PART II

The Euclidean Structures of $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$

In Chaps. 1–3, we developed the process of construction of spaces $\mathbb{R}^1, \mathbb{R}^2$ and $\mathbb{R}^3$, respectively, and investigated their aspects in the affine and linear structures. Euclidean concepts, such as length, angle, area, and volume, were spotted there in quite a few places in order to test *informally* the readers' knowledge and abilities of using "linear algebra (in particular, matrix operations)" in the treatment of geometry-oriented problems.

Now, in Part II, we will *formally* introduce the essence of the Euclidean structures of plane $\mathbb{R}^2$ and space $\mathbb{R}^3$. The concept and the algebraic operational properties of *inner product* $\langle\ ,\ \rangle$, which combines together the classical Euclidean concepts of lengths and angles, will be our main tool in the process of construction of the Euclidean structures.

Introduction

As for one-dimensional vector space $\mathbb{R}^1$ or simply $\mathbb{R}$, designate the *inner product* of two vectors x and y as

$$\langle x, y\rangle = xy,$$

which is the usual product of real numbers x and y, while the *length* of a vector x is defined as

$$|x| = \langle x, x\rangle^{1/2},$$

which is the absolute value of the real number x. Note that $|x| = \sqrt{x^2}$ and $|x+y| \leq |x| + |y|$ with equality if and only if $xy \geq 0$ holds as a scalar.

The whole Euclidean structure of two- or higher-dimensional space $\mathbb{R}^n$ will be based on the Pythagorean theorem, as explained below.

Orthogonality of two lines is one of the most prominent basic concepts in the classical Euclidean geometry. The right triangle ΔABC with AB as

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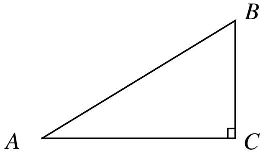

Fig. II.1

hypotenuse (see Fig. II.1) indicates, explicitly, the following most important facts:

(1) **Pythagorean theorem**

$$AB^2 = AC^2 + BC^2.$$

(2) The hypotenuse is *larger*, *in length*, *than* the two legs, i.e.

$$AB \geq AC \text{ and } AB \geq BC$$

with equality if and only if the triangle is degenerated into a line segment.

(3) **The orthogonal projection** Since legs AC and BC are perpendicular to each vector, leg AC may be viewed as the orthogonal projection of hypotenuse AB on the line generated by A and C. Similarly, leg BC is the orthogonal projection of AB on the line generated by B and C.

These three properties of a right triangle will lay the foundation of the whole Euclidean structure, and all other properties and results concerned can be deduced, directly or indirectly, from them.

Once again, it should be mentioned and remembered that we will always adopt rectangular coordinate systems in $\mathbb{R}^2$ and $\mathbb{R}^3$, unless otherwise stated. Therefore, we are able to start, from the very beginning, by using the Pythagorean theorem.

(1) If $\overrightharpoon{x} = (x_1, x_2) \in \mathbb{R}^2$, then the *length* of *vector* $\overrightharpoon{x}$ or the *distance* from $\overrightharpoon{0}$ to *point* $\overrightharpoon{x}$ is defined as

$$|\overrightharpoon{x}| = (x_1^2 + x_2^2)^{1/2}.$$

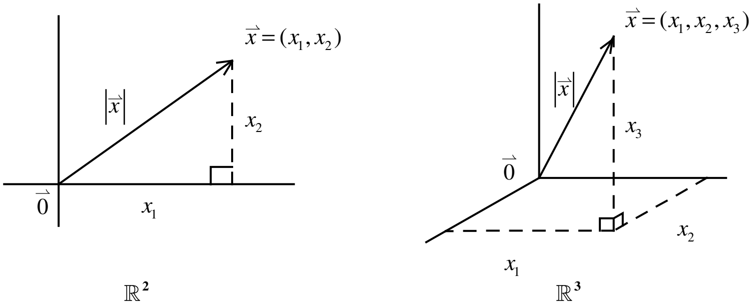

Fig. II.2

(2) If $\overrightarrow{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, then the *length* of *vector* $\overrightarrow{x}$ or the *distance* from $\overrightarrow{0}$ to *point* $\overrightarrow{x}$ is defined as

$$|\overrightarrow{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

See Fig. II.2.

How to Measure the Degree of Perpendicularity

Without using a protractor or other mechanical tools, how can one measure the angle between two intersecting lines in plane $\mathbb{R}^2$ or space $\mathbb{R}^3$? We are going to introduce a quantity and hence an operation between pairs of vectors, the so-called *natural inner product*, capable of measuring the degree of perpendicularity of two intersecting lines. Basic facts from the classical Euclidean geometry can be transformed into algebraic operational properties of the inner product so that they can be compatible with the linear structures, namely the scalar multiplication and addition of vectors, of the plane and the space.

On a gridded paper, draw the rectangular coordinate system and the graph of the straight line $3x_1 - 4x_2 = 0$ on which point (4, 3) is indicated (see Fig. II.3(a)). Then, find out point $(3, -4)$ which is formed by the coefficients 3 and -4 of the line equation, and a right triangular plate (or something like a geoliner) is placed with one of its legs along the straight line and its right angle vertex at the origin of the coordinate system (see Fig. II.3(b)). Now, it is quite obvious for everybody to watch that the other leg of the triangle passes through point $(3, -4)$, as it should be both intuitive and mathematics-oriented (see Fig. II.3(b)). Intuitively, one can try and investigate more straight lines, for example, $3x_1 + 4x_2 = 0$ with a pair

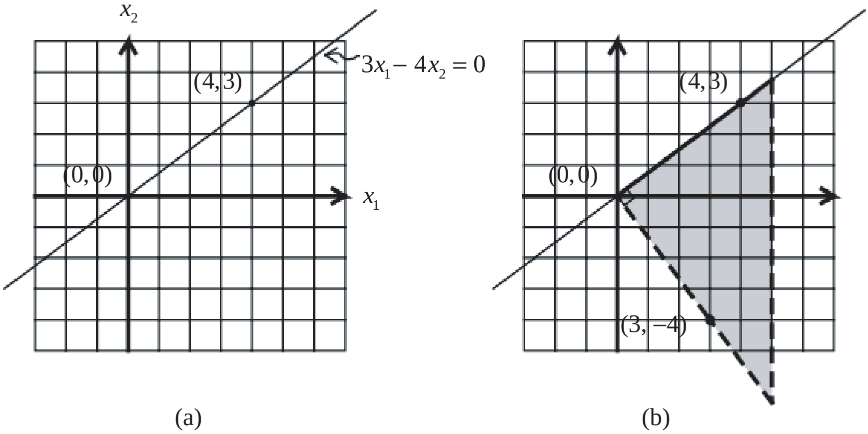

Fig. II.3

of points $(-4, 3)$ and (3, 4), $6x_1 - x_2 = 0$ with points (1, 6) and $(6, -1)$, etc. It is eventually hoped that one can observe that, in each case, *the sum of the componentwise multiplication of the indicated pair of points is equal to zero*, i.e.

$$4 \times 3 + 3 \times (-4) = 12 - 12 = 0,$$
$$(-4) \times 3 + 3 \times 4 = -12 + 12 = 0,$$
$$1 \times 6 + 6 \times (-1) = 6 - 6 = 0.$$

If one rotates the triangular plate around center (0, 0) and be able to assign coordinates (a_1, a_2) and (b_1, b_2), respectively, to any point on either of the two legs (see Fig. II.4(a)), then it is natural to believe that

$$a_1b_1 + a_2b_2 = 0 \tag{II.1}$$

is true, even without a rigorous mathematical reason or proof. Furthermore, if one rotates the triangular plate keeping the center at origin $(0, 0)$ but out of the plane and into the space, then it is reasonable to expect that, for any two points (a_1, a_2, a_3) and (b_1, b_2, b_3), each on one leg, the identity

$$a_1b_1 + a_2b_2 + a_3b_3 = 0 \tag{II.2}$$

must also be true (see Fig. II.4(b)).

What we have concluded in (II.1) and (II.2) are not accidental. They simply reflect, algebraically, the geometric fact that the lines passing through the origin and points (a_1, a_2) or (a_1, a_2, a_3) and (b_1, b_2) or (b_1, b_2, b_3)

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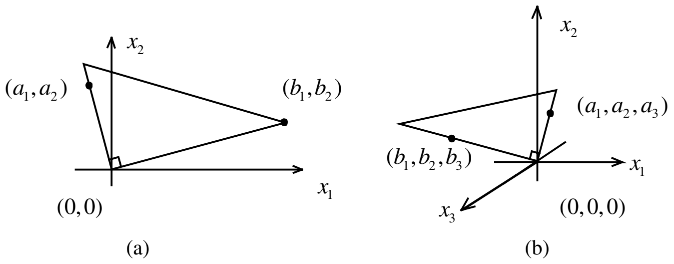

Fig. II.4

on each are perpendicular to each other. And the most important of all, it seems that quantity $a_1b_1 + a_2b_2$ or $a_1b_1 + a_2b_2 + a_3b_3$ can be used to measure the degree of perpendicularity of the two intersecting lines.

