \rightarrow 4$, $d \rightarrow 2$ and $N \rightarrow \infty$

A very nice feature of the effective average action formalism is that the one-loop results obtained in $d = 4 - \epsilon$ (with the ϕ^4 theory) and $d = 2 + \epsilon$ (with the non-linear sigma model for $N \geq 3$) are retrieved very simply while keeping in the *ansatz* for Γ_k only κ_k , λ_k and Z_k , Eqs. (203), (204), (206).^{28,42} The large N limit is also retrieved at leading order with the same *ansatz*. This is a very interesting property since it indicates that the same calculation leads to controlled results both in the upper and lower critical dimensions and at $N = \infty$. Needless to say that this is completely out of reach of the perturbative expansion.

It has also been shown – with the same *ansatz* – that the Kosterlitz-Thouless transition is qualitatively well reproduced. With the complete $O(\partial^2)$ approximation, a fairly good quantitative agreement is obtained.⁵²

The remarkable point here is that these results have been obtained without introducing by hand the vortices as is usually done otherwise.

Morris has also shown that the infinite sequence of multicritical fixed points of the Ising model in $d = 2$ can be retrieved. This is also a non-trivial result since it would be very complicated to obtain them perturbatively.⁵¹

10. Conclusion

In this introduction to the NPRG we have focused on its application to statistical mechanics and on some of its relations with perturbative renormalization. We have seen three important points.

First, at a conceptual level, the NPRG enables to understand how microphysics can be continuously related to macrophysics, something that is not possible in general within perturbative field theory. As a by-product, one can solve this way the paradox that a field theory involving infinitely many *interacting* degrees of freedom can be described in the infrared regime with only a finite (and small) number of coupling constants, precisely those that are called renormalizable within perturbation theory. This comes from the attractive character of the submanifold spanned by the renormalizable couplings in the space of coupling constants (the large river effect) and is the very meaning of universality.

Second, we have seen that contrary to common belief, it is possible to obtain qualitatively good results about the long distance physics with NPRG techniques from very short ansätze and even results that reproduce one-loop results around the upper and the lower critical dimensions and at large N . This is probably why the NPRG results obtained at finite N and for dimensions in between the upper and the lower critical dimension are reliable.

Third, the series obtained from the derivative expansion seem to converge rapidly, at least in dimension three for the Ising model. This makes the NPRG a quantitative tool for studying strongly correlated systems and not only, as often claimed, a qualitative one. Of course, this claim should be substantiated by calculations performed beyond the $O(\partial^4)$ and also in dimension two. It is nevertheless encouraging to see that critical exponents already converge at this order to the best known values without any resummation and that non universal quantities can be accurately computed.

Let us finally mention that a crucial drawback of the derivative expansion is its inadequacy to the calculation of the momentum dependence of the correlation functions. In fact, it can be shown that the derivative ex-

pansion makes sense only when the external momenta of the correlation functions are less than the running scale k . Thus, when $k \rightarrow 0$ only the infrared physics can be computed with the derivative expansion. Crucial improvements in the computation of the momentum dependence of $\Gamma^{(2)}$ and $\Gamma^{(4)}$ has been performed these last years^{47–49} and there is no doubt that if this method works it will be a new step in our possibility of computing new non perturbative phenomena in field theory.

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Appendix A. Definitions, conventions

- *Integrals in x and q spaces*

In real and Fourier spaces we define

$$\int_x = \int d^d x \quad , \quad \int_q = \int \frac{d^d q}{(2\pi)^d} \quad (\text{A.1})$$

- *Fourier transform*

$$f(x) = \int_q \tilde{f}(q) e^{iqx} \quad , \quad \tilde{f}(q) = \int_x f(x) e^{-iqx} . \quad (\text{A.2})$$

Depending on the context, we omit or not the tilde on the Fourier transform.

- *Definition of v_d*

$$\int_q f(q^2) = 2v_d \int_0^\infty dx x^{d/2-1} f(x) \quad \text{with} \quad v_d = \frac{1}{2^{d+1} \pi^{d/2} \Gamma(\frac{d}{2})}. \quad (\text{A.3})$$

- *Functional derivatives*

$$\frac{\delta}{\delta \tilde{\phi}_q} = \int_x \frac{\delta \phi(x)}{\delta \tilde{\phi}(q)} \frac{\delta}{\delta \phi(x)} = \int_x \frac{e^{iqx}}{(2\pi)^d} \frac{\delta}{\delta \phi_x}. \quad (\text{A.4})$$

- *Correlation functions*

$\Gamma[M]$ is a functional of $M(x)$. We define the 1PI correlation functions by

$$\Gamma^{(n)}[M(x), x_1, \dots, x_n] = \frac{\delta^n \Gamma[M]}{\delta M_{x_1} \dots \delta M_{x_n}} \quad (\text{A.5})$$

