Preface

Numerous papers have been written concerning ultradistribution spaces (see [127], [7], [8], [82]-[86], [87], [56], [44], [11] and references therein). Such spaces are related to the solvability and the regularity problems of partial differential equations. Because of this relation, the study of the structural problems as well as problems of various operations and integral transformations in this setting is interesting in itself. The unpublished book of Komatsu [86] and in general papers of Komatsu are the basis for our approach. Important results in the framework of ultradistribution and hyperfunction spaces (see [129]-[130], [77]) which will not be included in this book were obtained by D. Kim, S. Y. Chung and their collaborators (see [38]-[40], [41]-[42], [79]), Matsuzawa (see [98]-[100]), Vogt, Meise, Taylor, Petzsche and their collaborators (see [104], [105], [110], [111]), the Italian school with Rodino, Gramchev (see [124], [54]) and many others. A list of papers with results on various problems within ultradistribution spaces is given in the references which is, however, far from being complete.

This book is intended to be an analysis of various spaces of ultradistributions considered as boundary values of analytic functions having appropriate growth estimates, and deals, in particular, with the Cauchy and Poisson integrals in the ultradistribution spaces $\mathcal{D}'(*, L^s)$, with the convolution of ultradistributions and tempered ultradistributions, and with the integral transforms of tempered ultradistributions.

The problems of characterizing analytic functions whose boundary values are elements of the spaces of distributions, ultradistributions, hyperfunctions, infra-hyperfunctions and, conversely, of finding boundary value representations of elements of the quoted spaces of generalized functions by analytic functions have a long history; for references see e.g. [88]-[89], [139]-[140], [147], [6], [152], [27] and references therein.

Carmichael and his co-workers ([16]-[27], [33]-[35]) have studied the Cauchy and Poisson kernels in appropriate tube domains. By considering the Cauchy and Poisson integrals of distributions in appropriate subspaces of the Schwartz space $\mathcal{D}'$, they obtained characterizations of these subspaces by the a priori estimates of the corresponding analytic or harmonic functions in tube domains.

Boundary value characterizations for the spaces $\mathcal{D}'((M_p),\Omega)$, $\mathcal{D}'(\{M_p\}),\Omega)$ of ultradistributions and the spaces $\mathcal{E}'((M_p),\Omega)$, $\mathcal{E}'(\{M_p\},\Omega)$ of infra-hyperfunctions, related to a non-quasianalytic and quasianalytic sequence (M_p), respectively, are given in [127], [82], [111], [125].

The spaces $\mathcal{D}'((M_p),L^s)$ and $\mathcal{D}'((\{M_p\}),L^s)$ for $s \geq 1$ related to a non-quasianalytic sequence (M_p) are studied in papers by Carmichael and Pilipović. In this book, we investigate classes of analytic functions having boundary values in these spaces. For the analysis of Hardy type spaces of analytic functions, with bounds given by appropriate associated functions corresponding to the sequences (M_p), we apply the Cauchy and Poisson integrals as well as the Fourier transforms. The geometry of tube domains also is considered in this book. A complete boundary value characterization for the spaces $\mathcal{D}'(*,L^s)$ on $\mathbb{R}^n$, with $s \in (1,\infty)$, is given by means of almost analytic extensions, while in the cases $s=\infty$ and $s=1$ only partial results are obtained.

One of the most important operations in the theory of generalized functions (Schwartz distributions, ultradistributions of Beurling and Roumieu type, hyperfunctions, Mikusiński operators) is the convolution. In the literature, the convolution of two generalized functions is usually defined and considered only in the case where one of them is of compact support and the definition has many applications. For instance, this definition of the convolution was used in the study of convolution equations in the space of ultradistributions by various authors. However, such a definition is not sufficient in many situations where the convolution of generalized functions should be defined without any restrictions on the supports of the generalized functions involved.

General definitions of the convolution of this kind for distributions (and tempered distributions) were considered by many authors (see [36, 132, 57, 156, 135, 149, 151, 2, 45, 69]), and it appeared that most of the definitions are equivalent (see [135, 45, 69]). Similar general definitions of the convolution for ultradistributions were first introduced in [119]. Then other analogues of the definitions of the convolution of distributions and tempered distributions were discussed for ultradistributions and tempered ultradistributions in [92, 70, 72, 73], and their equivalence was proved in [70] and in [73], respectively. Moreover, various sufficient conditions for the existence of the convolution of ultradistributions were studied in [71] in terms of the supports of ultradistributions involved (so-called compatibility conditions) and in terms of the weighted ultradistributional $\mathcal{D}'^{(M_p)}_{L^q}$ spaces. The results of the quoted papers are collected and improved in the book.

The book is organized as follows.

In this Preface we give historical comments below to the material presented in the book.

In Chapter 1 we define some notions connected with cones in $\mathbb{R}^n$ as well as the Cauchy and Poisson kernels corresponding to tube domains. We present there results which will be used later in proving boundary value representations.