On the other hand, let $O = (0, 0)$ and $A = (a_1, a_2)$ be two points and OA the segment connecting O and A. With O as the center, rotate segment OA, either in the counterclockwise or in the clockwise direction, through an angle θ to a new position OB with $B = (b_1, b_2)$. See Fig. II.5.

This action of rotation through angle θ results in changing the algebraic quantity $a_1^2 + a_2^2 = a_1 \cdot a_1 + a_2 \cdot a_2$, the square of the length of OA, into two new quantities,

$$a_1b_1 + a_2b_2 \quad \text{and} \quad b_1 \cdot b_1 + b_2 \cdot b_2 = b_1^2 + b_2^2,$$

the latter being the square of the length of OB. Common sense or else might convince us to guess that $a_1b_1 + a_2b_2$ does have something to do with $a_1^2 + a_2^2$, $b_1^2 + b_2^2$ and θ. Surely, it is, as we will see in (II.4).

It is time now to proceed, rigorously and formally, to introduce the concept of inner product and develop its operational properties.

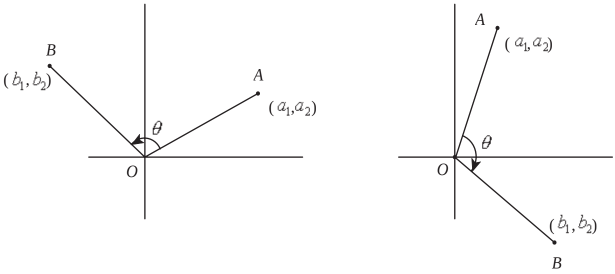

Fig. II.5

Fix three different points O, X, and Y in $\mathbb{R}^2$ or $\mathbb{R}^3$. Suppose, temporarily, that they are noncollinear so that vectors $\overrightarrow{OX}$ and $\overrightarrow{OY}$ are linearly independent.

In Fig. II.6, $OXAY$ denotes a parallelogram with AB perpendicular to OX or its extended line. Now

Are OX and OY perpendicular to each other?

$\Leftrightarrow$ (Pythagoras) Is $OA^2 = OX^2 + XA^2$ true or false?
$\Leftrightarrow$ Is $OA^2 - OX^2 - XA^2 = 0$ true or false?
$\Leftrightarrow$ (1) If $\vec{x} = \overrightarrow{OX} = (x_1, x_2)$ and $\vec{y} = \overrightarrow{OY} = (y_1, y_2)$ are in $\mathbb{R}^2$ then $\vec{x} + \vec{y} = \overrightarrow{OA} = (x_1 + y_1, x_2 + y_2)$. Is

$$\begin{aligned} &|\vec{x} + \vec{y}|^2 - |\vec{x}|^2 - |\vec{y}|^2 \\ &\quad = (x_1 + y_1)^2 + (x_2 + y_2)^2 - (x_1^2 + x_2^2) - (y_1^2 + y_2^2) \\ &\quad = 2(x_1y_1 + x_2y_2) \end{aligned}$$

equal to zero or not?

(2) If $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ are in $\mathbb{R}^3$, then $\vec{x} + \vec{y} = \overrightarrow{OA} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$. Is

$$|\vec{x} + \vec{y}|^2 - |\vec{x}|^2 - |\vec{y}|^2 = 2(x_1y_1 + x_2y_2 + x_3y_3)$$

equal to zero or not? (II.3)

Thus, the more the quantity $x_1y_1 + x_2y_2$ or $x_1y_1 + x_2y_2 + x_3y_3$ is closer to zero, the more the lines OX and OY are closer to be perpendicular to each other. And finally, $OX \perp$ (means perpendicular to) $OY \Leftrightarrow x_1y_1 + x_2y_2 = 0$ or $x_1y_1 + x_2y_2 + x_3y_3 = 0$.

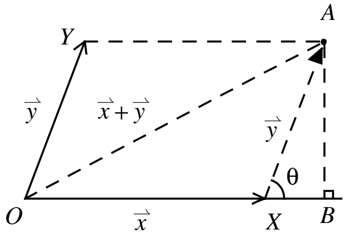

Fig. II.6

In triangle ΔOAB (see Fig. II.6), owing to fact that $AB \perp OB$, we have

$$\begin{aligned}
OA^2 &= OB^2 + AB^2 \\
&= (OX + XB)^2 + AB^2 = (OX + AX\cos\theta)^2 + (AX\sin\theta)^2 \\
&= OX^2 + AX^2 + 2OX \cdot AX \cdot \cos\theta \\
\Rightarrow OA^2 - OX^2 - AX^2 &= |\overrightarrow{x} + \overrightarrow{y}|^2 - |\overrightarrow{x}|^2 - |\overrightarrow{y}|^2 \\
&= 2|\overrightarrow{x}||\overrightarrow{y}|\cos\theta. \qquad \text{(II.4)}
\end{aligned}$$

This means that quantity $|\overrightarrow{x}||\overrightarrow{y}|\cos\theta$ can also be used to measure the perpendicularity of lines OX and OY. $OX \perp OY \Leftrightarrow \cos\theta = 0 \Leftrightarrow \theta = 90°$.

Summarize (II.3) and (II.4) as a

Quantity that measures the degree of perpendicularity

The degree of perpendicularity of two vectors $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^2$ (or $\mathbb{R}^3$) can be measured by the quantity, denoted by $\langle \overrightarrow{x}, \overrightarrow{y} \rangle$, as follows (see Fig. II.6):

$$\begin{aligned}
&\langle \overrightarrow{x}, \overrightarrow{y} \rangle \\
&\quad = \frac{1}{2}\left(|\overrightarrow{x} + \overrightarrow{y}|^2 - |\overrightarrow{x}|^2 - |\overrightarrow{y}|^2\right) \text{ (Pythagorean theorem)} \\
&\quad = \frac{1}{4}\left(|\overrightarrow{x} + \overrightarrow{y}|^2 - |\overrightarrow{x} - \overrightarrow{y}|^2\right) \text{ (the difference of two diagonals of a parellelogram)} \\
&\quad = |\overrightarrow{x}||\overrightarrow{y}|\cos\theta \text{ (signed orthogonal projection as explained in (II.8) below)} \\
&\quad = \begin{cases} x_1y_1 + x_2y_2, & \text{if } \overrightarrow{x} = (x_1, x_2) \text{ and } \overrightarrow{y} = (y_1, y_2) \text{ in } \mathbb{R}^2, \\ x_1y_1 + x_2y_2 + x_3y_3, & \text{if } \overrightarrow{x} = (x_1, x_2, x_3) \text{ and } \overrightarrow{y} = (y_1, y_2, y_3) \text{ in } \mathbb{R}^3. \end{cases}
\end{aligned}$$

Call $\langle \overrightarrow{x}, \overrightarrow{y} \rangle$ the *natural* or *standard inner product* of the vectors $\overrightarrow{x}$ and $\overrightarrow{y}$; see Remark 1 below. In particular,

1. $\overrightarrow{x}$ and $\overrightarrow{y}$ are *perpendicular* or *orthogonal* to each other, denoted by

$$\overrightarrow{x} \perp \overrightarrow{y} \Leftrightarrow \langle \overrightarrow{x}, \overrightarrow{y} \rangle = 0,$$

   and therefore, $\overrightarrow{0} \perp \overrightarrow{x}$ for any vector $\overrightarrow{x}$ in $\mathbb{R}^2$ (or $\mathbb{R}^3$).
2. The *length* of $\overrightarrow{x}$ is

$$|\overrightarrow{x}| = \langle \overrightarrow{x}, \overrightarrow{x} \rangle^{1/2}. \qquad \text{(II.5)}$$

The Natural or Standard Inner Product $\langle\,,\rangle$

In short, what we need in the sequel is the following definition of the *inner product* $\langle \overrightarrow{x}, \overrightarrow{y} \rangle$ of two vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ with angle θ between them:

$$\begin{aligned}\langle \overrightarrow{x}, \overrightarrow{y} \rangle &= |\overrightarrow{x}||\overrightarrow{y}|\cos\theta \\ &= \begin{cases} x_1y_1 + x_2y_2, & \overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^2, \\ x_1y_1 + x_2y_2 + x_3y_3, & \overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^3. \end{cases}\end{aligned} \tag{II.6}$$

See Fig. II.7 and the geometric interpretation of $|\overrightarrow{x}||\overrightarrow{y}|\cos\theta$ that follows. Temporarily suppose that $\overrightarrow{x}$ and $\overrightarrow{y}$ are nonzero vectors. Then, note that

(a) $\dfrac{\overrightarrow{x}}{|\overrightarrow{x}|}$ is a *unit vector* along $\overrightarrow{x}$.

(b) The signed length

$$\begin{aligned} OP &= \left\langle \frac{\overrightarrow{x}}{|\overrightarrow{x}|}, \overrightarrow{y} \right\rangle = \frac{\langle \overrightarrow{x}, \overrightarrow{y} \rangle}{|\overrightarrow{x}|} \\ &= |\overrightarrow{y}|\cos\theta \begin{cases} \geq 0, & \text{if } 0 \leq \theta \leq 90^\circ, \\ < 0, & \text{if } 90^\circ < \theta \leq 180^\circ \end{cases}\end{aligned}$$

is called the *signed orthogonal projection* (as a quantity) of vector $\overrightarrow{y}$ along $\overrightarrow{x}$ or on $\overrightarrow{x}$.