We also define the Fourier transform of $\Gamma^{(n)}$ by

$$\tilde{\Gamma}^{(n)}[M(x), q_1, \dots, q_n] = \int_{x_1 \dots x_n} e^{-i(q_1 x_1 + \dots + q_n x_n)} \Gamma^{(n)}[M(x), x_1, \dots, x_n] \quad (\text{A.6})$$

and

$$\bar{\Gamma}^{(n)}[M(x), q_1, \dots, q_n] = \frac{\delta^n \Gamma}{\delta \tilde{M}_{q_1} \dots \delta \tilde{M}_{q_n}}. \quad (\text{A.7})$$

The relation between $\Gamma^{(n)}$ and $\bar{\Gamma}^{(n)}$ follows from Eq. (A.4) :

$$\bar{\Gamma}^{(n)}[M(x), q_1, \dots, q_n] = (2\pi)^{-nd} \tilde{\Gamma}^{(n)}[M(x), q_1, \dots, q_n]. \quad (\text{A.8})$$

- *Cut-off function in x and q spaces*

$$\Delta H_k[\phi] = -\frac{1}{2} \int_q \tilde{\phi}_q \tilde{\phi}_{-q} \tilde{R}_k(q^2) = -\frac{1}{2} \int_x \phi(x) R_k(x-y) \phi(y). \quad (\text{A.9})$$

One should be careful about the fact that R_k is sometimes considered as a function of q and sometimes as a function of q^2 . It can be convenient to define a cut-off function with two entries by

$$\mathcal{R}_k(x, y) = R_k(x-y). \quad (\text{A.10})$$

Then

$$\tilde{\mathcal{R}}_k(q, q') = (2\pi)^d \delta^d(q+q') \tilde{R}_k(q) \quad (\text{A.11})$$

- *k -dependent anomalous dimension*

By definition:

$$k \partial_k Z_k = -\eta_k Z_k . \tag{A.12}$$

- *Threshold functions* l_n^d

$$l_n^d(w, \eta) = \frac{n + \delta_{n,0}}{2} \int_0^\infty dy y^{d/2-1} \frac{s(y)}{(y(1+r(y)) + w)^{n+1}} \tag{A.13}$$

where

$$R_k(q^2) = Z_k q^2 r(y) \quad \text{with} \quad y = \frac{q^2}{k^2} \tag{A.14}$$

and, by definition of $s(y)$

$$\begin{aligned} k \partial_k R_k(q^2) &= k \partial_k \left(Z_k q^2 r \left(\frac{q^2}{k^2} \right) \right) \\ &= Z_k k^2 (-\eta_k y r(y) - 2y^2 r'(y)) = Z_k k^2 s(y) . \end{aligned} \tag{A.15}$$

- *Threshold functions* m_{n_1, n_2}^d

$$m_{n_1, n_2}^d(w) = -\frac{1}{2} Z_k^{-1} k^{d-6} \int_0^\infty dx x^{d/2} \tilde{\partial}_t \frac{(\partial_x P)^2(x, 0)}{P^{n_1}(x, 0) P^{n_2}(x, w)} \tag{A.16}$$

with

$$P(x, w) = Z_k x + R_k(x) + w. \tag{A.17}$$

- *Universal value of* $l_n^{2n}(0, 0)$ *for* $n > 0$

For $n > 0$ and independently of the choice of cut-off function R_k :

$$l_n^{2n}(0, 0) = \frac{n}{2} \int_0^\infty dy (-2) \frac{r'(y)}{(1+r(y))^{n+1}} = 1. \tag{A.18}$$

- *Derivative of* l_n^d

$$\partial_w l_n^d(w, \eta) = -(n + \delta_{n,0}) l_{n+1}^d(w, \eta). \tag{A.19}$$

- *θ -cut-off*

A convenient cut-off function R_k that allows to compute analytically some threshold functions is

$$R_k(q) = Z_k (k^2 - q^2) \theta \left(1 - \frac{q^2}{k^2} \right). \tag{A.20}$$

With this cut-off we find

$$r(y) = \frac{1-y}{y} \theta(1-y). \tag{A.21}$$

- *Threshold functions l_n^d with the θ -cut-off*

With the cut-off function, Eq. (A.20), the l_n^d threshold functions can be computed analytically

$$l_n^d(w, \eta) = \frac{2}{d}(n + \delta_{n,0}) \left(1 - \frac{\eta_k}{d+2}\right) \frac{1}{(1+w)^{n+1}}. \quad (\text{A.22})$$