Chapter 2 contains the definitions and main properties of the spaces of ultradifferentiable test functions of Beurling and Roumieu type as well as of the corresponding spaces of ultradistributions. We are mainly interested in the spaces $\mathcal{D}(*, L^s)$ and $\mathcal{S}^*$ and their strong duals. After presenting basic properties of the sequences (M_p) and ultradifferential operators generating the respective ultradistribution spaces, we prove structural theorems for these spaces. We also give the definitions of the Fourier and Laplace transforms.

Chapter 3 is devoted to characterizations of bounded sets in the spaces $\mathcal{D}'(*, L^t)$ of L^t ultradistributions of Beurling and Roumieu type for $t \in [1, \infty]$ and in the spaces $\mathcal{S}'^*$ of tempered ultradistributions of Beurling and Roumieu type. The characterizations are given in terms of representations of elements of these sets in the form of infinite series of derivatives of certain functions of the class L^t (of the class L^2) whose norms satisfy the respective estimates as well as in terms of images of ultradifferentiable operators of bounded sets in the respective spaces of functions.

In Chapter 4, the Cauchy and Poisson kernels are studied as elements of the ultradifferentiable spaces $\mathcal{D}(*, L^r)$ (Section 4.1). The Cauchy and Poisson integrals of ultradistributions in $\mathcal{D}'(*, L^s)$ are defined in Sections 4.2 and 4.3. For $s \geq 2$ the use of Cauchy integrals gives a complete boundary value characterization of elements in $\mathcal{D}'(*, L^s)$. Notice that the Poisson integral of an element of the space $\mathcal{D}'(*, L^s)$, $s > 1$, converges to this element in the corresponding general ultradistribution space.

In Chapter 5, we deal with the boundary values of analytic functions in appropriate tube domains. Section 5.1 concerns the Fourier transform and suitable generalizations of Hardy spaces within ultradistribution classes for $r \in (1, 2]$. In Section 5.2, we show that elements of such spaces have boundary values in $\mathcal{D}'((M_p), L^1)$; while appropriate L^s bounds for $s \geq 2$ lead to boundary values in $\mathcal{D}'((M_p), L^r)$ for $r \in (1, 2]$. The extension of the results of Section 5.2 to the case $r > 2$ is given in Section 5.3 for appropriate cones. By means of almost analytic extensions and Stokes' theorem, we give in Section 5.4 the complete boundary value characterization for the spaces $\mathcal{D}'((M_p), L^s)$ and $\mathcal{D}'(\{M_p\}, L^s)$ with $s > 1$. The results given in Section 5.4 for ultradistributions on the real line are true also in the multidimensional case. In Section 5.5 the cases $s = \infty$ and $s = 1$ are considered. Due to the method of Komatsu (see [82]) appropriate L^∞ and L^1 estimates are obtained for the corresponding boundary values in the respective ultradistribution spaces.

In Chapter 6, we develop the theory of convolution of ultradistributions and tempered ultradistributions. Various general definitions of the convolution in the spaces of ultradistributions and tempered ultradistributions of Beurling type are considered in an analogous way to the classical definitions of the convolution in the theory of distributions given by Schwartz in [132], Chevalley in [36], Vladimirov in [149], Dierolf and Voigt in [45], and Kamiński in [69]. As in the case of distributions, the respective definitions of the convolution of ultradistributions are equivalent both in $\mathcal{D}'^{(M_p)}$ and $\mathcal{S}'^{(M_p)}$, although the proofs of the equivalence require new methods. Also various sufficient conditions for the existence of convolution of two ultradistributions are given: in terms of their supports (appropriate compatibility conditions) and in terms of various subspaces of ultradistributions on which the convolution is defined as a bilinear mapping. In particular, the weighted ultradistributional $\mathcal{D}'^{(M_p)}_{L^q}$ spaces are studied.

In Chapter 7, different types of integral transforms in the spaces of tempered ultradistributions of Beurling and Roumieu type are defined and discussed. Various characterizations concerning the Fourier and Laplace transforms, Bargmann transform, the Wigner distribution and the Hilbert transform of tempered ultradistributions are given. Singular integral operators are studied in the spaces of tempered ultradistributions of Beurling and Roumieu type. Moreover, the Hermite expansions of elements of the spaces of test functions and their duals can be considered as a generalized integral transform.

Historically, the representation of distributions and other generalized functions as boundary values of analytic functions has its direct foundations in the two papers [88] and [89] by G. Köthe.

Motivated by suggestions of Köthe, H.-G. Tillmann obtained in [139] extensions and generalizations of the work of Köthe and did so with the analysis being in n dimensions. Namely, Tillmann gave in [139] a characterization of the analytic functions which have the distributions with compact support as boundary values and extended in [142] these results to vector valued distributions. In 1961, Tillmann published two additional classical papers, [140] and [141], in which the analytic functions with distributional boundary values in $\mathcal{D}'_{L^p}$ and $\mathcal{S}'$, respectively, were characterized. When stated in one dimension, the principal characterization results of [140] and [141] can be described in the following way.

