## Chapter 1

# Introduction to electromagnetic theory

## 1.1 Introduction

Electromagnetism is a fundamental physical phenomena that is basic to many areas science and technology. This phenomenon is due to the interaction, called electromagnetic interaction, of electric and magnetic fields with the constituent particles of matter. This interaction is physically described in terms of electromagnetic fields, characterized by the electric field vector, $\vec{E}$ and the magnetic induction, $\vec{B}$. These field vectors are generally time-dependent as they are determined by the positions of the electric charges and their motions (currents) in a medium in which the electromagnetic field exists. The fields $\vec{E}$ and $\vec{B}$ are directly correlated by Ampère-Maxwell and Faraday-Henry laws that satisfy the requirements of special relativity. The time-dependent relations between the time-dependent vectors in these laws and Gauss' laws for electric and magnetic fields are given by Maxwell's equations that form the the basis of electromagnetic theory.

The electric charge and current distributions enter into these equations and are called the sources of the electromagnetic field, because if they are given Maxwell's equations may be solved for $\vec{E}$ and $\vec{B}$ under appropriate boundary conditions.

## 1.2 Maxwell's equations

In order to describe the effect of the electromagnetic field on matter, it is necessary to make use, apart from $\vec{E}$ and $\vec{B}$, of a set another three field vectors, viz., the magnetic vector, $\vec{H}$, the electric displacement vector, $\vec{D}$, and the electric current density, $\vec{J}$. The four Maxwell's equations may be written either in integral form or in differential form. In differential form, the Maxwell's equations are expressed as,

$$\nabla \times E(\vec{r},t) = -\frac{1}{c}\left[\frac{\partial B(\vec{r},t)}{\partial t}\right], \tag{1.1}$$

$$\nabla \times H(\vec{r},t) = \frac{1}{c}\left[4\pi J(\vec{r},t) + \frac{\partial D(\vec{r},t)}{\partial t}\right], \tag{1.2}$$

$$\nabla \cdot D(\vec{r},t) = 4\pi\rho(\vec{r},t) \qquad \text{and} \tag{1.3}$$

$$\nabla \cdot B(\vec{r},t) = 0. \tag{1.4}$$

In these equations, $c = 2.99,79 \times 10^8$meter(m)/second(s) is the velocity of light in free space, $\rho$ the volume charge density, and Gaussian units are used for expressing the vector quantities, and $\nabla$ represents a vector differential operator,

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

The unit of the electric field intensity, $\vec{E}$, is expressed as volt (V) $\mathrm{m}^{-1}$, and that for the magnetic flux density $|\vec{B}|$, tesla ($\mathrm{T} = \mathrm{Wb\,m}^{-2}$) in which $||$ stands for the modulus. Equations (1.1 -1.4) represent Faraday-Henry law of induction, Ampére's law with the displacement current introduced by Maxwell, known as Ampére-Maxwell law, Gauss' electric and magnetic laws respectively. It is further assumed that the space and time derivatives of the field vectors are continuous at every point $(\vec{r},t)$ where the physical properties of the media are continuous. In order to describe the interaction of light with matter at thermal equilibrium, the Maxwell's equations are substituted by the additional equations,

$$\vec{J} = \sigma\vec{E}, \tag{1.5}$$

$$\vec{B} = \mu\vec{H}, \tag{1.6}$$

$$\vec{D} = \epsilon_0\vec{E}, \tag{1.7}$$

where $\sigma$ is the specific conductivity, $\mu$ the permeability of the medium in which magnetic field acts, and $\epsilon_0 (= 8.8541 \times 10^{-12}$farads (F)/m) the permittivity or dielectric constant at vacuum.

Equations (1.5 - 1.7) describe the behavior of substances under the influence of the field. These relations are known as material equations. The electric and magnetic fields are also present in matter giving rise to the relations (in standard notation),

$$\vec{E}_m = \vec{E} + \frac{\vec{P}}{\epsilon_0}, \tag{1.8}$$

$$\vec{B}_m = \vec{B} + \mu_0 \vec{M}, \tag{1.9}$$

where $\vec{E}_m$ is the electric field corresponding to the dielectric displacement in $volts$(V) m$^{-1}$, $\vec{B}_m$ the magnetic field in the presence of medium, $\vec{P}$ the polarization susceptibility, $\vec{M}$ the magnetization, and $\mu_0 (= 4\pi k = 4\pi \times 10^{-7}$henrys (H)/m), the permeability in free space or in vacuum, and $k$ the constant of proportionality.

In a medium of free space, by using the integral form of Gauss' electric law,

$$\int_{\mathtt{S}} \vec{E} \cdot \vec{n} d\mathtt{S} = 4\pi q, \tag{1.10}$$

and the relation between $\vec{E}$ and $\varphi$, i.e.,

$$E(\vec{r}) = -\nabla \varphi(\vec{r}), \tag{1.11}$$

the Poisson (S. D. Poisson, 1781-1840) partial differential equation for $\varphi$ is obtained,

$$\nabla^2 \varphi = -4\pi \rho(\vec{r}), \tag{1.12}$$

in which the Lapacian operator, $\nabla^2$, in Cartesian coordinates reads,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{1.13}$$

The equation (1.12) relates the electric potential $\varphi(\vec{r})$ with its electric charge $\rho(\vec{r})$. In regions of empty of charge, this equation turns out to be homogeneous, i.e.,

$$\nabla^2 \varphi = 0. \tag{1.14}$$

This expression is known as the Laplace (P. S. de Laplace, 1749-1827) equation.

### 1.2.1 *Charge continuity equation*

Maxwell added the second term of the right hand side (RHS) of equation (1.2), which led to the continuity equation. By taking divergence on both sides of the said equation (1.2),

$$\nabla \cdot (\nabla \times H(\vec{r}, t)) = \frac{4\pi}{c} \nabla \cdot J(\vec{r}, t) + \frac{1}{c} \nabla \cdot \frac{\partial D(\vec{r}, t)}{\partial t}. \tag{1.15}$$

Using the vector equation, $\nabla \cdot (\nabla \times \vec{A}) = 0$ for any vector field $\vec{A}$, the equation (1.15) translates into,

$$\nabla \cdot J(\vec{r}, t) = -\frac{1}{4\pi} \nabla \cdot \frac{\partial D(\vec{r}, t)}{\partial t}. \tag{1.16}$$

By substituting the equation (1.3) into equation (1.16), the following relationship emerges,

$$\nabla \cdot \frac{\partial D(\vec{r}, t)}{\partial t} = \frac{\partial \rho(\vec{r}, t)}{\partial t}. \tag{1.17}$$

The volume charge density, $\rho$ and the current density, $J(\vec{r}, t)$ are the sources of the electromagnetic radiation[1]. The current density $\vec{J}$ associated with a charge density $\rho$ moving with a velocity $\vec{v}$ is $\vec{J} = \rho\vec{v}$.

On replacing the value of $\nabla \cdot \partial\vec{D}/\partial t$ from the equation (1.16) in equation (1.17), one obtains,

$$\nabla \cdot J(\vec{r}, t) = -\frac{\partial \rho(\vec{r}, t)}{\partial t}. \tag{1.18}$$

Thus the equation of continuity is derived as,

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0. \tag{1.19}$$

Equation (1.19) expresses the fact that the charge is conserved in the neighborhood of any point. By integrating this equation with the help of Gauss'