- *Threshold functions $m_{2,2}^d$ with the θ -cut-off*

$$m_{2,2}^d(w) = \frac{1}{(1+w)^2}. \quad (\text{A.23})$$

Appendix B. The Exact RG equations

For the sake of simplicity, we consider a scalar theory (e.g. Ising). We have by definition

$$Z_k[B] = \int \mathcal{D}\phi(x) \exp\left(-H[\phi] - \frac{1}{2} \int_q R_k(q) \phi_q \phi_{-q} + \int B\phi\right), \quad (\text{B.24})$$

$$W_k[B] = \log Z_k[B], \quad (\text{B.25})$$

$$\Gamma_k[M] + W_k[B] = \int_x BM - \frac{1}{2} \int_{x,y} M(x) R_k(x-y) M(y) \quad (\text{B.26})$$

with, by definition of $M(x)$:

$$\frac{\delta W_k}{\delta B(x)} = M(x) = \langle \phi(x) \rangle. \quad (\text{B.27})$$

When $B(x)$ is taken k -independent (as in $Z_k[B]$) then $M(x)$ computed from W_k is k -dependent. Reciprocally, if $M(x)$ is taken fixed (as in $\Gamma_k[M]$), then $B(x)$ computed from Eq. (B.31) becomes k -dependent.

Appendix B.1. RG equation for $W_k[B]$

$$\begin{aligned} \partial_k e^{W_k[B]} &= -\frac{1}{2} \int \mathcal{D}\phi \left(\int_{x,y} \phi_x R_k(x-y) \phi_y \right) \\ &\exp\left(-H[\phi] - \frac{1}{2} \int_q R_k(q) \phi_q \phi_{-q} + \int B\phi\right) = \end{aligned}$$

$$\begin{aligned}
 &= \left(-\frac{1}{2} \int_{x,y} \partial_k R_k(x-y) \frac{\delta}{\delta B_x} \frac{\delta}{\delta B_y} \right) e^{W_k[B]} \\
 &= -\frac{1}{2} \int_{x,y} \partial_k R_k(x-y) \left(\frac{\delta^2 W_k}{\delta B_x \delta B_y} + \frac{\delta W_k}{\delta B_x} \frac{\delta W_k}{\delta B_y} \right) e^{W_k[B]}. \quad (B.28)
 \end{aligned}$$

We therefore obtain for W_k :

$$\partial_k W_k[B] = -\frac{1}{2} \int_{x,y} \partial_k R_k(x-y) \left(\frac{\delta^2 W_k}{\delta B_x \delta B_y} + \frac{\delta W_k}{\delta B_x} \frac{\delta W_k}{\delta B_y} \right) \quad (B.29)$$

which is the Polchinski equation.

Appendix B.2. RG equation for $\Gamma_k[M]$

We first derive the reciprocal relation of Eq. (B.27). The Legendre transform is symmetric with respect to the two functions that are transformed. Here the Legendre transform of W_k is $\Gamma_k + 1/2 \int R_k M M$. Thus

$$\frac{\delta}{\delta M_x} \left(\Gamma_k + \frac{1}{2} \int_{x,y} M_x R_k(x-y) M_y \right) = B_x \quad (B.30)$$

and then

$$\frac{\delta \Gamma_k}{\delta M_x} = B_x - \int_y R_k(x-y) M_y. \quad (B.31)$$

In the Polchinski equation (B.29), the k -derivative is taken at fixed B_x . We must convert it to a derivative at fixed M :

$$\partial_k|_B = \partial_k|_M + \int_x \partial_k M_x|_B \frac{\delta}{\delta M_x}. \quad (B.32)$$

Acting on Eq. (B.26) with $\partial_k|_B$, we obtain:

$$\begin{aligned}
 \partial_k \Gamma_k[M]|_B + \partial_k W_k[B]|_B &= \int_x B \partial_k M|_B - \frac{1}{2} \int_{x,y} \partial_k R_k(x-y) M_x M_y \\
 &\quad - \int_{x,y} R_k(x-y) M_x \partial_k M_y|_B. \quad (B.33)
 \end{aligned}$$

Substituting Eqs. (B.31), (B.29), (B.32) into this equation we finally obtain

$$\partial_k \Gamma_k[M] = \frac{1}{2} \int_{x,y} \partial_k R_k(x-y) \frac{\delta^2 W_k}{\delta B_x \delta B_y}. \quad (B.34)$$

The last step consists in rewriting the right hand side of this equation in terms of Γ_k only. We start from (B.27) and act on it with $\delta/\delta M_z$:

$$\delta(x-z) = \frac{\delta^2 W_k}{\delta B_x \delta M_z} = \int_y \frac{\delta^2 W_k}{\delta B_x \delta B_y} \frac{\delta B_y}{\delta M_z}. \quad (B.35)$$

Now, using (B.31), we obtain

$$\delta(x - z) = \int_y \frac{\delta^2 W_k}{\delta B_x \delta B_y} \left(\frac{\delta^2 \Gamma_k}{\delta M_y \delta M_z} + R_k(y - z) \right). \tag{B.36}$$

