Chapter 1

Cones in $\mathbb{R}^n$ and Kernels

1.1 Notation

We present the n-dimensional notation which will be used throughout. For the origin in $\mathbb{R}^n$, the n-dimensional Euclidean space, we use the standard symbol 0 and it follows easily from the context if 0 denotes the number or the vector. Thus $0 = (0, \ldots, 0) \in \mathbb{R}^n$. The operations on vectors in $\mathbb{R}^n$ (in particular, in $\mathbb{N}^n$ and $\mathbb{N}_0^n$) and inequalities between them are meant coordinatewise which, in particular, simplifies summation symbols involving indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ in $\mathbb{N}_0^n$:

$$\sum_{0 \leq \beta \leq \alpha} a_\beta := \sum_{\beta_1=1}^{\alpha_1} \ldots \sum_{\beta_n=1}^{\alpha_n} a_{\beta_1,\ldots,\beta_n}.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of arbitrary reals (in particular, arbitrary integers). If $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, we define $t^\alpha := t_1^{\alpha_1} \ldots t_n^{\alpha_n}$; in particular, $t^\alpha := t^{\alpha_1 + \ldots + \alpha_n}$ for $t \in \mathbb{R}$, whenever the symbols t^{α_j} make sense. The symbol z^α for $z \in \mathbb{C}^n$ is defined analogously. For $\alpha, \beta \in \mathbb{N}_0^n$ with $\alpha \leq \beta$ we define $\bar{\alpha} := \alpha_1 + \ldots + \alpha_n$, $\alpha! := \alpha_1! \ldots \alpha_n!$ and

$$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \ldots \binom{\alpha_1}{\beta_1}.$$

Given two vectors $t = (t_1, \ldots, t_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ we use the symbol $\langle t, y \rangle$ for their scalar product, i.e.,

$$\langle t, y \rangle := t_1 y_1 + \ldots + t_n y_n.$$

A similar n-dimensional notation will be applied in the complex Euclidean space $\mathbb{C}^n$.

Let α denote an n-tuple of nonnegative integers, i.e., $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. The symbol $D^\alpha = D_t^\alpha$ with $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$

denotes the differential operator given by

$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n} \quad \text{with} \quad D_j^{\alpha_j} := -\frac{1}{2\pi i}\frac{\partial^{\alpha_j}}{\partial t_j^{\alpha_j}} \qquad \text{for} \quad j = 1, \dots, n. \tag{1.1}$$

On the other hand, the symbol $\partial^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ with $t \in \mathbb{R}^n$ denotes the partial differential operator defined analogously as in (1.1), but with the constant 1 instead of $-(2\pi i)^{-1}$. We also write $\varphi^{(\alpha)}(t)$ instead of $\frac{\partial^\alpha \varphi(t)}{\partial t^\alpha}$ for functions φ on $\mathbb{R}^n$. A similar convention is applied to the symbols D_z^α, $\frac{\partial^\alpha}{\partial z^\alpha}$ and $\varphi^{(\alpha)}(z)$ for $z \in \mathbb{C}^n$ and functions φ on $\mathbb{C}^n$.

For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and, in particular, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote

$$|z| := \big(\sum_{j=1}^n |z_j|^2\big)^{1/2}, \qquad |x| := \big(\sum_{j=1}^n |x_j|^2\big)^{1/2},$$

i.e., $|z|$ and $|x|$ denote the Euclidean norms of $z \in \mathbb{C}^n$ and $x \in \mathbb{R}^n$, respectively.

Functions and ultradistributions considered in the book can be treated as real- or complex-valued functions defined on (subsets of) $\mathbb{R}^n$ or $\mathbb{C}^n$. In general, we will try to distinguish a value of a function from the function itself, e.g. the symbols $\varphi(t), f(z) = f(x+iy)$ will mean the values of the functions φ, f at the points $t \in \mathbb{R}^n, z = x+iy \in T^C = \mathbb{R}^n + iC \subset \mathbb{C}^n$, where $C \subset \mathbb{R}^n$; consequently, for a given function $f\colon\ T^C \to \mathbb{C}$ and a fixed $y \in C$, the symbols $f(\cdot + iy)$, $\|f(\cdot + iy)\|_{L^s}$ will mean the function $g_y\colon\ \mathbb{R}^n \to \mathbb{C}$ defined as $g_y(x) := f(x+iy)$ for $x \in \mathbb{R}^n$ and $\|g_y\|_{L^s}$, respectively.