Theorem 1. (see [140]) *Every distribution* $U \in \mathcal{D}'_{L^p}$, $1 < p < \infty$, *is the boundary value of an analytic function* f *of variable* $z = x+iy$ *with* $\operatorname{Im} z \neq 0$ *for which*

(a) $\quad |f(z)| \leq M \max\{|y|^{-(p-1)/p}, |y|^{-m+1/p}\}, \qquad y = \operatorname{Im} z \neq 0;$

(b) $\quad g_\varepsilon = f(\cdot+i\varepsilon)-f(\cdot-i\varepsilon) \in \mathcal{D}'_{L^p}$ *for* $\varepsilon > 0$ *and the set* $\{g_\varepsilon :\ \varepsilon > 0\}$ *is bounded in* $\mathcal{D}'_{L^p}$;

(c) $\quad g_\varepsilon \to U$ *as* $\varepsilon \to 0+$ *in the strong topology of* $\mathcal{D}'_{L^p}$.

Conversely, every analytic function f *of variable* z *with* $\operatorname{Im} z \neq 0$ *which satisfies conditions* (a) *and* (b) *has an element* $U \in \mathcal{D}'_{L^p}$ *as boundary value in the sense of* (c).

The analytic function constructed from $U \in \mathcal{D}'_{L^p}$ in the proof of the sufficiency of Theorem 1 is the "indicatrix" of U or the Cauchy integral of U. In [96], Z. Łuszczki and Z. Zieleźny obtained results similar to those of [140] at about the same time. G. Bengel has extended in [6] the results for $\mathcal{D}'_{L^p}$ to vector valued distributions.

Theorem 2. (see [141]) *Every distribution* $U \in \mathcal{S}'$ *is the boundary value of an analytic function* f *of variable* $z = x + iy$ *with* $\operatorname{Im} z \neq 0$ *for which*

(a) $\quad |f(z)| \leq M(1 + |z|^2)^m |y|^{-1/2-r}, \qquad y = \operatorname{Im} z \neq 0;$

(b) $\quad g_\varepsilon = f(\cdot + i\varepsilon) - f(\cdot - i\varepsilon) \in \mathcal{S}', \qquad \varepsilon > 0;$

(c) $\quad g_\varepsilon \to U$ *as* $\varepsilon \to 0+$ *in the strong topology of* $\mathcal{S}'$.

Conversely, every analytic function f *of variable* z *with* $\operatorname{Im} z \neq 0$ *which satisfies conditions* (a) *and* (b) *has as boundary value an element* $U \in \mathcal{S}'$ *in the sense of* (c).

Associates of Tillmann, such as R. Meise (see [101], [102]), H. J. Petzsche (see [110], [111]), and D. Vogt (see [152]), have extended Theorem 2 and the related results. Several authors have continued the investigation of representing distributions and generalized functions as boundary values of analytic functions and associated analysis, such as recovery of the analytic functions from integrals of the boundary values. We mention the books of E. J. Beltrami and M. R. Wohlers [4], H. J. Bremermann [12], and B. W. Ross [126]. V. S. Vladimirov has extended much of the boundary value analysis to functions analytic in tube domains in n dimensional complex space and has discussed applications of this analysis to mathematical physics; we refer in particular to the two important books [147] and [151].

A more recent survey of distributional boundary value analysis and applications has been given in the book of R. D. Carmichael and D. Mitrović [30]. Here the analytic representation of distributions in $\mathcal{E}'$ and $\mathcal{O}'_\alpha$ in one dimension is given. Distributional Plemelj relations and representations of half plane analytic and meromorphic functions are obtained. The distributional boundary value results are applied to yield applications to boundary value problems and singular convolution equations. Analytic functions in tube domains in n dimensional complex space, as considered by Vladimirov, are studied with results being used to obtain the analytic representation of $\mathcal{E}'$ distributions, both in the scalar and vector valued case, $\mathcal{O}'_\alpha$ distributions, and $\mathcal{D}'_{L^p}$ distributions in terms of functions analytic in tube domains. Important in this analysis is the construction and properties of the Cauchy and Poisson kernel functions corresponding to tubes. Motivated by analysis of J. Sebastiõ e Silva (see [133], [134]) and H.-G. Tillmann (see [141]), the Cauchy integral of $\mathcal{S}'$ distributions is constructed and analyzed in n dimensions. The Hardy H^p functions in tubes are characterized in terms of the form of the boundary value as a subspace of those analytic functions which have $\mathcal{S}'$ boundary values.

A principal motivation of the present book is to present research concerning the representation of ultradistributions as boundary values of analytic functions and related topics that will aid in this study as the books [4], [12], [126], [147], [151], and [30] have done for distributions. In particular, the book [30] can be considered to be a companion book to the present one with the analysis there being for distributions as opposed to ultradistributions here.