We define

$$W_k^{(2)}(x, y) = \frac{\delta^2 W_k}{\delta B_x \delta B_y} \tag{B.37}$$

and thus

$$\delta(x - z) = \int_y W_k^{(2)}(x, y) \left(\Gamma_k^{(2)} + \mathcal{R}_k \right) (y, z). \tag{B.38}$$

$\Gamma_k^{(2)} + \mathcal{R}_k$ is therefore the inverse of $W_k^{(2)}$ in the operator sense. Note that this relation is valid for arbitrary M . The RG equation (B.34) can now be written in terms of Γ_k only:

$$\partial_k \Gamma_k[M] = \frac{1}{2} \int_{x,y} \partial_k R_k(x - y) \left(\Gamma_k^{(2)} + \mathcal{R}_k \right)^{-1} (x, y). \tag{B.39}$$

In Fourier space this equation becomes:

$$\partial_k \Gamma_k[M] = \frac{1}{2} \int_q \partial_k \tilde{R}_k(q) \left(\tilde{\Gamma}_k^{(2)} + \tilde{\mathcal{R}}_k \right)^{-1}_{q,-q}. \tag{B.40}$$

Appendix B.3. RG equation for the effective potential

The derivative expansion consists in expanding Γ_k as

$$\Gamma_k[M(x)] = \int_x \left(U_k(M^2(x)) + \frac{1}{2} Z_k(M^2(x)) (\nabla M)^2 + \dots \right) \tag{B.41}$$

where we have supposed that the theory is \mathbb{Z}_2 symmetric so that U_k, Z_k, \dots are functions of M^2 only. To compute the flow of these functions it is necessary to define them from Γ_k . The effective potential U_k coincides with Γ_k when it is evaluated for uniform field configurations M_{unif} :

$$\Gamma_k[M_{\text{unif}}] = \Omega U_k(M_{\text{unif}}^2) \tag{B.42}$$

where Ω is the volume of the system. It is easy to derive an RG equation from this definition of U_k if we use the local potential approximation (LPA) that consists in truncating Γ_k as in (B.41) with $Z_k(M) = 1$:

$$\Gamma_k^{\text{LPA}}[M(x)] = \int_x \left(U_k(M^2(x)) + \frac{1}{2} (\nabla M)^2 \right). \tag{B.43}$$

By acting on Eq. (B.42) with ∂_k we obtain:

$$\partial_k U_k(M) = \frac{1}{2\Omega} \int_q \partial_k R_k(q) \left(\Gamma_k^{(2)} \Big|_{M_{\text{unif.}}} + R_k \right)_{q,-q}^{-1} . \tag{B.44}$$

Thus, we have to invert $\Gamma_k^{(2)} + R_k$ for a uniform field configuration and within the LPA. From now on, we omit the superscript LPA on Γ_k . An elementary calculation leads to

$$\bar{\Gamma}_{k,q,q'}^{(2)} \Big|_{M_{\text{unif.}}} = \left(\frac{\partial^2 U_k}{\partial M^2} + q^2 \right) (2\pi)^{-d} \delta(q + q') . \tag{B.45}$$

Using $\delta(q = 0) = \Omega(2\pi)^{-d}$ we find

$$\partial_k U_k = \frac{1}{2} \int_q \frac{\partial_k R_k(q)}{q^2 + R_k(q) + \frac{\partial^2 U_k}{\partial M^2}} . \tag{B.46}$$

It is convenient to re-express this equation in terms of

$$\rho = \frac{1}{2} M^2 \tag{B.47}$$

which is the \mathbb{Z}_2 -invariant.

$$\partial_k U_k(\rho) = \frac{1}{2} \int_q \frac{\partial_k R_k(q)}{q^2 + R_k(q) + U'_k(\rho) + 2\rho U''_k(\rho)} \tag{B.48}$$

where $U'_k(\rho)$ and $U''_k(\rho)$ are derivatives of U_k with respect to ρ .

To obtain the RG equation for the dimensionless potential we have to perform the change of variables of Eq.160. We find

$$\partial_{t|_\rho} = \partial_{t|\bar{\rho}} + (2 - d - \eta) \bar{\rho} \frac{\partial}{\partial \bar{\rho}} \tag{B.49}$$

and

$$\partial_{t|_{q^2}} = \partial_{t|_y} - 2y \partial_y . \tag{B.50}$$

Inserting these relations together with Eq.(160) and with

$$\partial_t R_k(q^2) = -y(\eta_k r(y) + y r'(y)) Z_k k^2 \tag{B.51}$$

in Eq.(B.48) leads to the RG equation on \tilde{U}_t , Eq.(161).

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