[1]Electromagnetic radiation is emitted or absorbed when an atom or a molecule moves from one energy level to another. It has a continuous energy spectrum, a graph, that depicts the intensity of light being emitted over a range of energies. This radiation may be arranged in a spectrum according to its frequency ranging from very high frequencies to the lowest frequencies. The highest frequencies, known as gamma rays whose frequencies range between $10^{19}$ to $10^{21}$ Hz ($\lambda \sim 10^{-11} - 10^{-13}$ m), are associated with cosmic sources. The other sources are being the gamma decay of radioactive materials and nuclear fission. The frequency range for X-ray falls between $10^{17}$ to $10^{19}$ Hz ($\lambda \sim 10^{-9} - 10^{-11}$ m), which is followed by ultraviolet with frequencies between $10^{15}$ to $10^{17}$ Hz ($\lambda \sim 10^{-7} - 10^{-9}$ m). The frequencies of visible light fall between $10^{14}$ and $10^{15}$ Hz ($\lambda \sim 10^{-6} - 10^{-7}$ m). The infrared frequencies are $10^{11}$ to $10^{14}$ Hz ($\lambda \sim 10^{-3} - 10^{6}$ m); heat radiation is the source for infrared frequencies. The lower frequencies such as radio waves having frequencies $10^{4}$ to $10^{11}$ Hz ($\lambda \sim 10^{4} - 10^{-3}$ m) and microwave (short high frequency radio waves with wavelength 1 mm-30 cm) are propagated by commutated direct-current sources. Only the optical and portions of the infrared and radio spectrum can be observed at the ground.

theorem,

$$\frac{d}{dt}\int_{\mathrm{V}} \rho d\mathrm{V} + \int_{\mathrm{S}} \vec{J}\cdot\vec{n}d\mathrm{S} = 0. \quad (1.20)$$

The charged particle is a small body with a charge density $\rho$ and the total charge, $q = \int_{\mathrm{V}} \rho d\mathrm{V}$, contained within the domain can increase due to flow of electric current,

$$i = \int_{\mathrm{S}} \vec{J}\cdot\vec{n}d\mathrm{S}. \quad (1.21)$$

It is important to note that all the quantities that figure in the Maxwell's equations, as well as in the equation of continuity are evaluated in the rest frame of the observer and all surfaces and volumes are held fixed in that frame.

### 1.2.2 *Boundary conditions*

In free space, or vacuum, the vectors are $\vec{E}$ and $\vec{H}$, and the relations between the vectors $\vec{E}$, $\vec{B}$, $\vec{D}$, and $\vec{H}$ in a material are derived from the equations (1.6) and (1.7),

$$\begin{aligned} D(\vec{r},t) &= \epsilon E(\vec{r},t) = \epsilon_r\epsilon_0 E(\vec{r},t), \\ H(\vec{r},t) &= \frac{1}{\mu}B(\vec{r},t) = \frac{1}{\mu_r\mu_0}B(\vec{r},t), \end{aligned} \quad (1.22)$$

where $\epsilon$ is the permittivity of the medium in which the electric field acts, $\epsilon_r = \epsilon/\epsilon_0$, and $\mu_r = \mu/\mu_0$ the respective relative permittivity and permeability.

It is assumed that both $\epsilon$ and $\mu$ in equation (1.22) are independent of position ($\vec{r}$) and time ($t$), and that $\epsilon_r \geq 1, \mu_r \geq 1$. The field vectors can be determined in regions of space (Figure 1.1a) where both $\epsilon$ and $\mu$ are continuous functions of space from the set of Maxwell's equations, as well as from the material equations. From the Maxwell equation, $\nabla\cdot\vec{B} = 0$, one may write,

$$\int_{\mathrm{V}} \nabla\cdot\vec{B}d\mathrm{V} = 0. \quad (1.23)$$

Equation (1.23) implies the flux into the volume element is equal to the flux out of the volume. For a flat volume whose faces can be neglected, the integral form of Gauss' magnetic law may be written,

$$\oint_S \vec{B} \cdot \vec{n} d\mathrm{S} = 0. \tag{1.24}$$

Similarly, the other Maxwell equation $\nabla \cdot \vec{D} = \rho$, may also be used. With boundary conditions at the interface between two different media, i.e., when the physical properties of the medium are discontinuous, the electromagnetic fields within a bounded region are given by,

$$\vec{n} \cdot (\vec{B}_2 - \vec{B}_1) = 0, \tag{1.25}$$

$$\vec{n} \cdot (\vec{D}_2 - \vec{D}_1) = \rho, \tag{1.26}$$

in which $\vec{n}$ is the unit vector normal (a line perpendicular to the surface) to the surface of discontinuity directed from medium 1 to medium 2.

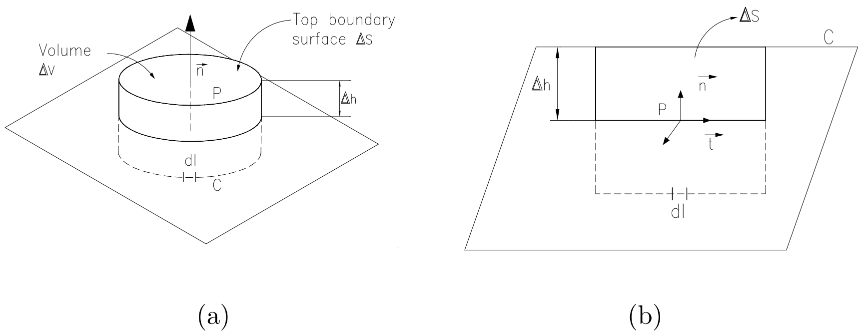


Fig. 1.1 Boundary conditions for (a) the normal components of the electromagnetic field, and (b) the tangential components of the said field.

Equations (1.25 and 1.26) may be written as,

$$B_{2n} - B_{1n} = 0, \tag{1.27}$$

$$D_{2n} - D_{1n} = \rho, \tag{1.28}$$

where $B_n = \vec{n} \cdot \vec{B}$ and the subscript $n$ signifies the component normal to the boundary surface.

Equations (1.27) and (1.28) are the boundary conditions for the normal components of $\vec{B}$ and $\vec{D}$, respectively. The normal component of the magnetic induction is continuous, while the normal component of the electric displacement changes across the boundary as a result of surface charges. From the Ampére-Maxwell law, the condition for $\vec{H}$ can also be derived.

Choosing the integration path in a way that the unit vector is tangential to the interface between the media (Figure 1.1b). The integral form of the equation after applying Stokes formula yields,

$$\vec{t} \times (\vec{H}_2 - \vec{H}_1) = \frac{4\pi}{c} \vec{J}_s, \tag{1.29}$$

where $\vec{t}$ signifies the unit vector tangential to the interface between the media, and $\vec{J}_s$ the surface density of current tangential to the interface, locally perpendicular to both $\vec{t}$ and $\vec{n}$.

Similarly, for a static case a corresponding equation for the tangent component of electric field, $\oint_C \vec{E} \cdot d\vec{l} \equiv 0$, is written as,

$$\vec{t} \times (\vec{E}_2 - \vec{E}_1) = 0. \tag{1.30}$$

Equations (1.29 and 1.30) demonstrate respectively that the tangential components of the electric field vector are continuous across the boundary and the tangential component of the magnetic vector changes across the boundary as a result of a surface current density.

Since $\vec{B} = \mu \vec{H}$, from the equation (1.25), one obtains,

$$\mu_1(\vec{H}_1 \cdot \vec{n}) = \mu_2(\vec{H}_2 \cdot \vec{n}), \tag{1.31}$$

and for the normal component,

$$(\vec{H}_1)_n = \frac{\mu_2}{\mu_1}(\vec{H}_2)_n. \tag{1.32}$$

In the case of the equation of continuity for electric charge (equation 1.19), the boundary condition is given by,

$$\vec{n} \cdot (\vec{J}_2 - \vec{J}_1) + \nabla_s \cdot \vec{J}_s = -\frac{\partial \rho_s}{\partial t}. \tag{1.33}$$

This is the surface equation of continuity for electric charge; it is a statement of conservation of charge at a point on the surface.

## 1.3 Energy flux of electromagnetic field

When a point charge $q$ moves with velocity, $\vec{v}$, in both electric and magnetic fields, $\vec{E}$ and $\vec{B}$, the total force exerted on charge, $q$, by the field is given by the Lorentz law,

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right). \tag{1.34}$$

The equation (1.34) describes the resultant force experienced by a particle of charge $q$ moving with velocity $\vec{v}$, under the influence of both an electric field, $\vec{E}$ and a magnetic field $\vec{B}$. The total force at a point within the particle is the applied field together with the field due to charge in the particle itself (self field). In practical situation, the self force is negligible, therefore the total force on the particle is approximately the applied force. The expression (equation 1.34) referred as Lorentz force density, provides the connection between classical mechanics and electromagnetism.