(c) By the way, vectors

$$\begin{aligned} \overrightarrow{OP} &= \left\langle \frac{\overrightarrow{x}}{|\overrightarrow{x}|}, \overrightarrow{y} \right\rangle \frac{\overrightarrow{x}}{|\overrightarrow{x}|} = \frac{\langle \overrightarrow{x}, \overrightarrow{y} \rangle}{|\overrightarrow{x}|^2}\overrightarrow{x} \quad \text{and} \\ \overrightarrow{PQ} &= \overrightarrow{y} - \frac{\langle \overrightarrow{x}, \overrightarrow{y} \rangle}{|\overrightarrow{x}|^2}\overrightarrow{x} \end{aligned}$$

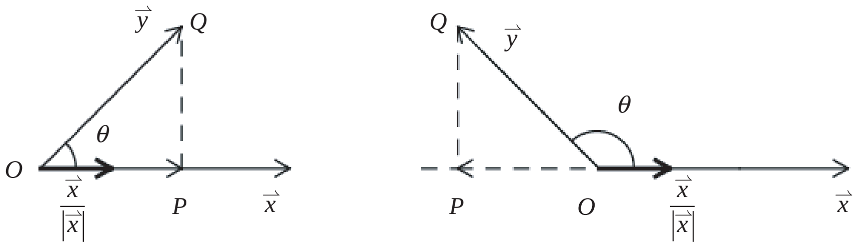

Fig. II.7

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are called, respectively, the *orthogonal projection* (as a vector) of $\overrightarrow{y}$ along $\overrightarrow{x}$ or on $\overrightarrow{x}$ and the *orthogonal projection* of $\overrightarrow{y}$ on the direction perpendicular to $\overrightarrow{x}$. (II.7)

Hence, geometrically,

$$\begin{aligned}\langle \overrightarrow{x}, \overrightarrow{y} \rangle &= |\overrightarrow{x}||\overrightarrow{y}|\cos\theta \\ &= \text{the } \textit{length} \text{ of } \overrightarrow{x} \ \textit{times} \text{ the } \textit{signed} \text{ orthogonal projection} \\ &\quad \text{of } \overrightarrow{y} \text{ along } \overrightarrow{x}. \end{aligned} \tag{II.8}$$

In case that at least one of $\overrightarrow{x}$ and $\overrightarrow{y}$ is equal to the zero vector, then θ can be chosen to be any value in formula $|\overrightarrow{x}||\overrightarrow{y}|\cos\theta$ which is equal to 0.

Remark 1

The prerequisite in the definition of $\langle \overrightarrow{x}, \overrightarrow{y} \rangle$ as $|\overrightarrow{x}||\overrightarrow{y}|\cos\theta$ is the concept of length and angle which are the essence of the classical Euclidean geometry. $\langle \overrightarrow{x}, \overrightarrow{y} \rangle$ links lengths $|\overrightarrow{x}|$ and $|\overrightarrow{y}|$ and angle θ together, and implicitly defines the concept of perpendicularity, and which is consistent with human being's intuition. Hence, we call $\langle \overrightarrow{x}, \overrightarrow{y} \rangle$ the *natural* or *standard inner product* of $\mathbb{R}^2$ or $\mathbb{R}^3$, in contrast to non-natural inner products to be presented in Secs. 4. 4 and 5. 4.

In the purely algebraic but equivalent definition of $\langle \overrightarrow{x}, \overrightarrow{y} \rangle$ as $x_1y_1+x_2y_2$ or $x_1y_1+x_2y_2+x_3y_3$, only the coordinates of $\overrightarrow{x}$ and $\overrightarrow{y}$ are needed. Hence, in a "space" on which the concepts of length and angle have not been defined beforehand, such as affine spaces or higher-dimensional vector space $\mathbb{R}^n$ ($n \geq 4$), an inner product can be defined formally via this algebraic method. For example, associated with any fixed coordinatized space $\mathbb{R}^2_{\Sigma(O;A_1,A_2)}$ of the affine plane $\mathbb{R}^2$, an inner product can be defined. As a consequence, the resulting concept of perpendicularity is, in general, no more so natural as we are used to. See Secs. 4.4 and 5.4.

The usefulness of the inner product depends solely on what good operational properties it might have.

Of course, it is quite easy to find basic operational properties as listed in (II.10) for the inner product via its algebraic definition. But, we prefer the geometric definition (II.8) to do so.

Once $\langle \overrightarrow{x}, \overrightarrow{y} \rangle$, $\overrightarrow{x}$, and $\overrightarrow{y}$ are known, (II.8) can be rewritten as

$$\cos\theta = \frac{\langle \overrightarrow{x}, \overrightarrow{y} \rangle}{|\overrightarrow{x}||\overrightarrow{y}|}, \quad \overrightarrow{x} \neq \overrightarrow{0}, \ \overrightarrow{y} \neq \overrightarrow{0}, \tag{II.9}$$

and the right-hand side can be used to define the cosine of *angle* θ between $\overrightarrow{x}$ and $\overrightarrow{y}$. Since $\cos(360° - \theta) = \cos\theta$ and $\cos(-\theta) = \cos\theta$, only the case pertaining to $0 \le \theta \le 180°$ will be discussed.

Owing to the fact that $|\overrightarrow{x}||\overrightarrow{y}| = |\overrightarrow{y}||\overrightarrow{x}|$, it is obvious that

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle = \langle \overrightarrow{y}, \overrightarrow{x} \rangle.$$

Geometrically, this means that two right triangles are similar if and only if two corresponding angles (no right angles) are equal. See Fig. II.8. Let $\overrightarrow{x} = \overrightarrow{OA}$ and $\overrightarrow{y} = \overrightarrow{OB}$. Draw the orthogonal projection of OB on OA and that of OA on OB. Because $\angle AOD = \angle BOC$, triangles ΔAOD and ΔBOC are similar. Thus, $OA{:}OB = OD{:}OC$; hence, $OA \cdot OC = OB \cdot OD$. But, the latter is equivalent to $|\overrightarrow{x}||\overrightarrow{y}|\cos\theta = |\overrightarrow{y}||\overrightarrow{x}|\cos\theta$ which is just the identity $\langle \overrightarrow{x}, \overrightarrow{y} \rangle = \langle \overrightarrow{y}, \overrightarrow{x} \rangle$.

When measuring an angle, the most fundamental fact is that its quantity is independent of the lengths of its both sides. See Fig. II.9. Let $\overrightarrow{x} = \overrightarrow{OA}$, $\overrightarrow{OC} = \alpha\overrightarrow{x}$, and $\overrightarrow{y} = \overrightarrow{OB}$. It is well known that $|\overrightarrow{OC}| = |\alpha\overrightarrow{x}| = |\alpha||\overrightarrow{x}|$ which is $\alpha\overrightarrow{x}$ if $\alpha > 0$ and $-\alpha|\overrightarrow{x}|$ if $\alpha < 0$. Then, in case $\alpha > 0$, one has

$$\cos\theta = \frac{\langle \overrightarrow{x}, \overrightarrow{y} \rangle}{|\overrightarrow{x}||\overrightarrow{y}|} = \frac{\langle \alpha\overrightarrow{x}, \overrightarrow{y} \rangle}{|\alpha\overrightarrow{x}||\overrightarrow{y}|} = \frac{\langle \alpha\overrightarrow{x}, \overrightarrow{y} \rangle}{\alpha|\overrightarrow{x}||\overrightarrow{y}|};$$

while in case $\alpha < 0$,

$$\cos(\pi - \theta) = \frac{\langle \alpha\overrightarrow{x}, \overrightarrow{y} \rangle}{|\alpha\overrightarrow{x}||\overrightarrow{y}|} = \frac{\langle \alpha\overrightarrow{x}, \overrightarrow{y} \rangle}{-\alpha|\overrightarrow{x}||\overrightarrow{y}|} = -\cos\theta = -\frac{\langle \overrightarrow{x}, \overrightarrow{y} \rangle}{|\overrightarrow{x}||\overrightarrow{y}|}.$$

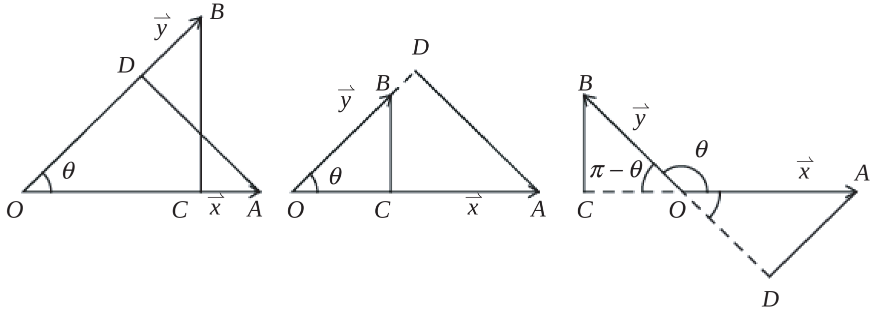


Fig. II.8

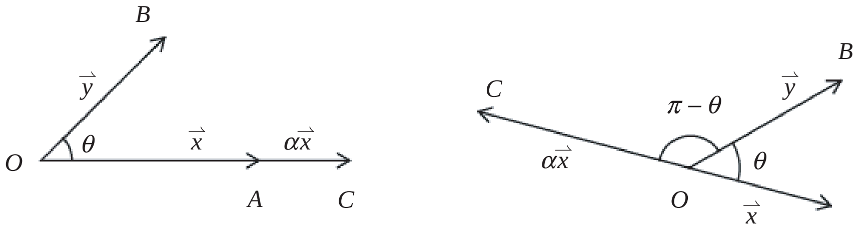


Fig. II.9

When simplifying, both identities reduce to the same identity

$$\langle\alpha\overrightarrow{x},\overrightarrow{y}\rangle=\alpha\langle\overrightarrow{x},\overrightarrow{y}\rangle,\quad \alpha\in\mathbb{R}.$$