Sometimes, however, it will be convenient to use the same symbol $f(x)$ for the function $f = f(\cdot)$ and its value $f(x)$ at the point x. For instance, in case of the Cauchy and Poisson kernels the traditional symbols $K(z-t)$ and $Q(z;t)$ will denote the values of the functions K and Q for concreate z and t, where $z = x + iy \in T^C = \mathbb{R}^n + iC$, for a given cone C in $\mathbb{R}^n$, and $t \in \mathbb{R}^n$, as well as the functions K and Q themselves. Similarly, if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, then the symbol x^α will mean both the power function which assigns to each $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the number $x^\alpha = x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$ and its value at the given point x. Moreover, for arbitrary $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$, we denote by $< x >^\beta$ both the function on $\mathbb{R}^n$ and its value at the point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ defined by the formula:

$$< x >^\beta := \prod_{j=1}^n [1 + (x_j)^2]^{\beta_j/2}. \tag{1.2}$$

We shall apply the following very convenient notation for exponents with two variables $z, \zeta \in \mathbb{C}^n$:

$$E_z(\zeta) := \exp[2\pi i\langle z, \zeta\rangle], \qquad z \in \mathbb{C}^n, \quad \zeta \in \mathbb{C}^n. \tag{1.3}$$

We also denote

$$e_y(t) := \exp[-2\pi\langle y, t\rangle], \qquad y \in \mathbb{R}^n, \quad t \in \mathbb{R}^n, \tag{1.4}$$

i.e., we have $E_{iy} = e_y$ for $y \in \mathbb{R}^n$. In particular, $e_y(t) = \exp(-2\pi yt)$ for $y, t \in \mathbb{R}$. The symbols $\tilde{g}$ and $\tilde{T}$ for a given real- or complex-valued function g and ultradistribution T in $\mathbb{R}^n$ are meant as follows:

$$\tilde{g}(x) := g(-x), \qquad x \in \mathbb{R}^n, \tag{1.5}$$

and

$$\langle \tilde{T}, \varphi\rangle := \langle T, \tilde{\varphi}\rangle \tag{1.6}$$

for every function φ from the respective space of test functions (see Section 2.3), where $\tilde{\varphi}$ is defined in (1.5).

The *Fourier transform* of a real- or complex-valued L^1 function φ, denoted by $\mathcal{F}[\varphi]$ or by $\hat{\varphi}$, is defined by (and used in Chapters 1-5)

$$\mathcal{F}[\varphi](x) = \hat{\varphi}(x) := \int_{\mathbb{R}^n} \varphi(t) e^{2\pi i\langle x,t\rangle}\, dt = \int_{\mathbb{R}^n} \varphi(t) E_x(t)\, dt \tag{1.7}$$

and the *inverse Fourier transform* of an L^1 function φ, denoted by $\mathcal{F}^{-1}[\varphi]$ or by $\check{\varphi}$ is defined by

$$\mathcal{F}^{-1}[\varphi](x) = \check{\varphi}(x) := \int_{\mathbb{R}^n} \varphi(t) e^{-2\pi i\langle x,t\rangle}\, dt = \int_{\mathbb{R}^n} \varphi(t) E_{-x}(t)\, dt. \tag{1.8}$$

In Chapter 7, another version of the Fourier and inverse Fourier transforms, $\mathcal{F}_0$ and $\mathcal{F}_0^{-1}$, will be convenient to be considered instead of $\mathcal{F}$ and $\mathcal{F}^{-1}$ defined in (1.7) and (1.8). This will not cause any misinterpretation, because both versions of the definitions differ from each other by constants, and all results remain true in both cases.