The concepts such as energy, linear and angular momentum[2] may be associated with the electromagnetic field through the expression that is derived above. In classical mechanics, a particle of mass $m$, moving with velocity $\vec{v}$ at position $\vec{r}$ in an inertial reference frame, has linear momentum $\vec{p}$ (Goldstein, 1980, Haliday et al. 2001),

$$\vec{p} = m\frac{d\vec{r}}{dt} = m\vec{v}. \tag{1.35}$$

The total force applied to the particle, according to the Newton's second law, is given by,

$$\begin{aligned}\vec{F} &= \frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} \\ &= m\frac{d^2\vec{r}}{dt^2} = m\vec{a},\end{aligned} \tag{1.36}$$

in which, $\vec{a}$ indicates the acceleration (the rate of change of velocity) of the particle.

If the particle has charge $e$, the force on the particle of mass $m$ due to electric field $\vec{E}$ is

$$\vec{F} = e\vec{E} = m\vec{a}. \tag{1.37}$$

The symbol $e$ is used to designate the charge of a particle, say electron ($e = 1.6 \times 10^{-19}$coulomb (C)), instead of $q$. Since the force $\vec{F}$ on the particle is equal to the charge of a particle that is placed in a uniform electric field, i.e., $\vec{F} = e\vec{E}$. The force is in the same direction as the field if the charge is positive, and the force become opposite to the field if the charge is negative. If the particle is rest and the field is applied, the particle is accelerated uniformly in the direction of the field.

[2]Angular momentum is defined as the product of moment of inertia and angular velocity of a body revolving about an axis.

The work done by the applied force on the particle when it moves through the displacement $\Delta\vec{r}$ is defined as,

$$\Delta W = \vec{F} \cdot \Delta\vec{r}. \tag{1.38}$$

The rate at which the work is done is the power $\mathcal{P}$,

$$\begin{aligned}\mathcal{P} &= \lim_{\Delta t \to 0} \left(\frac{\Delta W}{\Delta t}\right) \\ &= \lim_{\Delta t \to 0} \left(\vec{F} \cdot \frac{\Delta\vec{r}}{\Delta t}\right) = \vec{F} \cdot \vec{v}. \end{aligned} \tag{1.39}$$

The energy in the case of of a continuous charge configuration $\rho(\vec{r})$ is expressed as,

$$W = \frac{1}{2} \iint \frac{\rho(\vec{r}')\rho(\vec{r})}{|\vec{r} - \vec{r}'|} d\mathrm{V} d\mathrm{V}' = \frac{1}{2} \int \varphi(\vec{r})\rho(\vec{r}) d\mathrm{V}, \tag{1.40}$$

where the potential of a charge distribution is,

$$\varphi(\vec{r}) = \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d\mathrm{V}'.$$

In this equation (1.73), the integration extends over the point $\vec{r} = \vec{r}'$, so that the said equation contains self energy parts which become infinitely large for point charges. The amount of electrostatic energy stored in an electric field in a region of space is expressed as,

$$\begin{aligned} W &= \frac{1}{2} \int \varphi(\vec{r})\rho(\vec{r}) d\mathrm{V} = \frac{1}{2} \int \frac{1}{4\pi} \left[\nabla \cdot \vec{E}(\vec{r})\right] \varphi(\vec{r}) d\mathrm{V} \\ &= -\frac{1}{8\pi} \int E(\vec{r}) \cdot \nabla\varphi(\vec{r}) d\mathrm{V} = \frac{1}{8\pi} \int E^2(\vec{r}) d\mathrm{V}. \end{aligned} \tag{1.41}$$

The integrand represents the energy density of the electric field, i.e.,

$$w_e = \frac{1}{8\pi} \vec{E}^2. \tag{1.42}$$

The power can be determined in terms of the kinetic energy (KE) of the particle, $\mathcal{K}$ by invoking equation (1.39),

$$\begin{aligned} \mathcal{P} &= \vec{F} \cdot \vec{v} = m \frac{d\vec{v}}{dt} \cdot \vec{v} \\ &= \frac{d}{dt} \left(\frac{1}{2} m |\vec{v}|^2\right) = \frac{d\mathcal{K}}{dt}. \end{aligned} \tag{1.43}$$

Thus, the rate at which work is done by the applied force - the power - is equal to the rate of increase in KE of the particle. The mechanical force of electromagnetic origin acting on the charge and current for a volume V of free space at rest containing charge density, $\rho$ and current density, $\vec{J}$ is given by the Lorentz law,

$$\begin{aligned}\vec{F} &= \int_{\mathrm{V}} \left(\rho\vec{E} + \vec{J}\times\vec{B}\right) d\mathrm{V} \\ &= \int_{\mathrm{V}} \left(\rho\vec{E} + \rho\frac{\vec{v}}{c}\times\vec{B}\right) d\mathrm{V},\end{aligned} \tag{1.44}$$

where $\vec{J} = \rho\vec{v}$, and $\vec{v}$ is the velocity of the particle moving the current density within the particle.

The power $\mathcal{P}$ is deduced as,

$$\begin{aligned}\mathcal{P} &= \vec{v}\cdot\int_{\mathrm{V}} \left(\rho\vec{E} + \rho\frac{\vec{v}}{c}\times\vec{B}\right) d\mathrm{V} \\ &= \int_{\mathrm{V}} \left[\rho\vec{v}\cdot\vec{E} + \vec{v}\cdot\left(\rho\frac{\vec{v}}{c}\times\vec{B}\right)\right] d\mathrm{V}.\end{aligned} \tag{1.45}$$

Since the velocity is same at all points in the particle, $\vec{v}$ is moved under the integral sign. Because $\vec{v}\cdot\left(\vec{v}\times\vec{B}\right) = 0$, the magnetic field does no work on the charged particle. Thus the equation (1.45) is written as,

$$\mathcal{P} = \int_{\mathrm{V}} \vec{E}\cdot\vec{J} d\mathrm{V} = \frac{d\mathcal{K}}{dt}. \tag{1.46}$$

The equation (1.46) expresses the rate at which energy is exchanged between the electromagnetic field and the mechanical motion of the charged particle. When $\mathcal{P}$ is positive, the field supplies energy to the mechanical motion of the particle, and in the case of negative $\mathcal{P}$, the mechanical motion of the particle supplies energy to the field.

## 1.4 Conservation law of the electromagnetic field

The energy conservation law of the electromagnetic field was evolved by Poynting (John Henry Poynting, 1831-1879) in late Nineteenth century, from the Maxwell's equations (1.1 and 1.2), which results in

$$\vec{E}\cdot\left(\nabla\times\vec{H}\right) = \frac{4\pi}{c}\vec{E}\cdot\vec{J} + \frac{1}{c}\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}, \tag{1.47}$$

$$\vec{H} \cdot \left( \nabla \times \vec{E} \right) = -\frac{1}{c} \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}. \tag{1.48}$$

Equation (1.46) is applied to a general volume V. By subtracting equation (1.48) from equation (1.47), one gets,

$$\vec{E} \cdot \left( \nabla \times \vec{H} \right) - \vec{H} \cdot \left( \nabla \times \vec{E} \right) = \frac{4\pi}{c} \vec{E} \cdot \vec{J} + \frac{1}{c} \left( \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right). \tag{1.49}$$

The term $\vec{E} \cdot \vec{J}$ represents the work done by the field on the electric current density. By using the vector relation,

$$\vec{A} \cdot (\nabla \times \vec{B}) - \vec{B} \cdot (\nabla \times \vec{A}) = -\nabla \cdot (\vec{A} \times \vec{B}),$$

the left hand side (LHS) quantity of the equation (1.82) can be written as,

$$\vec{E} \cdot (\nabla \times \vec{H}) - \vec{H} \cdot (\nabla \times \vec{E}) = -\nabla \cdot (\vec{E} \times \vec{H}). \tag{1.50}$$

Therefore, the equation (1.82) turns out to be,

$$\frac{4\pi}{c} \vec{E} \cdot \vec{J} + \frac{1}{c} \left( \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right) + \nabla \cdot (\vec{E} \times \vec{H}) = 0. \tag{1.51}$$