Intuitively, it is obvious that the sum of the signed orthogonal projections (see (b) in (II.7)) of two vectors along the same line is equal to the signed orthogonal projection of their sum vector along that line. See Fig. II.10. Let $\overrightarrow{x}=\overrightarrow{OA}$, $\overrightarrow{y}=\overrightarrow{OB}$, and $\overrightarrow{x}+\overrightarrow{y}=\overrightarrow{OC}$, and take a third vector $\overrightarrow{z}=\overrightarrow{OD}$. Draw the respective orthogonal projection of OA, AC (parallel to OB), and OC along the line OD, denoted by OE, EF, and OF, respectively.

In case (a), $OF=OE+EF$, and then

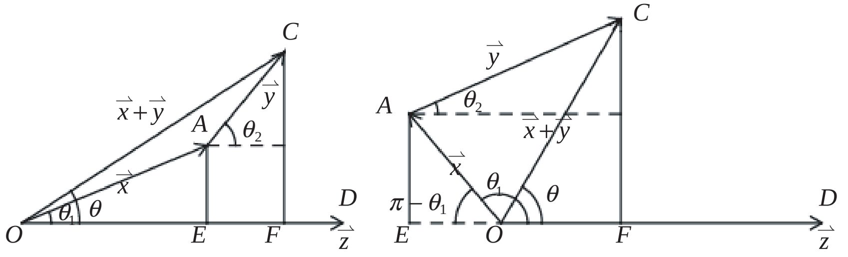
$$|\overrightarrow{x}+\overrightarrow{y}|\cos\theta=|\overrightarrow{x}|\cos\theta_1+|\overrightarrow{y}|\cos\theta_2;$$

while in case (b), $OF=EF-OE$, and then

$$|\overrightarrow{x}+\overrightarrow{y}|\cos\theta=|\overrightarrow{y}|\cos\theta_2-|\overrightarrow{x}|\cos(\pi-\theta_1)=|\overrightarrow{x}|\cos\theta_1+|\overrightarrow{y}|\cos\theta_2.$$

After multiplying both sides by $|\overrightarrow{z}|$, the above two identities reduce to the single one

$$\langle\overrightarrow{x}+\overrightarrow{y},\overrightarrow{z}\rangle=\langle\overrightarrow{x},\overrightarrow{y}\rangle+\langle\overrightarrow{y},\overrightarrow{z}\rangle.$$



(a) (b)

Fig. II.10

Now, we summarize the above three identities as

The operational properties of inner product

The natural inner product $\langle , \rangle$ in $\mathbb{R}^2$ (or $\mathbb{R}^3$) has the following properties.

(1) (Positive definite) For any $\overrightarrow{x} \in \mathbb{R}^2$ (or $\mathbb{R}^3$),

$$\langle \overrightarrow{x}, \overrightarrow{x} \rangle \geq 0, \text{ and}$$
$$\langle \overrightarrow{x}, \overrightarrow{x} \rangle = 0 \Leftrightarrow \overrightarrow{x} = \overrightarrow{0}.$$

(2) (Symmetry) For any $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^2$ (or $\mathbb{R}^3$),

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle = \langle \overrightarrow{y}, \overrightarrow{x} \rangle.$$

(3) (Bilinearity) For any $\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z} \in \mathbb{R}^2$ (or $\mathbb{R}^3$) and $\alpha \in \mathbb{R}$,

$$\langle \alpha \overrightarrow{x}, \overrightarrow{y} \rangle = \alpha \langle \overrightarrow{x}, \overrightarrow{y} \rangle \quad \text{and} \quad \langle \overrightarrow{x} + \overrightarrow{y}, \overrightarrow{z} \rangle = \langle \overrightarrow{x}, \overrightarrow{z} \rangle + \langle \overrightarrow{y}, \overrightarrow{z} \rangle.$$

The vector space $\mathbb{R}^2$ (or $\mathbb{R}^3$) with this specific inner product $\langle , \rangle$ is called the *standard two-* (*or three-*) *dimensional inner product space*, or simply *the Euclidean plane* (or *space*). (II.10)

The bilinearity in (3) will be explained further as follows. For a fixed $\overrightarrow{y} \in \mathbb{R}^2$ (or $\mathbb{R}^3$), define the mapping

$$\langle \cdot, \overrightarrow{y} \rangle : \mathbb{R}^2 \text{ (or } \mathbb{R}^3) \to \mathbb{R} \text{ by } \langle \cdot, \overrightarrow{y} \rangle(\overrightarrow{x}) = \langle \overrightarrow{x}, \overrightarrow{y} \rangle. \tag{II.11}$$

Then, (3) indicates that $\langle \cdot, \overrightarrow{y} \rangle$ is compatible with the linear structures (i.e. $\alpha \overrightarrow{x}$ and $\overrightarrow{x}_1 + \overrightarrow{x}_2$) of the vector space $\mathbb{R}^2$ (or $\mathbb{R}^3$) and is called a *linear functional* from $\mathbb{R}^2$ (or $\mathbb{R}^3$) to $\mathbb{R}$. Similarly, the mapping

$$\langle \overrightarrow{x}, \cdot \rangle : \mathbb{R}^2 \text{ (or } \mathbb{R}^3) \to \mathbb{R} \text{ defined by } \langle \overrightarrow{x}, \cdot \rangle(\overrightarrow{y}) = \langle \overrightarrow{x}, \overrightarrow{y} \rangle \tag{II.11$'$}$$

is also called a *linear functional* from $\mathbb{R}^2$ (or $\mathbb{R}^3$) to $\mathbb{R}$. Combined together, we say that the inner product $\langle , \rangle$ has the *bilinear* property, usually stated as

$$\begin{aligned}&\langle \alpha_1 \overrightarrow{x}_1 + \alpha_2 \overrightarrow{x}_2, \beta_1 \overrightarrow{y}_1 + \beta_2 \overrightarrow{y}_2 \rangle \\ &\quad = \alpha_1\beta_1 \langle \overrightarrow{x}_1, \overrightarrow{y}_1 \rangle + \alpha_1\beta_2 \langle \overrightarrow{x}_1, \overrightarrow{y}_2 \rangle + \alpha_2\beta_1 \langle \overrightarrow{x}_2, \overrightarrow{y}_1 \rangle + \alpha_2\beta_2 \langle \overrightarrow{x}_2, \overrightarrow{y}_2 \rangle.\end{aligned} \tag{II.12}$$

Remark 2

As shown above in (II.5) and (II.10), inner product combines the concepts of length and angle together into a single form, and possesses nice operational properties. Hence, as might be expected, it can be employed as a concept or a method to handle classical Euclidean geometric problems, and conversely, results obtained purely by the method of inner product can always be interpreted in the realm of Euclidean geometry (see Secs. 4.1, 4.2, 5.1 and 5.2).

In short, the concepts of length and angle and the resulting Euclidean geometry can be algebraicallized, even simpler, into the essence of the concept of inner product and its operational properties: positive definite, symmetric, and bilinear. Hence, $\mathbb{R}^2$ (or $\mathbb{R}^3$), endowed with the natural inner product $\langle\, ,\, \rangle$, is also called the *Euclidean plane* (or *space*).

Some Identities and Inequalities

Finally, we are going to establish some basic identities and inequalities concerning inner products.

To start with, we try to give inner product another geometric interpretation which seems to be not so natural but is crucial in the geometric definition of determinants to be presented in Sec. 4.3. Given two nonzero vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ with angle θ between them, and then construct a third vector $\overrightarrow{x}^{\perp}$ *perpendicular to* $\overrightarrow{x}$ *in the anticlockwise direction but of the same length as* $\overrightarrow{x}$ (refer to (4.1.1)). See Fig. II.11.