We assume familiarity on the part of the reader with properties of the Fourier transform on L^r, $1 \le r \le 2$, the corresponding inverse Fourier transform, and the associated Plancherel theory for the Fourier transform.

The *inclusion* of two sets will be denoted by means of the symbol $A \subseteq B$ and the *proper inclusion* by the symbol $A \subset B$. The *closure* of the set $A \subseteq \mathbb{R}^n$ (in the sense of the Euclidean topology) will be denoted by $\bar{A}$.

By the *support* of 1° a given function in $\mathbb{R}^n$; 2° a given ultradistribution T in $\mathbb{R}^n$, denoted by 1° $\operatorname{supp} g$, 2° $\operatorname{supp} T$ we will mean 1° the set

$$\operatorname{supp} g := \overline{\{t \in \mathbb{R}^n : \; g(t) \neq 0\}}; \tag{1.9}$$

2° the smallest closed set $A \subseteq \mathbb{R}^n$ such that $\langle T, \varphi\rangle = 0$ for all functions φ from the respective space of test functions such that $\operatorname{supp}\varphi \subseteq A^c$.

1.2 Cones in $\mathbb{R}^n$

We introduce the definitions and notation associated with cones in $\mathbb{R}^n$ and tubes in $\mathbb{C}^n$ (cf. [147], [151]).

A set $C \subseteq \mathbb{R}^n$ is a *cone* (with vertex at zero) if $y \in C$ implies $\lambda y \in C$ for all positive reals λ. The intersection of the cone C with the unit sphere $\{y \in \mathbb{R}^n\colon\ |y| = 1\}$ is called the *projection* of C and is denoted by $\mathrm{pr}(C)$. If C_1 and C_2 are cones such that $\mathrm{pr}(\overline{C_1}) \subset \mathrm{pr}(C_2)$, the cone C_1 will be called a *compact subcone* of C_2 and we will write then $C_1 \subset\subset C_2$. An open convex cone C such that $\overline{C}$ does not contain any entire straight line will be called a *regular cone*. The set

$$C^* := \{t \in \mathbb{R}^n\colon\ \langle t, y\rangle \geq 0 \quad \text{for all } y \in C\}$$

is the *dual cone* of the cone C. A cone is called *self dual* if $C^* = \overline{C}$. For every cone C, the dual cone C^* is closed and convex. We have $C^* = \overline{C}^* = (O(C))^*$ and $C^{**} = \overline{O(C)}$, where $O(C)$ denotes the convex hull of C. The function u_C defined by

$$u_C(t) := \sup_{y \in \mathrm{pr}(C)} (-\langle t, y\rangle), \qquad t \in \mathbb{R}^n$$

is said to be the *indicatrix* of the cone C.

We have $C^* = \{t \in \mathbb{R}^n\colon\ u_C(t) \leq 0\}$. Moreover, $u_C(t) \leq u_{O(C)}(t)$ for all $t \in \mathbb{R}^n$ and $u_C(t) = u_{O(C)}(t)$ for $t \in C^*$.

Given a cone C, put $C_* := \mathbb{R}^n \setminus C^*$. The number

$$\rho_C := \sup_{t \in C_*} u_{O(C)}(t)/u_C(t)$$

characterizes the convexity of C. Notice that a cone C is convex if and only if $\rho_C = 1$. Further, if a cone is open and consists of a finite number of components, then $\rho_C < +\infty$.

We give some examples of cones and their dual cones. If $C = (0, \infty)$, then $C^* = [0, \infty)$, $u_C(t) = -t$ and $\rho_C = 1$. The case $C = (-\infty, 0)$ is analogous. If $C = \mathbb{R}^n$, then $C^* = \{0\}$, $u_C(t) = |t|$ and $\rho_C = 1$.