Integrating equation (1.51) all through an arbitrary volume, and using Gauss' divergence theorem, $\int_{\mathrm{V}} \nabla \cdot \vec{A} d\mathrm{V} = \oint_{\mathrm{S}} \vec{n} \cdot \vec{A} d\mathrm{S}$, one finds

$$\int_{\mathrm{V}} \vec{E} \cdot \vec{J} d\mathrm{V} + \frac{1}{4\pi} \int_{\mathrm{V}} \left( \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right) d\mathrm{V} + \frac{c}{4\pi} \oint_{\mathrm{S}} (\vec{E} \times \vec{H}) \cdot d\vec{S} = 0. \tag{1.52}$$

The equation (1.52) represents the energy law of electromagnetic field. Let

$$S(\vec{r}, t) = \frac{c}{4\pi} \left[ E(\vec{r}, t) \times H(\vec{r}, t) \right], \tag{1.53}$$

then term, $\vec{S}$, is called the energy flux density of the electromagnetic field in the direction of propagation. It is known as the Poynting vector, or power surface density. The Poynting vector $\vec{S}$ has the units of energy per unit area per unit time (joule (J) $\mathrm{m}^{-2}\mathrm{s}^{-1}$) or power per unit area watt (W)$\mathrm{m}^{-2}$. Its magnitude $|\vec{S}|$ is equal to the rate of flow per unit area element perpendicular to $\vec{S}$.

Thus far the expression obtained above is for the energy associated with the motion of a charged particle. In what follows, an expression for the energy that applies to the general volume distribution of charge, $\rho$ and current $\vec{J}$ is derived. Let the equation (1.52) be written in the form,

$$\int_{\mathrm{V}} \vec{E}\cdot\vec{J}d\mathrm{V} + \frac{1}{4\pi}\int_{\mathrm{V}}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)d\mathrm{V} + \oint_{\mathrm{S}}\vec{S}\cdot d\vec{S} = 0. \tag{1.54}$$

This relation is known as Poynting theorem. The power carried away from a volume bounded by a surface S by the electromagnetic field is given by the term, $\oint_{\mathrm{S}}\vec{S}\cdot d\vec{S}$. This is equal to the rate at which electromagnetic energy is leaving volume by passing through its surfaces.

On using material equations $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$, the second term of the Poynting theorem (equation 1.52) can be simplified. For the electric term, one gets,

$$\frac{1}{4\pi}\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} = \frac{1}{4\pi}\vec{E}\cdot\frac{\partial}{\partial t}\left(\epsilon\vec{E}\right) = \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon\vec{E}^2\right) = \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\vec{E}\cdot\vec{D}\right). \tag{1.55}$$

Similarly, for the magnetic term one may derive as,

$$\frac{1}{4\pi}\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\mu\vec{H}^2\right) = \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\vec{H}\cdot\vec{B}\right). \tag{1.56}$$

Thus, the second term of the equation (1.54) is rewritten as,

$$\int_{\mathrm{V}}\left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)d\mathrm{V} = \frac{1}{8\pi}\frac{\partial}{\partial t}\int_{\mathrm{V}}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right)d\mathrm{V}. \tag{1.57}$$

For an electrostatic field in a simple material, the energy stored in the electric field, as well as for a magnetostatic field in a simple material, the stored energy in the magnetic field are respectively given by,

$$w_e = \frac{1}{8\pi}\vec{E}\cdot\vec{D}; \qquad w_m = \frac{1}{8\pi}\vec{H}\cdot\vec{B}, \tag{1.58}$$

where $w_e$ and $w_m$ are the electric and magnetic energy densities respectively.

From the expressions (equations 1.57, 1.58), the equation (1.51) is cast as,

$$\frac{4\pi}{c}\vec{E}\cdot\vec{J} + \nabla\cdot(\vec{E}\times\vec{H}) = \frac{\partial}{\partial t}(w_e + w_m). \tag{1.59}$$

This expression (1.92) describes the transfer of energy during a decrease of the total energy density of the electromagnetic field in time. The Poynting theorem (equation 1.54) takes the form,

$$-\frac{dW}{dt} = \frac{d}{dt}\int_{\mathrm{V}}(w_e + w_m)d\mathrm{V} = \int_{\mathrm{V}} \vec{E}\cdot\vec{J}d\mathrm{V} + \oint_{\mathrm{S}} \vec{S}\cdot d\vec{S}, \tag{1.60}$$

in which

$$W = \int_{\mathrm{V}}(w_e + w_m)d\mathrm{V}. \tag{1.61}$$

is total electric and magnetic energy.

The equation (1.60) represents the energy conservation law of electrodynamics. The term $dW/dt$ is interpreted as the time rate of change of the total energy contained within the volume, V. Let the Lorentz law given by equation (1.34) be recalled, and assuming that all the charges $e_k$ are displaced by $\delta\vec{x}_k$ (where $k = 1, 2, 3, \cdots$) in time $\delta t$, therefore the total work done is given by,

$$\begin{aligned}\delta A &= \sum_k e_k\left[\vec{E}_k + \frac{1}{c}\vec{v}_k \times \vec{B}\right]\cdot\delta\vec{x}_k \\ &= \sum_k e_k\vec{E}_k\cdot\delta\vec{x}_k = \sum_k e_k\vec{E}_k\cdot\vec{v}_k\delta t,\end{aligned} \tag{1.62}$$

with $\delta\vec{x}_k = \vec{v}_k\delta t$.

On introducing the total charge density $\rho$, one obtains,

$$\frac{\delta A}{\delta t} = \int_{\mathrm{V}} \rho\vec{v}\cdot\vec{E}d\mathrm{V}. \tag{1.63}$$

The current density, $\vec{J}$, is may be split into two parts,

$$\vec{J} = \vec{J}_c + \vec{J}_v, \tag{1.64}$$

where $\vec{J}_c = \sigma\vec{E}$ is the conduction current density, and $\vec{J}_v = \rho\vec{v}$ the convection current density.

Thus for an isothermal conductor, the energy is irreversibly transferred to a heat reservoir as Joule's heat (James Brescott Joule, 1818 - 1889), then one writes,

$$\mathcal{Q} = \int_{\mathrm{V}} \vec{E}\cdot\vec{J}_c d\mathrm{V} = \int_{\mathrm{V}} \sigma\vec{E}^2 d\mathrm{V}. \tag{1.65}$$

Here $\mathcal{Q}$ represents resistive dissipation of energy called Joule's heat in a conductor ($\sigma \neq 0$).

When the motion of the charge is instantaneously supplying energy to the electromagnetic field throughout the volume, the volume density of current due to the motion of the charge $\vec{J}_v$ is given by,

$$\frac{\delta A}{\delta t} = \int_{\mathrm{V}} \vec{E} \cdot \vec{J}_v d\mathrm{V}. \tag{1.66}$$

From the equations (1.63) and (1.64), one finds,

$$\int_{\mathrm{V}} \vec{J} \cdot \vec{E} d\mathrm{V} = \mathcal{Q} + \int_{\mathrm{V}} \vec{J}_v \cdot \vec{E} d\mathrm{V} = \mathcal{Q} + \frac{\delta A}{\delta t}. \tag{1.67}$$

Thus, equation (1.60) translates into,

$$\frac{dW}{dt} = -\mathcal{Q} - \frac{\delta A}{\delta t} - \oint_{\mathrm{S}} \vec{S} \cdot d\vec{S}. \tag{1.68}$$

where $\delta A/\delta t$ is the rate at which electromagnetic energy is being stored.

The interpretation of such a relation as a statement of conservation of energy within the volume, V, stands. Finally, in a nonconducting medium ($\sigma = 0$) where no mechanical work is done ($A = 0$), the energy law may be written in the hydrodynamical continuity equation for non-compressible fluids,

$$\frac{\partial w}{\partial t} + \nabla \cdot \vec{S} = 0, \tag{1.69}$$

with $w = w_e + w_m$.