In case $0 \le \theta \le \frac{\pi}{2}$,

$$\begin{aligned}\langle \overrightarrow{x}, \overrightarrow{y} \rangle &= |\overrightarrow{x}||\overrightarrow{y}|\cos\theta \\ &= |\overrightarrow{x}||\overrightarrow{y}|\sin\left(\frac{\pi}{2}-\theta\right) = |\overrightarrow{x}^{\perp}||\overrightarrow{y}|\sin\left(\frac{\pi}{2}-\theta\right) \\ &= \text{the area of the parallelogram } \square\, \overrightarrow{y}\,\overrightarrow{x}^{\perp} \text{ generated} \\ &\quad\ \text{by side vectors } \overrightarrow{y} \text{ and } \overrightarrow{x}^{\perp}. \end{aligned} \tag{II.13}$$

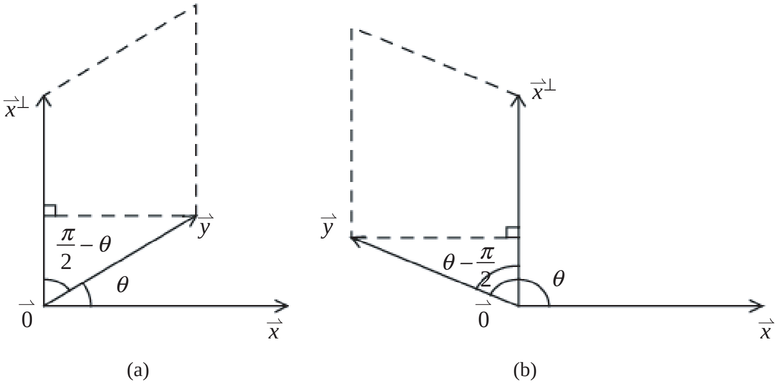


Fig. II.11

In case $\frac{\pi}{2} \le \theta \le \pi$,

$$\begin{aligned}
\langle \overrightarrow{x}, \overrightarrow{y} \rangle &= |\overrightarrow{x}||\overrightarrow{y}|\cos\theta \\
&= -|\overrightarrow{x}||\overrightarrow{y}|\sin\left(\theta - \frac{\pi}{2}\right) = -|\overrightarrow{x}^{\perp}||\overrightarrow{y}|\sin\left(\theta - \frac{\pi}{2}\right) \\
&= \text{the negative of the area of the parallelogram} \\
&\qquad \text{▱}\,\overrightarrow{y}\,\overrightarrow{x}^{\perp} \text{ generated by side vectors } \overrightarrow{y} \text{ and } \overrightarrow{x}^{\perp}. \qquad \text{(II.13)}'
\end{aligned}$$

By using (II.13) and (II.13)′, one can give a geometric proof of the so-called *Cauchy–Schwarz inequality* as follows. Even by intuition, it is well known that the area of a parallelogram is not larger than the product of lengths of two consecutive sides, with equality if and only if it is a rectangle. Therefore, in case $0 \le \theta \le \frac{\pi}{2}$,

$$\begin{aligned}
\langle \overrightarrow{x}, \overrightarrow{y} \rangle &\le |\overrightarrow{x}^{\perp}||\overrightarrow{y}| = |\overrightarrow{x}||\overrightarrow{y}|, \quad \text{and} \\
&= |\overrightarrow{x}^{\perp}||\overrightarrow{y}| = |\overrightarrow{x}||\overrightarrow{y}| \Leftrightarrow \frac{\pi}{2} - \theta = \frac{\pi}{2}, \quad \text{or } \theta = 0 \\
&\Leftrightarrow \overrightarrow{x} \text{ and } \overrightarrow{y} \text{ are collinear in the same direction,} \qquad \text{(II.14)}
\end{aligned}$$

while, in case $\frac{\pi}{2} \le \theta \le \pi$;

$$\begin{aligned}
-\langle \overrightarrow{x}, \overrightarrow{y} \rangle &\le |\overrightarrow{x}^{\perp}||\overrightarrow{y}| = |\overrightarrow{x}||\overrightarrow{y}|, \quad \text{and} \\
&= |\overrightarrow{x}^{\perp}||\overrightarrow{y}| = |\overrightarrow{x}||\overrightarrow{y}| \Leftrightarrow \theta - \tfrac{\pi}{2} = \tfrac{\pi}{2}, \quad \text{or } \theta = \pi \\
&\Leftrightarrow \overrightarrow{x} \text{ and } \overrightarrow{y} \text{ are collinear but in opposite directions.} \qquad \text{(II.14)}'
\end{aligned}$$

See Figs. II.12(a) and II.12(b), respectively.

Observing that $|\cos\theta| \le 1$, of course, it is much easier to prove this inequality by using the very definition of inner product. Thus,

$$|\langle \overrightarrow{x}, \overrightarrow{y} \rangle| = |\cos\theta||\overrightarrow{x}||\overrightarrow{y}| \le |\overrightarrow{x}||\overrightarrow{y}|$$

with $\langle \overrightarrow{x}, \overrightarrow{y} \rangle = |\overrightarrow{x}||\overrightarrow{y}| \Leftrightarrow \cos\theta = 1$ or $\theta = 0$ and $\langle \overrightarrow{x}, \overrightarrow{y} \rangle = -|\overrightarrow{x}||\overrightarrow{y}| \Leftrightarrow \cos\theta = -1$ or $\theta = \pi$.

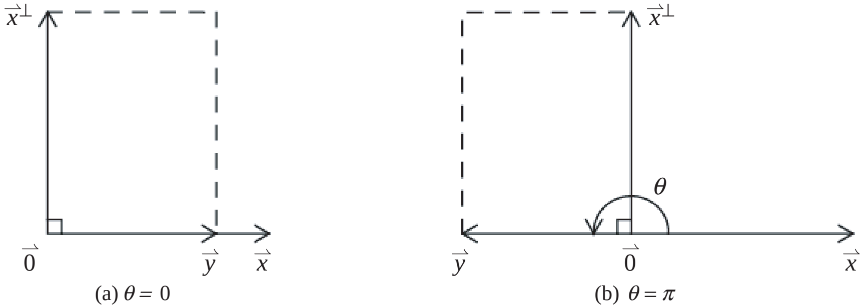


Fig. II.12

In the long run, especially in abstract inner product spaces, it is worthwhile to prove the inequality by using identity (II.12). For scalar t,

$$\begin{aligned}|\overrightarrow{x}+t\overrightarrow{y}|^2 &= \langle\overrightarrow{x}+t\overrightarrow{y},\overrightarrow{x}+t\overrightarrow{y}\rangle\\ &= |\overrightarrow{x}|^2+2t\langle\overrightarrow{x},\overrightarrow{y}\rangle+t^2|\overrightarrow{y}|^2\\ &= |\overrightarrow{y}|^2\left(t+\frac{\langle\overrightarrow{x},\overrightarrow{y}\rangle}{|\overrightarrow{y}|^2}\right)^2+\frac{|\overrightarrow{x}|^2|\overrightarrow{y}|^2-\langle\overrightarrow{x},\overrightarrow{y}\rangle^2}{|\overrightarrow{y}|^2} \quad (\text{if } \overrightarrow{y}\neq\overrightarrow{0})\\ &\geq 0 \text{ for all } t\in\mathbb{R} \qquad\qquad (\text{II.15})\end{aligned}$$

$\Rightarrow$ the discriminant $\langle\overrightarrow{x},\overrightarrow{y}\rangle^2 \leq |\overrightarrow{x}|^2|\overrightarrow{y}|^2$ holds, i.e.

$$\begin{aligned}\langle\overrightarrow{x},\overrightarrow{y}\rangle &\leq |\overrightarrow{x}||\overrightarrow{y}|, \quad \text{and}\\ &= \Leftrightarrow \overrightarrow{x}+t\overrightarrow{y}=\overrightarrow{0}.\end{aligned}$$

This proves the inequality. By the way, take $t = 1$ or -1 in (II.15); some identities would result (see (II.16)).

Summarize as

Some identities and inequalities about inner product
Let $\overrightarrow{x},\overrightarrow{y}\in\mathbb{R}^2$ (or $\mathbb{R}^3$).

(1) $|\overrightarrow{x}\pm\overrightarrow{y}|^2 = |\overrightarrow{x}|^2\pm 2\langle\overrightarrow{x},\overrightarrow{y}\rangle+|\overrightarrow{y}|^2$ and

$|\overrightarrow{x}+\overrightarrow{y}|^2+|\overrightarrow{x}-\overrightarrow{y}|^2 = 2(|\overrightarrow{x}|^2+|\overrightarrow{y}|^2)$,

which means that the sum of the squares of two diagonals of a parallelogram is equal to the sum of the squares of its four sides (see Fig. II.13).