Let Θ be the set of all n-tuples whose entries are -1 or 1, i.e.,

$$\Theta := \{u = (u_1, \ldots, u_n) \in \mathbb{R}^n\colon\ \ u_j \in \{-1, 1\} \ \text{ for } \ j = 1, \ldots, n\}. \tag{1.10}$$

Fix $u = (u_1, \ldots, u_n) \in \Theta$, an arbitrary element of the 2^n elements of Θ. Then

$$C_u := \{y \in \mathbb{R}^n\colon\ u_j y_j > 0 \ \text{ for } \ j = 1, \ldots, n\} \tag{1.11}$$

is a self dual cone in $\mathbb{R}^n$ for every $u \in \Theta$. Each of the 2^n sets C_u with $u \in \Theta$ will be called an *n-rant* in $\mathbb{R}^n$.

Each n-rant C_u in $\mathbb{R}^n$ $(u \in \Theta)$ defined in (1.11) is an example of a regular cone. The forward and backward light cones, defined by

$$\Gamma^+ := \{y \in \mathbb{R}^n\colon\ y_1 > (y_2^2 + \ldots + y_n^2)^{1/2}\},$$
$$\Gamma^- := \{y \in \mathbb{R}^n\colon\ y_1 < -(y_2^2 + \ldots + y_n^2)^{1/2}\},$$

respectively, are important self dual cones in mathematical physics.

For an arbitrary cone C in $\mathbb{R}^n$ the set

$$T^C := \mathbb{R}^n + iC = \{z = x + iy\colon\ x \in \mathbb{R}^n,\ y \in C\}$$

will be called a *tube* in $\mathbb{C}^n$. The set $\{z = x + iy :\ \ x \in \mathbb{R}^n,\ y = 0\}$ is called the *distinguished boundary* of the tube T^C, while $\mathbb{R}^n + i\partial C$, with ∂C denoting the boundary of C, is the *topological boundary* of T^C.

We now present two important lemmas concerning cones and dual cones which will be of particular use in the construction and analysis of the Cauchy and Poisson kernel functions below. The lemmas are proved in [147], Section 25; we give here a separate proof of the second lemma.

Lemma 1.2.1. *Let C be an open connected cone in $\mathbb{R}^n$. The closure $\overline{O(C)}$ of $O(C)$ contains an entire straight line if and only if the dual cone C^* lies in some $(n-1)$-dimensional plane.*

Lemma 1.2.2. *Let C be an open (not necessarily connected) cone in $\mathbb{R}^n$. For every $y \in O(C)$ there exists a positive δ (depending on y) such that*

$$\langle y, t\rangle \geq \delta\, |y|\, |t|, \qquad t \in C^*. \tag{1.12}$$

Further, if C' is an arbitrary compact subcone of $O(C)$, then there exists a $\delta > 0$ (depending only on C' and not on $y \in C'$) such that (1.12) holds for all $y \in C'$ and all $t \in C^*$.

Proof. Since $u_C(t) = u_{O(C)}(t)$ for $t \in C^*$, we have $\langle y, t\rangle \geq 0$ for all $y \in O(C)$ and all $t \in C^*$. For an arbitrary $y \in O(C)$, we have

$$\tilde{y} := y/|y| \in \mathrm{pr}(O(C)) \subset O(C),$$

since $O(C)$ is a cone. Moreover, $O(C)$ is open, because C is open. Thus there exists a $\delta = \delta_y > 0$ such that

$$N(\tilde{y}, 2\delta) := \{y'\colon\ |y' - \tilde{y}| < 2\delta\} \subset O(C).$$

Hence

$$\tilde{y} - (t/|t|)\,\delta \in N(\tilde{y}, 2\delta) \subset O(C)$$

and thus

$$\langle \tilde{y} - (t/|t|)\,\delta, t\rangle \geq 0$$

for every $t \in C^*$, but this implies (1.12). Now, let C' be an arbitrary compact subcone of $O(C)$. Let d be the distance from $\text{pr}(C')$ to the complement of $O(C)$ in $\mathbb{R}^n$, that is,

$$d := \inf\{|y_1 - y_2| : \; y_1 \in \text{pr}(C'), \; y_2 \notin O(C)\}.$$

Obviously, d is positive and depends only on C' and not on $y \in C'$. Define now $\delta = d/2$. The preceding considerations show that (1.12) holds for all $y \in C'$ and $t \in C^*$. The proof is complete. □