The physical meaning of the equation (1.69) is that the decrease in the time rate of change of electromagnetic energy density within a volume is equal to the flow of energy out of the volume.

## 1.5 Electromagnetic wave equations

Consider the propagation of light in a medium, in which the charges or currents are absent, i.e., $\vec{J} = 0$ and $\rho = 0$, and therefore, the first two Maxwell's equations can be cast into the forms,

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t},$$

$$\nabla \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t}. \tag{1.70}$$

To proceed further, $\vec{B}$ is replaced with $\mu\vec{H}$ (equation 1.6) in the first equation (1.70), so that,

$$\nabla \times \vec{E} = -\frac{\mu}{c}\frac{\partial \vec{H}}{\partial t}, \tag{1.71}$$

or,

$$\frac{1}{\mu}\left(\nabla \times \vec{E}\right) = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}. \tag{1.72}$$

The curl of the equation (1.72) gives,

$$\nabla \times \left[\frac{1}{\mu}\left(\nabla \times \vec{E}\right)\right] = \nabla \times \left[-\frac{1}{c}\frac{\partial \vec{H}}{\partial t}\right] = -\frac{1}{c}\nabla \times \frac{\partial \vec{H}}{\partial t}. \tag{1.73}$$

Similarly, by replacing $\vec{D}$ with $\epsilon\vec{E}$ (equation 1.7) from the second equation (1.70), one writes,

$$\nabla \times \vec{H} = \frac{\epsilon}{c}\frac{\partial \vec{E}}{\partial t}. \tag{1.74}$$

Differentiating both sides of equation (1.74) with respect to time, and interchanging differentiation with respect to time and space, one gets,

$$\nabla \times \frac{\partial \vec{H}}{\partial t} = \frac{\epsilon}{c}\frac{\partial^2 \vec{E}}{\partial t^2}. \tag{1.75}$$

Substituting (1.75) in equation (1.73), the following relationship emerges,

$$\nabla \times \left[\frac{1}{\mu}\left(\nabla \times \vec{E}\right)\right] = -\frac{\epsilon}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}, \tag{1.76}$$

By using the vector triple product identity,

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A},$$

we may write,

$$\nabla \times \left[\frac{1}{\mu}\left(\nabla \times \vec{E}\right)\right] = \nabla\left(\frac{1}{\mu}\nabla \cdot \vec{E}\right) - \frac{1}{\mu}\nabla^2\vec{E}. \tag{1.77}$$

When light propagates in vacuum, use of the Maxwell's equation $\nabla \cdot \vec{E} = 0$, in equation (1.77) yields,

$$\nabla \times \left[\frac{1}{\mu}\left(\nabla \times \vec{E}\right)\right] = -\frac{1}{\mu}\nabla^2\vec{E}. \tag{1.78}$$

Invoking equation (1.76), this equation (1.78) takes the form,

$$\frac{1}{\mu}\nabla^2\vec{E} = \frac{\epsilon}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}, \tag{1.79}$$

or, on rearranging this equation (1.79),

$$\left(\nabla^2 - \frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0. \tag{1.80}$$

Similarly, one derives for $\vec{H}$,

$$\left(\nabla^2 - \frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{H} = 0. \tag{1.81}$$

The above expressions (equations 1.80-1.81) are known as the electromagnetic wave equations, which indicate that electromagnetic disturbances (waves) are propagated through the medium. This result gives rise to Maxwell's electromagnetic theory of light. The propagation velocity $v$ of the waves obeying the wave equations is given by,

$$v = \frac{c}{\sqrt{\epsilon\mu}}, \tag{1.82}$$

therefore, one may express the wave equation (1.80) as,

$$\left(\nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0. \tag{1.83}$$

For a scalar wave $E$ propagating in the $z$-direction, the equation (1.83) is simplified to,

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2} = 0. \tag{1.84}$$

The permittivity constant $\epsilon_0$ and the permeability constant $\mu_0$ in a vacuum are related to the speed of light $c$,

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 2.99, 79 \times 10^8 \mathrm{m\,s^{-1}}. \tag{1.85}$$

### 1.5.1 *The Poynting vector and the Stokes parameter*

It is evident from Maxwell's equations that the electromagnetic radiation is transverse wave motion, where the electric and magnetic fields oscillate perpendicular to each other and also perpendicular to the direction of propagation denoted by $\vec{\kappa}$ (see Figure 1.2). These variations are described by the harmonic wave equations in the form,

$$E(\vec{r}, t) = E_0(\vec{r}, \omega)e^{i(\vec{\kappa} \cdot \vec{r} - \omega t)}, \tag{1.86}$$

$$B(\vec{r}, t) = \vec{B}_0(\vec{r}, \omega)e^{i(\vec{\kappa} \cdot \vec{r} - \omega t)}, \tag{1.87}$$

in which $E_0(\vec{r}, \omega)$ and $B_0(\vec{r}, \omega)$ are the amplitudes[3] of the electric and magnetic field vectors respectively, $\vec{r}(= x, y, z)$ the position vector, $\omega(= 2\pi\nu)$ is the angular frequency, $\nu = 1/T$ represents the number of complete cycles of waves per unit time, called frequency, (the shorter the wavelength[4], the higher the frequency) and $T$ the period[5] of motion, and

$$\vec{\kappa} \cdot \vec{r} = \kappa_x x + \kappa_y y + \kappa_z z, \tag{1.88}$$

represents planes in a space of constant phase (any portion of the wave cycle), and

$$\vec{\kappa} = \kappa_x \vec{i} + \kappa_y \vec{j} + \kappa_z \vec{k}. \tag{1.89}$$

The Cartesian components of the wave travel with the same propagation vector $\vec{\kappa}$ and frequency $\omega$. The cosinusoidal fields are,

$$\begin{aligned} E(\vec{r}, t) &= \Re\left[E_0(\vec{r}, \omega)e^{i(\vec{\kappa} \cdot \vec{r} - \omega t)}\right] = E_0(\vec{r}, \omega)\cos(\vec{\kappa} \cdot \vec{r} - \omega t), \\ B(\vec{r}, t) &= \Re\left[\vec{B}_0(\vec{r}, \omega)e^{i(\vec{\kappa} \cdot \vec{r} - \omega t)}\right] = B_0(\vec{r}, \omega)\cos(\vec{\kappa} \cdot \vec{r} - \omega t). \end{aligned} \tag{1.90}$$

Assuming that $\vec{E}_0$ is constant, hence the divergence of the equation (1.86) becomes,

$$\begin{aligned} \nabla \cdot \vec{E} &= \vec{E}_0 \cdot \nabla\left(e^{i[\vec{\kappa} \cdot \vec{r} - \omega t]}\right) \\ &= \vec{E}_0 \cdot (i\vec{\kappa})e^{i[\vec{\kappa} \cdot \vec{r} - \omega t]} = (i\vec{\kappa}) \cdot \vec{E}. \end{aligned} \tag{1.91}$$

[3]An amplitude of a wave defined as the maximum magnitude of the displacement from the equilibrium position during one wave cycle.

[4]Wavelength is defined as the least distance between two points in same phase in a periodic wave motion

[5]Period is defined by the shortest interval in time between two instants when parts of the wave profile that are oscillating in phase pass a fixed point and any portion of the wave cycle is called a phase. When two waves of equal wavelength travel together in the same direction they are said to be in phase if they are perfectly aligned in their cycle, and out of phase if they are out of step.