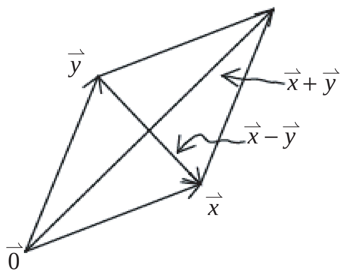


Fig. II.13

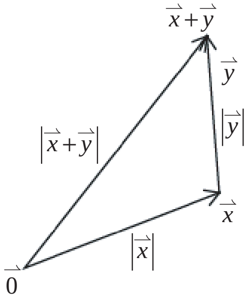


Fig. II.14

(2) *Cauchy–Schwarz inequality:*

$$|\langle \vec{x}, \vec{y} \rangle| \le |\vec{x}||\vec{y}| \quad \text{or} \ -|\vec{x}||\vec{y}| \le \langle \vec{x}, \vec{y} \rangle \le |\vec{x}||\vec{y}|$$

with

$$\langle \vec{x}, \vec{y} \rangle = |\vec{x}||\vec{y}| \Leftrightarrow \vec{y} = \lambda \vec{x} \quad \text{or} \ \vec{x} = \lambda \vec{y}, \lambda \ge 0 \text{ and}$$
$$\langle \vec{x}, \vec{y} \rangle = -|\vec{x}||\vec{y}| \Leftrightarrow \vec{y} = \lambda \vec{x} \quad \text{or} \ \vec{x} = \lambda \vec{y}, \lambda \le 0.$$

(3) *Lengths* $|\vec{x}|$ of vectors $\vec{x}$ have the following properties:

1. $|\vec{x}| \ge 0$, and $= 0 \Leftrightarrow \vec{x} = \vec{0}$.
2. $|\vec{x} - \vec{y}| = |\vec{y} - \vec{x}|$.
3. $|\alpha \vec{x}| = |\alpha||\vec{x}|, \alpha \in \mathbb{R}$.
4. *Triangle inequality* (see Fig. II.14):

$$||\vec{x}| - |\vec{y}|| \le |\vec{x} + \vec{y}| \le |\vec{x}| + |\vec{y}|$$

with

$$\begin{aligned}|\overrightarrow{x}+\overrightarrow{y}| = |\overrightarrow{x}|+|\overrightarrow{y}| &\Leftrightarrow \overrightarrow{x} = \overrightarrow{0} \text{ or } \\ &\quad \overrightarrow{y} = \lambda\overrightarrow{x},\ \lambda \ge 0, \text{ and} \\ |\overrightarrow{x}+\overrightarrow{y}| = ||\overrightarrow{x}|-|\overrightarrow{y}|| &\Leftrightarrow \overrightarrow{x} = \overrightarrow{0} \quad \text{or} \quad \overrightarrow{y} = \lambda\overrightarrow{x}, \lambda \le 0.\end{aligned} \tag{II.16}$$

Exercises

⟨A⟩

1. For each following pair of vectors $\overrightarrow{x}$ and $\overrightarrow{y}$,
   (a) $\overrightarrow{x} = (2,3)$, $\overrightarrow{y} = (6,-8)$,
   (b) $\overrightarrow{x} = (1,-5,4)$, $\overrightarrow{y} = (3,3,3)$,
   (c) $\overrightarrow{x} = (-2,2,3)$, $\overrightarrow{y} = (1,7,4)$,
   (d) $\overrightarrow{x} = (2,4,-8)$, $\overrightarrow{y} = (5,3,7)$,
   (e) $\overrightarrow{x} = (-6,0,4)$, $\overrightarrow{y} = (3,1,6)$,
   (f) $\overrightarrow{x} = (3,0,4)$, $\overrightarrow{y} = (2,3,3)$,
   (g) $\overrightarrow{x} = (3,-2,6)$, $\overrightarrow{y} = (1,2,-7)$,

   do the following questions:

   (1) Compute $\langle\overrightarrow{x},\overrightarrow{y}\rangle$, $|\overrightarrow{x}|$, and $|\overrightarrow{y}|$, and decide angle θ between them. Is θ a right angle?
   (2) Compute the orthogonal projections (both as a scalar and as a vector) of $\overrightarrow{y}$ along $\overrightarrow{x}$?
   (3) Compute the orthogonal projections of $\overrightarrow{x}$ along $\overrightarrow{y}$ and compare with results in (2).

2. For each of the following sets of points $\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}$,
   (a) $\overrightarrow{x} = (3,-1,2), \overrightarrow{y} = (0,-4,2), \overrightarrow{z} = (-3,2,1)$,
   (b) $\overrightarrow{x} = (3,-2,1), \overrightarrow{y} = (7,6,9), \overrightarrow{z} = (9,1,-5)$,
   (c) $\overrightarrow{x} = (4,-1,4), \overrightarrow{y} = (0,7,-4), \overrightarrow{z} = (3,1,-2)$,
   (d) $\overrightarrow{x} = (1,3,2), \overrightarrow{y} = (4,0,2), \overrightarrow{z} = (4,3,-1)$,
   (e) $\overrightarrow{x} = (3,-1,1), \overrightarrow{y} = (4,7,5), \overrightarrow{z} = (7,-5,8)$,
   (f) $\overrightarrow{x} = (5,4,-1), \overrightarrow{y} = (3,6,7), \overrightarrow{z} = (4,2,6)$,
   (g) $\overrightarrow{x} = (4,-8,1), \overrightarrow{y} = (7,4,4), \overrightarrow{z} = (-1,11,11)$,

   do the following questions:

   (1) Show that $\overrightarrow{x}, \overrightarrow{y}$, and $\overrightarrow{z}$ are coplanar but not collinear points.

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(2) Determine the three interior angles of triangle $\Delta \overrightarrow{x}\,\overrightarrow{y}\,\overrightarrow{z}$ with $\overrightarrow{x}$, $\overrightarrow{y}$, and $\overrightarrow{z}$ as vertices. Is $\Delta \overrightarrow{x}\,\overrightarrow{y}\,\overrightarrow{z}$ an acute, right, or obtuse triangle?

(3) Find the three altitudes of $\Delta \overrightarrow{x}\,\overrightarrow{y}\,\overrightarrow{z}$.

(4) Find the area of $\Delta \overrightarrow{x}\,\overrightarrow{y}\,\overrightarrow{z}$.

3. In $\mathbb{R}^2$ (or $\mathbb{R}^3$), suppose the vector $\overrightarrow{x}_0$ is perpendicular to all vectors $\overrightarrow{x}$, i.e.

$$\langle \overrightarrow{x}_0, \overrightarrow{x} \rangle = 0, \quad \overrightarrow{x} \in \mathbb{R}^2(\text{or } \mathbb{R}^3).$$

Prove that $\overrightarrow{x}_0 = \overrightarrow{0}$.

4. Suppose vectors $\overrightarrow{x}$, $\overrightarrow{y}$, and $\overrightarrow{z}$ are in $\mathbb{R}^2$ (or $\mathbb{R}^3$) and satisfy

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle = \langle \overrightarrow{x}, \overrightarrow{z} \rangle.$$

Explain graphically what this means and find an example to show that $\overrightarrow{y} = \overrightarrow{z}$ is not necessarily true.

5. *Gram–Schmidt orthogonalization* in $\mathbb{R}^3$. Let $\{\overrightarrow{y}_1, \overrightarrow{y}_2, \overrightarrow{y}_3\}$ be linearly independent in $\mathbb{R}^3$. Define $\overrightarrow{x}_1 = \overrightarrow{y}_1$ and (see (II.7))

$$\overrightarrow{x}_2 = \overrightarrow{y}_2 - \frac{\langle \overrightarrow{y}_2, \overrightarrow{x}_1 \rangle}{|\overrightarrow{x}_1|^2}\overrightarrow{x}_1,$$

$$\overrightarrow{x}_3 = \overrightarrow{y}_3 - \frac{\langle \overrightarrow{y}_3, \overrightarrow{x}_1 \rangle}{|\overrightarrow{x}_1|^2}\overrightarrow{x}_1 - \frac{\langle \overrightarrow{y}_3, \overrightarrow{x}_2 \rangle}{|\overrightarrow{x}_2|^2}\overrightarrow{x}_2.$$

Show that $\overrightarrow{x}_1$, $\overrightarrow{x}_2$, and $\overrightarrow{x}_3$ are orthogonal to each other. Geometric meaning?

⟨B⟩

1. Let

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}.$$

Define a mapping $\langle\ ,\ \rangle_A : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle_A = \langle \overrightarrow{x}, \overrightarrow{y} A \rangle = x_1y_1 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_1 + \frac{1}{3}x_2y_2,$$

where $\overrightarrow{x} = (x_1, x_2)$ and $\overrightarrow{y} = (y_1, y_2)$ are vectors in $\mathbb{R}^2$.

(a) Show that $\langle\,,\,\rangle_A$ is an inner product on $\mathbb{R}^2$ so that $(\mathbb{R}^2, \langle\,,\,\rangle_A)$ is an inner product space (refer to Sec. B.9).

(b) Is (1, 0) a unit vector, i.e. $\langle(1,0),(1,0)\rangle_A = 1$ in $\langle\,,\,\rangle_A$? How about $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$? If not, try to normalize it so that it becomes a unit vector.

(c) Determine the vectors that are perpendicular to (1, 0). Which one is $(1,0)^\perp$ (see Fig. II.11) among these vectors?

(d) Describe the set of unit vectors, i.e.

$$\{\overrightarrow{x} \in \mathbb{R}^2 | \langle\overrightarrow{x}, \overrightarrow{x}\rangle_A = 1\}.$$

(e) How does the *unit circle* $S = \{\overrightarrow{x} \in \mathbb{R}^2 | \langle\overrightarrow{x}, \overrightarrow{x}\rangle = 1\}$ in $\mathcal{N}$ look like in $\langle\,,\,\rangle_A$?

(f) State and verify both the Cauchy–Schwarz inequality and the triangle inequality.

(g) Try to find a basis $\mathcal{B}$ for $\mathbb{R}^2$ so that

$$\langle\overrightarrow{x}, \overrightarrow{y}\rangle_A = \langle[\overrightarrow{x}]_{\mathcal{B}}, [\overrightarrow{y}]_{\mathcal{B}}\rangle,$$

where $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^2$ and $[\overrightarrow{x}]_{\mathcal{B}}$ is the coordinate vector of $\overrightarrow{x}$ with respect to $\mathcal{B}$.