For C being an open connected cone in $\mathbb{R}^n$, we denote the distance from $y \in C$ to the topological boundary ∂C of C by

$$d(y) := \inf\{|y - y_1| : \; y_1 \in \partial C\}.$$

It has been shown in [151], p. 159, that

$$d(y) = \inf_{t \in \text{pr}(C^*)} \langle t, y\rangle, \qquad y \in C. \tag{1.13}$$

Let C' be an arbitrary compact subcone of C. It follows from Lemma 1.2.2 and (1.13) that there exists a $\delta = \delta(C') > 0$, depending only on C' and not on $y \in C'$, such that

$$0 < \delta|y| \leq d(y) \leq |y|, \qquad y \in C' \subset\subset C. \tag{1.14}$$

Let C be an open connected cone in $\mathbb{R}^n$. We make the following convention concerning the notation $y \to 0, \; y \in C$, which normally means that y varies arbitrarily within C while $y \to 0$. But frequently the above symbol will mean that $y \to 0, y \in C'$ for every compact subcone C' of C. We shall distinguish between these two convergences only when necessary; in most relevant situations the analysis clearly shows which of the interpretations of the symbol $y \to 0, y \in C$, is used in a given case.

Let V be an ultradistribution and let f be a function of the variable $z = x + iy \in T^C$ for a given cone C. By $f(\cdot + iy) \to V$ in the weak topology of the ultradistribution space as $y \to 0, y \in C$, we mean the convergence

$$\langle f(\cdot + iy), \varphi\rangle \to \langle V, \varphi\rangle$$

as $y \to 0, y \in C$, for each fixed element φ in the corresponding test function space. By $f(\cdot + iy) \to V$ in the strong topology of the ultradistribution space as $y \to 0, y \in C$, we mean

$$\langle f(\cdot + iy), \varphi\rangle \to \langle V, \varphi\rangle$$

as $y \to 0, \; y \in C$, where the convergence is uniform for an arbitrary bounded set of functions φ in the corresponding test function space. Then V is called the weak or strong, respectively, ultradistributional boundary value of f and is defined on the distinguished boundary of the tube T^C.

1.3 Cauchy and Poisson kernels

Let C be a regular cone in $\mathbb{R}^n$, that is C is an open convex cone such that $\overline{C}$ does not contain any entire straight line. The *Cauchy kernel* corresponding to the tube T^C, denoted traditionally by $K(z-t)$, is defined as a function of variables $z = x + iy \in T^C$ and $t \in \mathbb{R}^n$ by the formula

$$K(z-t) := \int_{C^*} E_{z-t}(u)\, du = \int_{C^*} \exp\left[2\pi i\langle z-t, u\rangle\right] du, \qquad z \in T^C,\ \ t \in \mathbb{R}^n. \tag{1.15}$$

Note that the Cauchy kernel $K(z-t)$ is well defined for $z = x + iy \in T^C$ and $t \in \mathbb{R}^n$, because $\langle y, u\rangle \geq 0$ for $y \in C$ and $u \in C^*$ and

$$|E_{z-t}(u)| = |\exp\left[2\pi i\langle x-t, u\rangle\right] \exp\left[-2\pi\langle y, u\rangle\right]| = \exp\left[-2\pi\langle y, u\rangle\right].$$

Moreover, denoting by I_{C^*} the characteristic function of C^* and using the definition (1.8) of the inverse Fourier transform $\mathcal{F}^{-1}$, we can write formula (1.15) in the form:

$$K(z-t) := \int_{C^*} \exp\left[-2\pi i t\right] E_z(u)\, du = \mathcal{F}^{-1}[I_{C^*} E_z](t), \qquad z \in T^C,\ \ t \in \mathbb{R}^n. \tag{1.16}$$

In case $C = C_u$ is any of the 2^n n-rants in $\mathbb{R}^n$, the Cauchy kernel $K(z-t) = K_u(z-t)$ takes the classical form

$$K(z-t) := \frac{(-1)^u}{(2\pi i)^n} \prod_{j=1}^{n} (t_j - z_j)^{-1}, \qquad z \in \mathbb{R}^n + iC_u,\ \ t \in \mathbb{R}^n,$$

since $C_u^* = \overline{C}_u$ in this case.