The curl of the electric field is derived as,

$$\begin{aligned}\nabla \times \vec{H} &= \frac{\epsilon}{c}\frac{\partial \vec{E}}{\partial t} = \frac{\epsilon}{c}\vec{E}_0 \frac{\partial}{\partial t}\left(e^{i(\vec{\kappa}\cdot\vec{r} - \omega t)}\right) \\ &= -\frac{i\omega\epsilon}{c}\vec{E}_0 e^{i(\vec{\kappa}\cdot\vec{r} - \omega t)} = -\frac{i\omega\epsilon}{c}\vec{E}. \end{aligned} \tag{1.92}$$

Replacing $\nabla$ to $i\vec{\kappa}$ and $\partial/\partial t$ to -$i\omega$, this equation (1.92) is recast as,

$$\vec{\kappa} \times \vec{H} = -\frac{\epsilon\omega}{c}\vec{E}. \tag{1.93}$$

Similarly, from the Maxwell's equation (1.1) one derives,

$$\vec{\kappa} \times \vec{E} = \frac{\omega\mu}{c}\vec{H}, \tag{1.94}$$

After rearranging equations (1.93, 1.94),

$$\vec{E} = -\frac{c}{\epsilon\omega}\vec{\kappa} \times \vec{H} = -\frac{1}{\omega}\sqrt{\frac{\mu}{\epsilon}}\vec{\kappa} \times \vec{H}, \tag{1.95}$$

$$\vec{H} = \frac{c}{\omega\mu}\vec{\kappa} \times \vec{E} = \frac{1}{\omega}\sqrt{\frac{\epsilon}{\mu}}\vec{\kappa} \times \vec{E}. \tag{1.96}$$

with $c = \sqrt{\epsilon\mu}$ and $i = \sqrt{-1}$.

In vacuum, $\rho$ is assumed to be zero, therefore, the Maxwell equation for the electric field is written as, $\nabla \cdot \vec{E} = 0$. Hence from the equation (1.91), one finds,

$$\vec{\kappa} \cdot \vec{E} = 0. \tag{1.97}$$

Similarly, from the divergence of the magnetic field, i.e., $\nabla \cdot \vec{B} = 0$, one derives,

$$\vec{\kappa} \cdot \vec{B} = 0. \tag{1.98}$$

Scalar multiplication with $\vec{\kappa}$ provides us,

$$\vec{E} \cdot \vec{\kappa} = \vec{H} \cdot \vec{\kappa} = 0, \tag{1.99}$$

This shows that the electric and magnetic field vectors lie in planes normal to the direction of propagation. From the equation (1.99) one gets,

$$\sqrt{\mu}|\vec{H}| = \sqrt{\epsilon}|\vec{E}|. \tag{1.100}$$

The magnitude of a real vector $|\vec{E}|$ for a general time dependent electromagnetic field, $\vec{E}(\vec{r}, t)$ is represented by $\sqrt{\vec{E} \cdot \vec{E}}$. In Cartesian coordinates the quadratic term, $\vec{E} \cdot \vec{E}$, is written out as,

$$\vec{E} \cdot \vec{E} = E_x E_x + E_y E_y, \tag{1.101}$$

Thus, the Maxwell's theory leads to quadratic terms associated with the flow of energy, that is intensity (or irradiance), $I$, which is defined as the time average of the amount of energy carried by the wave across the unit area perpendicular to the direction of the energy flow in unit time, therefore, the time averaged intensity of the optical field. The unit of intensity is expressed as the joule per square meter per second, ($\text{J m}^{-2}\text{s}^{-1}$), or watt per square meter, ($\text{W m}^{-2}$).

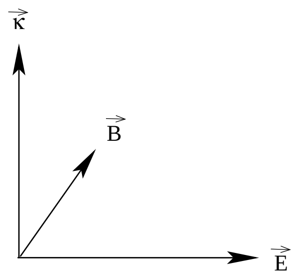


Fig. 1.2 The orthogonal triad of vectors.

It is observed from the equations (1.91-1.94) that in an electromagnetic wave, the field intensities $\vec{E}$, $\vec{H}$, and the unit vector in the propagation direction of the wave $\vec{\kappa}$ form a right handed orthogonal triad of vectors. To be precise, if an electromagnetic wave travels in the positive $x-$axis, the electric and magnetic fields would oscillate parallel to the $y-$ and $z-$axis respectively.

The energy crossing an element of area in unit time is perpendicular to the direction of propagation. In a cylinder with unit cross-sectional area, whose axis is parallel to $\vec{s}$, the amount of energy passing the base of the cylinder in unit time is equal to the energy that is contained in the portion of the cylinder of length $v$. Therefore, the energy flux is equal to $vw$, where

$$w = \frac{\epsilon}{4\pi} \left| \vec{E} \right|^2 = \frac{\mu}{4\pi} \left| \vec{H} \right|^2, \tag{1.102}$$

is the energy density. Hence the energy densities of both electric and magnetic fields are equal everywhere along an electromagnetic wave. The equation (1.102) is derived by considering the equations (1.58), and (1.100).

Thus, the Poynting vector is expressed as,

$$\begin{aligned}\vec{S} &= \frac{c}{4\pi}(\vec{E} \times \vec{H}) = \frac{c}{4\pi}\frac{\vec{\kappa}}{\omega}|\vec{E}||\vec{H}| \\ &= \frac{c}{4\pi}\sqrt{\frac{\epsilon}{\mu}}\frac{\vec{\kappa}}{\omega}|\vec{E}|^2 = \frac{c}{4\pi}\sqrt{\frac{\mu}{\epsilon}}\frac{\vec{\kappa}}{\omega}|\vec{H}|^2. \end{aligned} \tag{1.103}$$

Equation (1.103) relates that the electric and magnetic fields are perpendicular to each other in electromagnetic wave. By combining the two equations (1.102) and (1.103), one finds,

$$\vec{S} = \frac{c}{\sqrt{\epsilon\mu}}\frac{\vec{\kappa}}{\omega}w = \frac{\vec{\kappa}}{\omega}vw, \tag{1.104}$$

with $v = c/\sqrt{\epsilon\mu}$.

The Poynting vector represents the flow of energy, both with respect to its magnitude and direction of propagation. Expressing $\vec{E}$ and $\vec{H}$ in complex terms, then the time-averaged flux of energy is given by the real part of the Poynting vector,

$$\vec{S} = \frac{1}{2}\frac{c}{4\pi}(\vec{E} \times \vec{H}^*), \tag{1.105}$$

in which $^*$ represents for the complex conjugate of ‘ ’.

Thus one may write,

$$\vec{S} = \frac{c}{8\pi}\sqrt{\frac{\epsilon}{\mu}}\frac{\vec{\kappa}}{\omega}(\vec{E} \cdot \vec{E}^*). \tag{1.106}$$

In order to describe the strength of a wave, the amount of energy carried by the wave in unit time across unit area perpendicular to the direction of propagation is used. This quantity, known as intensity of the wave, according to the Maxwell's theory is given in equation (1.101). From the relationship that described in equation (1.103), one may derive the intensity as,

$$\begin{aligned}I = v\langle w \rangle &= \frac{c}{4\pi}\sqrt{\frac{\epsilon}{\mu}}\left\langle \vec{E}^2 \right\rangle \\ &= \frac{c}{4\pi}\sqrt{\frac{\mu}{\epsilon}}\left\langle \vec{H}^2 \right\rangle, \end{aligned} \tag{1.107}$$

where $\langle\,\rangle$ stands for the time average of the quantity.

The Poynting vector, $\vec{S}$, in terms of spherical coordinates is written as,

$$\vec{S} = \frac{c}{8\pi}\sqrt{\frac{\epsilon}{\mu}}\frac{\vec{\kappa}}{\omega}\left(E_\theta E_\theta^* + E_\phi E_\phi^*\right), \tag{1.108}$$

The quantity within the parentheses represents the total intensity of the wave field, known as the first Stokes parameter $I$. Thus the Poynting vector is directly proportional to the first Stokes parameter.

### 1.5.2 *Harmonic time dependence and the Fourier transform*

The Maxwell's equations for an electromagnetic field with time dependence are simplified by specifying a field with harmonic dependence (Smith, 1997). The harmonic time dependent electromagnetic fields are given by,

$$E(\vec{r}, t) = \Re\left[E_0(\vec{r}, \omega)e^{i\omega t}\right], \tag{1.109}$$

$$B(\vec{r}, t) = \Re\left[B_0(\vec{r}, \omega)e^{i\omega t}\right], \tag{1.110}$$

in which $\vec{E}_0$ is a complex vector with Cartesian rectangular components,

$$\begin{aligned} \vec{E}_{0x} &= a_1(\vec{r}, \omega)e^{i\psi_1(\vec{r}, \omega)}, \\ \vec{E}_{0y} &= a_2(\vec{r}, \omega)e^{i\psi_2(\vec{r}, \omega)}, \\ \vec{E}_{0z} &= a_3(\vec{r}, \omega)e^{i\psi_3(\vec{r}, \omega)}, \end{aligned} \tag{1.111}$$

where $a_j(\vec{r}, \omega)$ is the amplitude of the electric wave, $\vec{\kappa}$ the propagation vector, and $j = 1, 2, 3$.