Note One may use the LDL^*-decomposition of A (see Sec. 2.7.5).

2. $\mathcal{B} = \{(1,2), (-2,3)\}$ is a basis for $\mathbb{R}^2$. Define $\langle\,,\,\rangle_{\mathcal{B}}$ on $\mathbb{R}^2$ as $\langle\overrightarrow{x}, \overrightarrow{y}\rangle_{\mathcal{B}} = \langle[\overrightarrow{x}]_{\mathcal{B}}, [\overrightarrow{y}]_{\mathcal{B}}\rangle$.

(a) Show that $(\mathbb{R}^2, \langle\,,\,\rangle_{\mathcal{B}})$ is an inner product space.

(b) Are $(1,2)$ and $(-2,3)$ perpendicular to each other? Are they of unit length?

(c) Let $\overrightarrow{x}_0 = (1,0)$. Determine its *orthogonal complement*

$$\langle\langle\overrightarrow{x}_0\rangle\rangle^\perp_{\mathcal{B}} = \{\overrightarrow{x} \in \mathbb{R}^2 | \langle\overrightarrow{x}, \overrightarrow{x}_0\rangle_{\mathcal{B}} = 0\}.$$

(d) Describe the *unit circle*

$$S_{\mathcal{B}} = \{\overrightarrow{x} \in \mathbb{R}^2 | \langle\overrightarrow{x}, \overrightarrow{x}\rangle_{\mathcal{B}} = 1\}.$$

How does it look like in $(\mathbb{R}^2, \langle\,,\,\rangle)$?

(e) How does the *unit disk* $D = \{\overrightarrow{x} \in \mathbb{R}^2 | \langle\overrightarrow{x}, \overrightarrow{x}\rangle < 1\}$ look like in $(\mathbb{R}^2, \langle\,,\,\rangle_{\mathcal{B}})$?

(f) State and verify both the Cauchy–Schwarz inequality and the triangle inequality.

(g) Try to find a square matrix $A_{2\times 2}$ so that

$$\langle \overrightarrow{x}, \overrightarrow{y}\rangle_{\mathcal{B}} = \langle \overrightarrow{x}, \overrightarrow{y}\rangle_A,\ \overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^2,$$

where $\langle\ ,\ \rangle_A$ is defined as in Ex. 1. Is this A unique?

3. Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Define $\langle \overrightarrow{x}, \overrightarrow{y}\rangle_A = \langle \overrightarrow{x}, \overrightarrow{y} A\rangle$ for $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^3$. Do the same questions as in Ex. 1. Also, let

$$\overrightarrow{y}_1 = (1,-3,2),\quad \overrightarrow{y}_2 = (4,1,0),\ \text{ and }\ \overrightarrow{y}_3 = (0,2,-1).$$

Try to use Ex. ⟨A⟩5 to orthogonalize these three vectors in $\langle\ ,\ \rangle_A$.

4. Let $\mathcal{B} = \{\overrightarrow{x}_1, \overrightarrow{x}_2, \overrightarrow{x}_3\}$, where $\overrightarrow{x}_1 = (1,2,-1), \overrightarrow{x}_2 = (1,0,2)$, and $\overrightarrow{x}_3 = (2,1,1)$. Then $\mathcal{B}$ is a basis for $\mathbb{R}^3$. Define $\langle \overrightarrow{x}, \overrightarrow{y}\rangle_{\mathcal{B}} = \langle [\overrightarrow{x}]_{\mathcal{B}}, [\overrightarrow{y}]_{\mathcal{B}}\rangle$ for $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^3$. Do the same questions as in Ex. 2. Also, let

$$\overrightarrow{y}_1 = (1,0,-1),\quad \overrightarrow{y}_2 = (2,5,1),\ \text{ and }\ \overrightarrow{y}_3 = (0,-4,3).$$

Try to use Ex. ⟨A⟩5 to orthogonalize these three vectors in $\langle\ ,\ \rangle_{\mathcal{B}}$.

⟨C⟩ Abstraction and generalization

Read Sec. B.9 and do Exs. 1 and 2 there. Also do the following problems:

1. In $\mathbb{R}^n$, where $n = 1,2,3,\ldots$, let $\mathcal{B} = \{\overrightarrow{x}_1, \overrightarrow{x}_2, \ldots, \overrightarrow{x}_n\}$ be a basis for $\mathbb{R}^n$. For

$$\overrightarrow{x} = [\overrightarrow{x}]_{\mathcal{B}} = (\alpha_1,\ldots,\alpha_n)\ \text{ and }\ \overrightarrow{y} = [\overrightarrow{y}]_{\mathcal{B}} = (\beta_1,\ldots,\beta_n),$$

define

$$\langle \overrightarrow{x}, \overrightarrow{y}\rangle_{\mathcal{B}} = \langle [\overrightarrow{x}]_{\mathcal{B}}, [\overrightarrow{y}]_{\mathcal{B}}\rangle.$$

Try to do some questions as in Ex. ⟨B⟩2. State similar results in $\mathbb{C}^n$ and verify them. Do as far as you can.

2. Let
$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1-i}{2} & 1+i \end{bmatrix}.$$
Show that
$$H = A + \overline{A}^* = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$$
is a Hermitian matrix (i.e. $\overline{H}^* = H$) with $\det H > 0$. Define
$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle_H = \langle \overrightarrow{x}, \overrightarrow{y} H \rangle = x_1\overline{(y_1 - iy_2)} + x_2\overline{(iy_1 + 2y_2)},$$
where $\overrightarrow{x} = (x_1, x_2)$ and $\overrightarrow{y} = (y_1, y_2) \in \mathbb{C}^2$. Do all the questions as in Ex. ⟨B⟩1.

   Also, let
$$\overrightarrow{y}_1 = (1, 2+i) \quad \text{and} \quad \overrightarrow{y}_2 = (3, i).$$
Use the Gram–Schmidt process (refer to Ex. ⟨A⟩5) to orthogonalize $\overrightarrow{y}_1$ and $\overrightarrow{y}_2$.
3. Let
$$A = \begin{bmatrix} 1 & 1+i & 3+i \\ 0 & 1 & 1-i \\ 2 & 1 & 8+i \end{bmatrix}.$$
Do the same questions as in Ex. 2.
4. Prove Ex. 26 in Sec. B.4.
5. In $M(2;\mathbb{R})$, let
$$\mathcal{N} = \left\{ E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$
and
$$\mathcal{B} = \left\{ F_{11} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, F_{12} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, F_{21} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, F_{22} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$
Both $\mathcal{N}$ and $\mathcal{B}$ are bases for $M(2;\mathbb{R})$. For $A, B \in M(2;\mathbb{R})$, define
$$\langle A, B\rangle_{\mathcal{N}} = \operatorname{tr}(AB^*) = \langle [A]_{\mathcal{N}}, [B]_{\mathcal{N}} \rangle, \quad \text{and} \quad \langle A, B\rangle_{\mathcal{B}} = \langle [A]_{\mathcal{B}}, [B]_{\mathcal{B}} \rangle,$$
where $\langle\, ,\, \rangle$ is the natural inner product in $\mathbb{R}^4$.
   (a) Show that both $(M(2;\mathbb{R}), \langle\, ,\, \rangle_{\mathcal{N}})$ and $(M(2;\mathbb{R}), \langle\, ,\, \rangle_{\mathcal{B}})$ are inner product spaces.

(b) Let

$$A = \begin{bmatrix} -1 & 2 \\ 4 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Compute $\langle A, B\rangle_{\mathcal{N}}$ and the lengths in $\langle\,,\,\rangle_{\mathcal{N}}$, namely $|A|_{\mathcal{N}}$ and $|B|_{\mathcal{N}}$. Try to define an angle θ between A and B. What is θ?

(c) Compute the *orthogonal complement* of A in $M(2;\mathbb{R})$, i.e.

$$\{X \in M(2;\mathbb{R}) | \langle X, A\rangle_{\mathcal{N}} = 0\}.$$

(d) Describe the *unit sphere* in $M(2;\mathbb{R})$ with $\langle\,,\,\rangle_{\mathcal{N}}$, i.e.

$$\{X \in M(2;\mathbb{R}) | \langle X, X\rangle_{\mathcal{N}} = 1\}.$$

(e) Try to find possible connection between $\langle\,,\,\rangle_{\mathcal{N}}$ and $\langle\,,\,\rangle_{\mathcal{B}}$. More precisely, try to find invertible matrices $P_{4\times 4}$ and $Q_{4\times 4}$ so that

$$\langle A, B\rangle_{\mathcal{B}} = \langle [A]_{\mathcal{N}}, [B]_{\mathcal{N}} P\rangle \quad \text{and} \quad \langle A, B\rangle_{\mathcal{N}} = \langle [A]_{\mathcal{B}}, [B]_{\mathcal{B}} Q\rangle.$$

How are P and Q related?

(f) Use (e) to redo (b), (c), and (d) for $\langle\,,\,\rangle_{\mathcal{B}}$.