The *Poisson kernel* Q corresponding to the tube T^C is the function of variables $z \in T^C$ and $t \in \mathbb{R}^n$ given by

$$Q(z;t) := \frac{K(z-t)\overline{K(z-t)}}{K(2iy)}, \qquad z = x + iy \in T^C, \quad t \in \mathbb{R}^n. \tag{1.17}$$

In case $C = C_u$ is any of the n-rants, the Poisson kernel $Q(z;t) = Q_u(z;t)$ reduces to the classical form

$$Q(z;t) = \frac{(-1)^u}{\pi^n} \prod_{j=1}^{n} \frac{y_j}{(t_j - x_j)^2 + y_j^2}, \qquad z = x + iy \in \mathbb{R}^n + iC_u, \quad t \in \mathbb{R}^n.$$

If the cone C above had been assumed to be open and connected but not necessarily convex, we would have defined the kernels $K(z-t)$ and $Q(z;t)$ for $z \in T^{O(C)}$ and would obtain all the properties concerning the kernels for $z \in T^{O(C)}$. Thus we have assumed that C is convex without loss of generality. From Lemma 1.2.1, the dual cone C^* will lie in an $(n-1)$-dimensional plane if $\overline{C}$ contains an entire straight line. In this case the Lebesgue measure of C^* would be zero. Hence the Cauchy kernel $K(z-t)$ would be zero and the Poisson kernel $Q(z;t)$ would be undefined. To avoid this situation we have to assume that $\overline{C}$ does not contain any entire straight line. Therefore we consider regular cones unless explicitly stated otherwise.

We conclude this section with several technical lemmas which will be used in our analysis concerning the Cauchy and Poisson kernels.

Lemma 1.3.1. *Let C be an open connected cone in $\mathbb{R}^n$.*

I. Fix arbitrarily $z \in T^C = \mathbb{R}^n + iC$ and denote by I_{C^*} the characteristic function of C^*. Then $E_z I_{C^*} \in L^p$ for all p, $1 \le p \le \infty$.

II. Assume that g is a continuous function on $\mathbb{R}^n$ with support in C^* such that, for arbitrary $m > 0$ and compact subcone C' of C,

$$|g(t)| \le M(C', m) \exp[2\pi(\langle w, t\rangle + \sigma|w|)], \qquad t \in \mathbb{R}^n, \tag{1.18}$$

whenever $\sigma > 0$ and $w \in C' \setminus (C' \cap \overline{N}(0,m)))$, where $\overline{N}(0,m)$ is the closure of the ball with center at 0 and radius m and $M(C', m)$ is a constant. Then, for an arbitrary y in C, $y \ne 0$, we have $e_y g \in L^p$, whenever $1 \le p < \infty$.

Proof. To prove part I fix z in T^C and let $y = \operatorname{Im} z$. Applying Lemma 1.2.2, we find a $\delta = \delta_y > 0$ such that

$$|E_z(t)|\, I_{C^*}(t) = e_y(t)\, I_{C^*}(t) \le e_{\delta\,|y|}(|t|)\, I_{C^*}(t) \le 1 \tag{1.19}$$

for all $z = x + iy \in T^C$ and all $t \in \mathbb{R}^n$, since $I_{C^*}(t) = 0$ for $t \notin C^*$. Part I of the lemma for $p = \infty$ follows from (1.19). For $1 \le p < \infty$, we use (1.19) and integration by parts $n-1$ times (or the gamma function after the change of variable for $v = 2\pi\delta p|y|r)$) to get

$$\int_{\mathbb{R}^n} |E_z(t) I_{C^*}(t)|^p\, dt \le \int_{\mathbb{R}^n} e_{p\delta|y|}(|t|)\, dt$$

$$= \Omega_n \int_0^\infty r^{n-1} e_{p\delta|y|}(r)\, dr = (n-1)!\, \Omega_n (2\pi\delta p|y|)^{-n}, \tag{1.20}$$

where Ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. The estimate in (1.20) proves part I of the lemma for $1 \le p < \infty$.