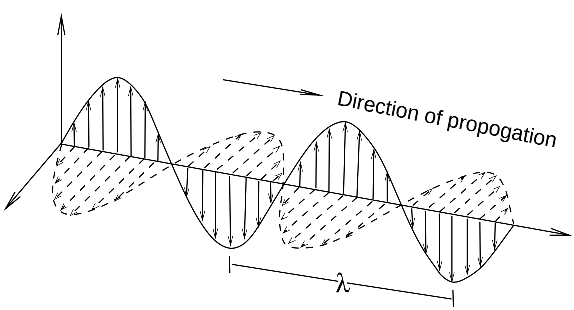


Fig. 1.3 Propagation of a plane electromagnetic wave; the solid and dashed lines represent respectively the electric and magnetic fields.

Figure (1.3) depicts the propagation of a plane electromagnetic wave. For a homogeneous plane wave, the amplitudes, $a_j(\vec{r},\omega)$'s, are constant. Each component of the vector phasor $\vec{E}_0$ has a modulus $a_j$ and argument $\psi_j$ which depend on the position $\vec{r}$ and the parameter $\omega$. The unit of this vector phasor $E_0(\vec{r},\omega)$ for harmonic time dependence is $\mathrm{Vm}^{-1}$. By differentiating the equation (1.109) with respect to the temporal variables, the Maxwell's equation (1.1) turns out to be,

$$\begin{aligned}\nabla\times E(\vec{r},t) &= \nabla\times\Re\left[E_0(\vec{r},\omega)e^{i\omega t}\right]\\ &= -\frac{\partial}{\partial t}\Re\left[E_0(\vec{r},\omega)e^{i\omega t}\right] = \Re\left[-i\omega B_0(\vec{r},\omega)e^{i\omega t}\right].\end{aligned} \tag{1.112}$$

By rearranging this equation (1.112),

$$\nabla\times E_0(\vec{r},\omega)e^{i\omega t} = -i\omega B_0(\vec{r})e^{i\omega t}, \tag{1.113}$$

or,

$$\nabla\times E_0(\vec{r},\omega) = -i\omega B_0(\vec{r},\omega). \tag{1.114}$$

Similarly, the other Maxwell's equations may also be derived,

$$\nabla\times H_0(\vec{r},\omega) = J_0(\vec{r},\omega) + i\omega D_0(\vec{r},\omega), \tag{1.115}$$

$$\nabla\cdot D_0(\vec{r},\omega) = \rho(\vec{r},\omega), \tag{1.116}$$

$$\nabla\cdot B_0(\vec{r},\omega) = 0, \tag{1.117}$$

$$\nabla\cdot J_0(\vec{r},\omega) = -i\omega\rho(\vec{r},\omega). \tag{1.118}$$

These equations (1.115-1.118) are known as the Maxwell's equations for the frequency domain. The Maxwell's equations for the complex vector phasors, $E_0(\vec{r},\omega)$, $B_0(\vec{r},\omega)$, etc., are applied to electromagnetic systems in which the constitutive relations for all materials are time-invariant and linear. The Maxwell's equation with a cosinusoidal excitation are solved to obtain the vector phasors for the electromagnetic field $E(\vec{r},t)$, $B(\vec{r},t)$. For harmonic time dependence, $E(\vec{r},t) = \Re[E_0(\vec{r},\omega)e^{i\omega t}]$, the Hermitian magnitude of a complex vector is, $|\vec{E}_0| = [\vec{E}_0\cdot\vec{E}_0^*]^{1/2}$. If the electromagnetic is harmonic in time, the instantaneous rate at which energy is exchanged between the field and the mechanical motion of the charge is the product of,

$$E(\vec{r},t)\cdot J(\vec{r},t) = \Re\left[E_0(\vec{r},\omega)e^{i\omega t}\right]\cdot\Re\left[J_0(\vec{r},\omega)e^{i\omega t}\right]. \tag{1.119}$$

The real terms are written as,

$$\begin{aligned}\Re\left[E_0(\vec{r},\omega)e^{i\omega t}\right] &= \frac{1}{2}\left[E_0(\vec{r},\omega)e^{i\omega t} + E_0^*(\vec{r},\omega)e^{-i\omega t}\right],\\ \Re\left[J_0(\vec{r},\omega)e^{i\omega t}\right] &= \frac{1}{2}\left[J_0(\vec{r},\omega)e^{i\omega t} + J_0^*(\vec{r},\omega)e^{-i\omega t}\right].\end{aligned} \tag{1.120}$$

The scalar product of these two terms provides,

$$\begin{aligned}E(\vec{r},t)\cdot J(\vec{r},t) &= \frac{1}{4}\left[E_0(\vec{r},\omega)\cdot J_0^*(\vec{r},\omega) + E_0^*(\vec{r},\omega)\cdot J_0(\vec{r},\omega)\right.\\ &\quad \left. + E_0(\vec{r},\omega)\cdot J_0(\vec{r},\omega)e^{2i\omega t} + E_0^*(\vec{r},\omega)\cdot J_0^*(\vec{r},\omega)e^{-2i\omega t}\right]\\ &= \frac{1}{2}\left\{\Re\left[E_0(\vec{r},\omega)\cdot J_0^*(\vec{r})\right] + \Re\left[E_0(\vec{r},\omega)\cdot J_0(\vec{r},\omega)e^{2i\omega t}\right]\right\}.\end{aligned} \tag{1.121}$$

Since the optical frequencies are very large, one can observe their time average[6] over a period of oscillation, $T = 2\pi/\omega$. Hence the time average of the product $\langle E(\vec{r},t)\cdot J(\vec{r},t)\rangle$ is expressed as,

$$\langle E(\vec{r},t)\cdot J(\vec{r},t)\rangle = \frac{1}{2}\Re\left(E_0(\vec{r},\omega)\cdot J_0^*(\vec{r},\omega)\right). \tag{1.122}$$

Similarly, the time average value of the Poynting vector product may also be derived as,

$$\begin{aligned}\langle S(\vec{r},t)\rangle &= \frac{1}{T}\int_0^T \frac{c}{4\pi}\left[E(\vec{r},t)\times H(\vec{r},t)\right]dt\\ &= \frac{c}{4\pi}\frac{1}{T}\int_0^T \frac{1}{4}\left[E_0(\vec{r},\omega)\times H_0^*(\vec{r},\omega) + E_0^*(\vec{r},\omega)\times H_0(\vec{r},\omega)\right.\\ &\quad \left. + E_0(\vec{r},\omega)\times H_0(\vec{r},\omega)e^{2i\omega t} + E_0^*(\vec{r},\omega)\times H_0^*(\vec{r},\omega)e^{-2i\omega t}\right]dt\\ &\simeq \frac{c}{16\pi}\left[E_0(\vec{r},\omega)\times H_0^*(\vec{r},\omega) + E_0^*(\vec{r},\omega)\times H_0(\vec{r},\omega)\right]\\ &= \frac{c}{8\pi}\Re\left[E_0(\vec{r},\omega)\times H_0^*(\vec{r},\omega)\right] = \Re[S_c(\vec{r},\omega)].\end{aligned} \tag{1.123}$$

---

[6] The time average over a time that is large compared with the inverse frequency of the product of the two harmonic time-independent functions $\vec{a}$ and $\vec{b}$, of the same frequency is given by,

$$\left\langle \vec{a}(t)\cdot\vec{b}(t)\right\rangle = \frac{1}{T}\int_0^T \frac{1}{4}\left[\vec{a}e^{i\omega t} + \vec{a}^*e^{-i\omega t}\right]\cdot\left[\vec{b}e^{i\omega t} + \vec{b}^*e^{-i\omega t}\right]dt = \frac{1}{2}\Re\;\; \vec{a}\cdot\vec{b}^* \;\; .$$