6. $M(2;\mathbb{C})$ with $\langle A, B\rangle_{\mathcal{N}} = \operatorname{tr}(A\overline{B}^*)$ is a complex inner product space. Let

$$A = \begin{bmatrix} 2-i & 1 \\ 3 & -i \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}.$$

(a) Compute $\langle A, B\rangle_{\mathcal{N}}$ and the lengths, i.e. $|A|_{\mathcal{N}}$ and $|B|_{\mathcal{N}}$.

(b) Determine the orthogonal complement of A in $\langle\,,\,\rangle_{\mathcal{N}}$, i.e.

$$\{X \in M(2;\mathbb{C}) | \langle X, A\rangle_{\mathcal{N}} = 0\}.$$

(c) Let S be the linear subspace of $M(2;\mathbb{C})$, generated by A and B. Determine the orthogonal complement of S in $\langle\,,\,\rangle_{\mathcal{N}}$.

7. Bear in mind that $\mathbb{C}^k[a,b]$ is the vector space of *complex valued* functions, defined on the bounded and closed interval $[a,b]$, which are k times continuously differentiable. $\mathbb{C}^0[a,b]$ is simply denoted as $\mathbb{C}[a,b]$.

(a) Show that $\mathbb{C}[a,b]$ with

$$\langle f, g\rangle = \int_a^b f(t)\overline{g(t)}dt$$

is a complex inner product space. Write out the Cauchy–Schwarz inequality and the triangle inequality.

(b) Fix a point c so that $a \leq c < b$. Is

$$\langle f, g\rangle = \int_a^c f(t)\overline{g(t)}dt$$

an inner product on $\mathbb{C}[a,b]$? Why?

(c) Is

$$\langle f, g\rangle = \int_a^b f'(t)\overline{g(t)}dt$$

an inner product on $\mathbb{C}^1[a,b]$? Why?

8. Let $(V, \langle,\rangle)$ be an inner product space over field $\mathbb{F}$, where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. Prove the following *polar identities*: for $\overrightarrow{x}, \overrightarrow{y} \in V$,

$$\langle \overrightarrow{x}, \overrightarrow{y}\rangle = \begin{cases} \frac{1}{4}(|\overrightarrow{x}+\overrightarrow{y}|^2 - |\overrightarrow{x}-\overrightarrow{y}|^2), & \text{if } \mathbb{F} = \mathbb{R}, \\ \frac{1}{4}\sum_{n=1}^{4} i^n|\overrightarrow{x} + i^n\,\overrightarrow{y}|^2, & \text{if } \mathbb{F} = \mathbb{C}, \end{cases}$$

where $i = \sqrt{-1}$. Also, prove that

$$\langle \overrightarrow{x}, \overrightarrow{y}\rangle = \begin{cases} \frac{1}{2}(|\overrightarrow{x}+\overrightarrow{y}|^2 - |\overrightarrow{x}|^2 - |\overrightarrow{y}|^2), & \text{if } \mathbb{F} = \mathbb{R}, \\ \frac{1}{2}(|\overrightarrow{x}+\overrightarrow{y}|^2 + i|\overrightarrow{x}+i\,\overrightarrow{y}|^2) - \frac{1+i}{2}(|\overrightarrow{x}|^2 + |\overrightarrow{y}|^2), & \text{if } \mathbb{F} = \mathbb{C}. \end{cases}$$

⟨D⟩ Applications in analysis

Suppose V is a vector space over field $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. A function

$$\|\ \| : V \to \mathbb{R}$$

satisfying the following conditions, for all $\overrightarrow{x}, \overrightarrow{y} \in V$ and $\alpha \in \mathbb{F}$:

1. $\|\overrightarrow{x}\| \geq 0$ and $\|\overrightarrow{x}\| = 0 \Leftrightarrow = \overrightarrow{0}$,
2. $\|\alpha\overrightarrow{x}\| = |\alpha|\|\overrightarrow{x}\|$,
3. $\|\overrightarrow{x}+\overrightarrow{y}\| \leq \|\overrightarrow{x}\| + \|\overrightarrow{y}\|$,

is called a *norm* on V. V endowed with a norm $\|\ \|$ is called a *normed space* and is denoted as (V, $\|\ \|$) or simply as V. An inner product space $(V, \langle\ ,\ \rangle)$ is always a normed space if its norm is provided by $\|\overrightarrow{x}\| = |\overrightarrow{x}|$, but not conversely (see Exs. 1 and 2 below).

A sequence $\{\overrightarrow{x}_n\}$ or $\overrightarrow{x}_n$ in a normed space $(V, \|\ \|)$ is called *Cauchy* if the real sequence $\lim_{m,n\to\infty}\|\overrightarrow{x}_m - \overrightarrow{x}_n\| = 0$ holds. $(V, \|\|)$ is called a

Banach space if each Cauchy sequence in V always converges to a vector belonging to it, i.e. $\lim_{n\to\infty} \|\overrightarrow{x}_n - \overrightarrow{x}_0\| = 0$ holds for some $\overrightarrow{x}_0 \in V$. An inner product space $(V, \langle,\rangle)$ is called a *Hilbert space* if the induced normed space $(V, \|\ \|)$ is a Banach space.

Do the following problems:

1. Let $(V, \|\ \|)$ be a real normed space such that the norm $\|\ \|$ satisfies the parallelogram law (see Fig. II.13)

$$\|\overrightarrow{x} + \overrightarrow{y}\|^2 + \|\overrightarrow{x} - \overrightarrow{y}\|^2 = 2(\|\overrightarrow{x}\|^2 + \|\overrightarrow{y}\|^2) \quad \text{for } \overrightarrow{x}, \overrightarrow{y} \in V.$$

Define $\langle\ ,\ \rangle$ on V by

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle = \frac{1}{4}(\|\overrightarrow{x} + \overrightarrow{y}\|^2 - \|\overrightarrow{x} - \overrightarrow{y}\|^2).$$

Show that $(V, \langle\ ,\ \rangle)$ is the real inner product space with $|\overrightarrow{x}| = \|\overrightarrow{y}\|$.

2. Let $(V, \|\ \|)$ be a complex normed space such that

$$\|\overrightarrow{x} + \overrightarrow{y}\|^2 + \|\overrightarrow{x} - \overrightarrow{y}\|^2 = 2(\|\overrightarrow{x}\|^2 + \|\overrightarrow{y}\|^2) \quad \text{for } \overrightarrow{x}, \overrightarrow{y} \in V.$$

Define $\langle\ ,\ \rangle$ on V by

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle = \frac{1}{4}\sum_{n=1}^{4} i^n \|\overrightarrow{x} + i^n \overrightarrow{y}\|^2.$$

Show that $(V, \langle\ ,\ \rangle)$ is a complex inner product space with $|\overrightarrow{x}| = \|\overrightarrow{x}\|$.

3. For $A = [a_{ij}]_{m\times n} \in M(m, n; \mathbb{R})$, define for $\overrightarrow{x} \in \mathbb{R}^m$

$$\|A\| = \max_{|\overrightarrow{x}|=1} |\overrightarrow{x}A| = \sup_{\overrightarrow{x}\neq\overrightarrow{0}} \frac{|\overrightarrow{x}A|}{|\overrightarrow{x}|}.$$

   (a) Show that $\|A\| = \sqrt{\lambda}$, where λ is the largest eigenvalue of AA^*.
   (b) Show that $(M(m, n; \mathbb{R}), \|\ \|)$ is a Banach space. In case $m = n$, $(M(n; \mathbb{R}), \|\ \|)$ is a Banach algebra with $\|AB\| \le \|A\|\ \|B\|$.
   (c) Define $\|\ \|_2$ on $M(m, n; \mathbb{R})$ as $\|A\|_2 = (\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2)^{1/2}$. Show that $(M(m, n; \mathbb{R}), \|\ \|_2)$ is a Banach space. Is it possible to be a Hilbert space?

Note For other norms, see Ex. ⟨C⟩3 of Sec. 3.7.6 and Ex. ⟨C⟩14 of Sec. 3.7.7.

4. Let $\langle\ ,\ \rangle$ be the inner product for $\mathbb{C}[a, b]$ as in Ex. ⟨C⟩7. Define $\|\ \|_\infty$ as

$$\|f\|_\infty = \max_{a\le x\le b} |f(x)| \quad \text{for } f \in \mathbb{C}[a, b].$$

(a) Show that $(\mathbb{C}[a,b], \|\ \|_\infty)$ is a Banach space.
(b) Show that $(\mathbb{C}[a,b], \langle\ ,\ \rangle)$ is not a Hilbert space. What is its completion space?

5. For a bounded real or complex sequence $\overrightarrow{x} = \{x_n\}$ and $1 \le p \le \infty$, define

$$\| \overrightarrow{x} \|_p = \begin{cases} \max\limits_{1 \le n < \infty} |x_n|, & \text{if } p = \infty, \\ \left(\sum\limits_{n=1}^{\infty} |\mathrm{x}_n|^p \right)^{1/p}, & \text{if } 1 \le p < \infty. \end{cases}$$

(a) Show that $l^p = \{\overrightarrow{x} = \{x_n\} \mid \| \overrightarrow{x} \|_p < \infty\}$ is a Banach space with $\|\ \|_p$.
(b) Show that l^2 is a Hilbert space.

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