To prove part II fix a point y in C. Since C is open, there exists a compact subcone C' of C and a $m > 0$ such that $y \in C' \setminus (C' \cap \overline{N}(0,m))$. Since $y \notin \overline{N}(0,m))$, we have $|y| > m$. Choose $w := \lambda y$, where λ is an arbitrary number such that $m/|y| < \lambda < 1$. Since C' is a cone, $y \in C'$ and $\lambda|y| > m$, we have $w = \lambda y \in C' \setminus (C' \cap \overline{N}(0, m))$, i.e., the estimate given by (1.18) is true for w just chosen. Since $C' \subset\subset C$, it follows from Lemma 1.2.2 that there is a $\delta = \delta(C') > 0$, not depending on $y \in C'$, such that (1.12) holds for all $t \in C^*$. Hence, denoting $A(\sigma, \lambda, y) := M(C', m) \exp[2\pi\sigma\lambda|y|]$, we have

$$|e_y(t)g(t)| \leq A(\sigma, \lambda, y)e_{(1-\lambda)y}(t) \leq A(\sigma, \lambda, y)e_{(1-\lambda)\delta|y|}(|t|)$$

for $t \in C^*$. Integrating by parts (or using the gamma function) yields

$$\begin{aligned}\int_{\mathbb{R}^n} |E_{iy}(t)g(t)|^p \, dt &\leq \int_{C^*} e_{p(1-\lambda)\delta|y|}(|t|) \, dt \\ &= \Omega_n[A(\sigma, \lambda, y)]^p \int_0^\infty r^{n-1} e_{p(1-\lambda)\delta|y|}(r) \, dr \\ &= (n-1)! \, \Omega_n[A(\sigma, \lambda, y)]^p [2\pi p(1-\lambda)\delta|y|]^{-n} < \infty,\end{aligned}$$

since $\operatorname{supp} g \subseteq C^*$. This completes the proof of part II and the lemma. □

Lemma 1.3.2. *Let C be a regular cone. The Cauchy kernel $K(z-t)$ is an analytic function of the variable $z \in T^C$ for each fixed $t \in \mathbb{R}^n$.*

Proof. Let I_{C^*} denote the characteristic function of C^*. By the proof of Lemma 1.3.1, $I_{C^*}E_{z-t} \in L^1$ for fixed $z \in T^C$ and $t \in \mathbb{R}^n$. Let K be an arbitrary compact subset of T^C and let $z \in K \subset T^C$. There exists a compact subcone C' of C such that $y = \operatorname{Im} z \in C'$ and y has positive distance (say k) from 0. By Lemma 1.2.2, there is a $\delta = \delta(C') > 0$, depending only on C', such that

$$|I_{C^*}(u)E_{z-t}(u)| = I_{C^*}(u) \exp[-2\pi\langle y, u\rangle] \leq I_{C^*}(u) \exp[-2\pi\delta k|u|] \quad (1.21)$$

for $t \in \mathbb{R}^n$ and $u \in C^*$. The right side of (1.21) is an L^1 function of variable $u \in \mathbb{R}^n$ for arbitrary $z \in K$ and $t \in \mathbb{R}^n$, according to the proof of Lemma 1.3.1, and the function $z \mapsto I_{C^*}(u)E_{z-t}(u)$ is analytic in $z \in T^C$ for each fixed $t \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$. To conclude the assertion it remains to use a well known theorem concerning integrals involving a parameter (see e.g. [15], pp. 295-296). □

Lemma 1.3.3. *Let C be a regular cone and fix $w = u + iv \in T^C$. The function*

$$K(z+w) := \int_{C^*} E_{z+w}(u) \, du, \qquad z \in T^C,$$

is analytic in $z \in T^C$ *and*

$$|K(z+w)| \le M_v < \infty, \qquad z \in T^C,$$

where M_v *is a constant which depends only on* $v = \operatorname{Im} w$.