Thus the complex Poynting vector is deduced as,

$$S_c(\vec{r},\omega) = \frac{1}{2}\left[E_0(\vec{r},\omega) \times H_0^*(\vec{r},\omega)\right]. \tag{1.124}$$

The real part of this Poynting vector is known as the time average of the Poynting vector. The law of conservation of energy takes a simple form. The complex Poynting theorem is given by,

$$\int_{\mathrm{V}} \left(\frac{1}{2}\vec{E}_0 \cdot \vec{J}_0^*\right) d\mathrm{V} - i\omega \int_{\mathrm{V}} \frac{1}{2}\left(\vec{E}_0 \cdot \vec{D}_0^* - \vec{H}_0^* \cdot \vec{B}_0\right) d\mathrm{V} + \oint_{\mathrm{S}} \vec{S}_c \cdot d\vec{S} = 0. \tag{1.125}$$

For non-conducting medium ($\sigma = 0$), where no mechanical work is done, the time average of equation (1.69) turns out to be,

$$\nabla \cdot \langle S(\vec{r},t)\rangle = 0. \tag{1.126}$$

By integrating this equation (1.127) over an arbitrary volume which contains no absorber or radiator of energy, one obtains after applying Gauss' theorem,

$$\oint_{\mathrm{S}} \langle S(\vec{r},t)\rangle \cdot \vec{n} d\vec{S} = 0, \tag{1.127}$$

in which $\vec{n}$ is the outward normal to the surface.

Thus the averaged total flux of energy through any closed surface is zero. The time average of the electric energy density is derived as,

$$\begin{aligned} \langle w_e \rangle &= \frac{1}{T}\int_0^T \frac{\epsilon}{8\pi}\vec{E}^2 dt \\ &= \frac{\epsilon}{8\pi}\frac{1}{T}\int_0^T \frac{1}{4}\left[\vec{E}_0^2 e^{2i\omega t} + \vec{E}_0 \cdot \vec{E}_0^* + \vec{E}_0{}^{*2} e^{-2i\omega t}\right] dt. \end{aligned} \tag{1.128}$$

Since $T$ is assumed to be large, the integrals involving the exponentials are neglected. Therefore, one gets,

$$\langle w_e \rangle = \frac{\epsilon}{16\pi}\vec{E}_0 \cdot \vec{E}_0^* = \frac{\epsilon}{16\pi}|\vec{E}_0|^2. \tag{1.129}$$

Similarly, the time average of the magnetic energy density is also derived as,

$$\langle w_m \rangle = \frac{\mu}{16\pi}\vec{H}_0 \cdot \vec{H}_0^* = \frac{\mu}{16\pi}|\vec{H}_0|^2. \tag{1.130}$$

therefore, the difference in the time-average energies stored in the electric and magnetic fields with the volume,

$$\langle w_e \rangle - \langle w_m \rangle = \frac{\epsilon}{16\pi}|\vec{E}_0|^2 - \frac{\mu}{16\pi}|\vec{H}_0|^2 = 0. \tag{1.131}$$

Hence $\langle w_e \rangle = \langle w_m \rangle$. The total energy $W$ is given by $W = 2\langle w_e \rangle = 2\langle w_m \rangle$.

A time dependent field is a linear superposition of fields that vary harmonically with time at different frequencies. This relationship is known as Fourier transform (FT; Bracewell, 1965). The free wave equation is a linear homogeneous differential equation, therefore any linear combination of its solution is a solution as well. From the Fourier equation (see Appendix B) associated with the harmonic function of frequency $\omega$, the FT of an electric field is expressed as,

$$\widehat{E}_0(\vec{r},\omega) = \int_{-\infty}^{\infty} E(\vec{r},t)e^{-i\omega t}dt, \tag{1.132}$$

in which, the functions satisfy the relation, $f(t) \leftrightarrow \widehat{f}(\omega)$ and the notation, $\widehat{}$ stands for a Fourier transform (FT) of a particular physical quantity,

The Fourier transform of $\widehat{E}_0(\vec{r},\omega)$ is a complex function of the variable $\omega$ and has the units of the electric field (Vm$^{-1}$) per unit frequency, i.e., $(V/m)/Hz$. By invoking the principle of Fourier transform of a temporal derivative of a function, $df(t)/dt \leftrightarrow i\omega\widehat{f}(\omega)$, the curl of the equation (1.133) is applied on Maxwell's equation, thus,

$$\begin{aligned}\nabla\times\widehat{E}_0(\vec{r},\omega) &= \int_{-\infty}^{\infty} \nabla\times E(\vec{r},t)e^{-i\omega t}dt \\ &= \int_{-\infty}^{\infty}\left[\frac{\partial}{\partial t}B(\vec{r},t)\right]e^{-i\omega t}dt.\end{aligned} \tag{1.133}$$

The Fourier transform of the magnetic field is given by,

$$\widehat{B}_0(\vec{r},\omega) = \int_{-\infty}^{\infty} B(\vec{r},t)e^{-i\omega t}dt, \tag{1.134}$$

therefore, the equation (1.134) turns out to be,

$$\nabla\times\widehat{E}_0(\vec{r},\omega) = -i\omega\widehat{B}_0(\vec{r},\omega). \tag{1.135}$$

Similar operations may be applied with other Maxwell's equations in order to derive their Fourier transforms as well,

$$\nabla\times\widehat{H}_0(\vec{r},\omega) = \widehat{J}_0(\vec{r},\omega) + i\omega\widehat{D}_0(\vec{r},\omega), \tag{1.136}$$

$$\nabla\cdot\widehat{D}_0(\vec{r},\omega) = \widehat{\rho}(\vec{r},\omega), \tag{1.137}$$

$$\nabla \cdot \widehat{B}_0(\vec{r}, \omega) = 0, \tag{1.138}$$

$$\nabla \cdot \widehat{J}_0(\vec{r}, \omega) = -i\omega\widehat{\rho}(\vec{r}, \omega). \tag{1.139}$$

These are the Maxwell's equations for the Fourier transform of the electromagnetic fields, $E_0(\vec{r}, \omega)$, $B_0(\vec{r}, \omega)$, etc.

Let the integral of the complex Poynting vector be examined,

$$\frac{2}{\pi}\int_0^\infty \vec{n} \cdot \Re[S_c(\vec{r}, \omega)]d\omega = \frac{2}{\pi}\int_0^\infty \vec{n} \cdot \left\{\frac{c}{8\pi}\Re\left[\widehat{E}_0(\vec{r}, \omega) \times \widehat{H}_0^*(\vec{r}, \omega)\right]\right\} d\omega. \tag{1.140}$$

By using Parseval's theorem (see appendix II), one finds,

$$\begin{aligned}\int_{-\infty}^\infty \vec{n} \cdot S(\vec{r}, t)dt &= \int_{-\infty}^\infty \vec{n} \cdot \frac{c}{4\pi}[E(\vec{r}, t) \times H(\vec{r}, t)]dt \\ &= \frac{2}{\pi}\int_0^\infty \vec{n} \cdot \left\{\frac{c}{8\pi}\Re\left[\widehat{E}_0(\vec{r}, \omega) \times \widehat{H}_0^*(\vec{r}, \omega)\right]\right\} d\omega \\ &= \frac{2}{\pi}\int_0^\infty \vec{n} \cdot \left\{\Re\left[\widehat{S}_c(\vec{r}, \omega)\right]\right\} d\omega. \end{aligned} \tag{1.141}$$

The LHS of equation (1.142) is the total electromagnetic energy passing through a unit area of surface with the unit normal $\vec{n}$, while the integrand on the RHS,

$$\frac{2}{\pi}\vec{n} \cdot \left\{\Re\left[\widehat{\vec{S}}_c(\vec{r}, \omega)\right]\right\},$$

is the energy passing through a unit area of this surface per unit frequency, $(\text{J/m}^2)$/Hz and is known as energy spectral density (Lang and Kohn, 1971).