Proof. The proof that $K(z+w)$ is analytic in $z \in T^C$ is the same as in the proof of Lemma 1.3.2. We have $\langle y, u\rangle \ge 0$ for $y \in C$ and $u \in C^*$. By Lemma 1.2.2, there is a $\delta = \delta_v > 0$ such that $\langle v, u\rangle \ge \delta|v||u|$ for $v \in C$ and $u \in C^*$. The assertion now follows by similar analysis as in (1.20). □

Lemma 1.3.4. *Let* $h \in L^p$, $1 \le p \le 2$ *and let* $g := \mathcal{F}^{-1}[h]$ *in the sense of the space* L^p. *Assume that* $gE_z \in L^1$ *for* $z \in T^C$ *and* $\operatorname{supp} g \subseteq C^*$ *almost everywhere. We have*

$$\int_{C^*} g(u)E_z(u)\,du = \int_{\mathbb{R}^n} h(t)K(z-t)\,dt, \qquad z \in T^C. \tag{1.22}$$

Proof. Let $z \in T^C$. Let $1 \le p \le 2$ and $1/p + 1/q = 1$. As a result of the remarks below, $K(z-t)$ as a function of $t \in \mathbb{R}^n$ belongs to L^q, i.e. $K(z-\cdot) \in L^q$, for every $z \in T^C$. Therefore the integral on the right side of (1.22) is well defined. First consider $p = 1$. By Lemma 1.3.3, Fubini's theorem and definition (1.8) of the inverse Fourier transform $\mathcal{F}^{-1}[h]$ of h, we have

$$\begin{aligned}\int_{\mathbb{R}^n} h(t)K(z-t)\,dt &= \int_{\mathbb{R}^n} h(t)\,dt \int_{C^*} e^{2\pi i<z-t,u>}\,du \\ &= \int_{C^*} e^{2\pi i<z,u>}\,du \int_{\mathbb{R}^n} h(t)E_{-u}(t)\,dt = \int_{C^*} g(u)E_z(u)\,du,\end{aligned} \tag{1.23}$$

which proves (1.22) for $p = 1$. In case $1 < p \le 2$, the function g is the limit in the L^q norm of the sequence (g_k) of the functions

$$g_k(u) := \int_{|t|\le k} h(t)E_{-u}(t)\,dt \qquad (k \in \mathbb{N})$$

and so, by Hölder's inequality,

$$\int_{C^*} |gE_z - g_kE_z|\,du \le \|g - g_k\|_{L^q}\|e_y\|_{L^p} \to 0$$

as $k \to \infty$ for every $z \in T^C$ (i.e. $y \in C$). Consequently, applying Fubini's theorem, we conclude from (1.23) that

$$\begin{aligned}\int_{C^*} g(u)E_z(u)\;du &= \lim_{k\to\infty} \int_{C^*} g_k(u)E_z(u)\;du \\ &= \lim_{k\to\infty} \int_{|t|\le k} h(t)\,dt \int_{C^*} E_{z-t}(u)\;du = \int_{\mathbb{R}^n} h(t)K(z-t)\,dt\end{aligned}$$

for $z \in T^C$, which shows (1.22) in the cases $1 < p \leq 2$ and Lemma 1.3.4 is thus proved. $\square$

The Poisson kernel defined in (1.17) has been known for some time to be an approximate identity. We state this in the following lemma (see [137], p. 105):

Lemma 1.3.5. *Let C be a regular cone, let $z \in T^C$ and $t \in \mathbb{R}^n$. The Poisson kernel $Q(z;t)$ has the following properties:*

$$(i) \qquad Q(z;t) \geq 0, \qquad z \in T^C, \quad t \in \mathbb{R}^n;$$

$$(ii) \qquad \int_{\mathbb{R}^n} Q(z;t)\,dt = 1, \qquad z \in T^C;$$

$$(iii) \qquad \lim_{z \to t_0, z \in T^C} \int_{|t-t_0|>\delta} Q(z;t)\,dt = 0, \qquad \delta > 0$$

uniformly for all $t_0 \in \mathbb{R}^n$.

We shall prove later that the Cauchy and Poisson kernels are in certain ultradifferentiable function spaces.