MODULI SPACE OF KILLING HELICES OF LOW ORDERS ON A COMPLEX SPACE FORM

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We give a report on the moduli space of helices of proper order less than 5 which are generated by some Killing vector fields on a complex space form from the viewpoint of the length spectrum.

1. Introduction

In this note we give a summary of my work concerning essential Killing helices of low orders on a non-flat complex space form, which is either a complex projective space or a complex hyperbolic space. A smooth curve γ parameterized by its arclength on a Riemannian manifold M is said to be a *helix of proper order* d if it satisfies the following system of ordinary differential equations

$$\nabla_{\dot{\gamma}} Y_j = -\kappa_{j-1} Y_{j-1} + \kappa_j Y_{j+1}, \quad 1 \leqq j \leqq d, \tag{1.1}$$

with positive constants $\kappa_1, \ldots, \kappa_{d-1}$ and an orthonormal system $\{Y_1 = \dot{\gamma}, Y_2, \ldots, Y_d\}$ of vector fields along γ. Here $\kappa_0 = \kappa_d = 0$, and Y_0, Y_{d+1} are null vector fields along γ. These constants $\kappa_1, \ldots, \kappa_{d-1}$ are called the *geodesic curvatures* of γ and the system $\{Y_i\}$ the *Frenet frame* of γ. We call a helix *Killing* if it is generated by some Killing vector field on M. On real space forms, which are standard spheres, Euclidean spaces and real hyperbolic spaces, all helices are Killing and lengths of closed helices are given by their geodesic curvatures. But on a complex space form the situation is different. We study the difference on laminations on the moduli spaces of Killing helices which are induced by the length spectrum.

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- Proceedings of the 8th International Workshop on Complex Structures and Vector Fields

2. Moduli space of Killing helices and length spectrum

We say two helices γ_1, γ_2 on a Riemannian manifold M are *congruent* to each other if there exist an isometry φ of M and a constant t_0 satisfying $\gamma_2(t) = \varphi \circ \gamma_1(t + t_0)$ for all t. We denote by $\mathcal{K}_d(M)$ the set of all congruence classes of Killing helices of proper order d on M. We put $\mathcal{K}(M) = \bigcup_{d=1}^{\infty} \mathcal{K}_d(M)$ and call it the *moduli space* of Killing helices. On a real space form $\mathbb{R}M^n$, as helices are classified by their geodesic curvatures, we see $\mathcal{K}_d(\mathbb{R}M^n)$ is bijective to $(0,\infty)^{d-1}$, the $(d-1)$ product of half lines, when $d \leq n$. But on a non-flat complex space form, as isometries are either holomorphic or anti-holomorphic, the moduli space of Killing helices is not so simple. For a helix on a Kähler manifold (M, J) with Frenet frame $\{Y_i\}$, we define its *complex torsions* τ_{ij} $(1 \leq i < j \leq d)$ by $\tau_{ij} = \langle Y_i, JY_j \rangle$. As was pointed out in [8], on a non-flat complex space form $\mathbb{C}M^n$ a helix γ is Killing if and only if all its complex torsions are constant along γ.

We call a helix γ *closed* if there is positive t_c with $\gamma(t + t_c) = \gamma(t)$ for all t. The minimum positive t_c with this property is called the length of γ and is denoted by length(γ). When γ is not closed we say it is *open* and put length$(\gamma) = \infty$. The *length spectrum* $\mathcal{L} : \mathcal{K}(M) \to (0, \infty]$ is defined by $\mathcal{L}\big([\gamma]\big) = \text{length}(\gamma)$, where $[\gamma]$ denotes the congruence class containing a helix γ. For the sake of simplicity we denote a restriction of $\mathcal{L}$ onto a subset of $\mathcal{K}(M)$ also by $\mathcal{L}$.

3. Moduli space of helices on a real space form

For the sake of comparison, we here show some properties on length spectrum of helices on a real space form $\mathbb{R}M^n(c)$ of constant sectional curvature c. The length spectrum $\mathcal{L} : \mathcal{K}_2\big(\mathbb{R}M^n(c)\big) \cong (0,\infty) \to (0,\infty]$ of circles of positive geodesic curvature, which are helices of proper order 2, is given as $\mathcal{L}(\kappa) = 2\pi/\sqrt{\kappa^2 + c}$, where we read it infinity when $\kappa^2 + c \leq 0$. Thus if we induce the canonical Euclidean differential structure on $\mathcal{K}_2\big(\mathbb{R}M^n(c)\big)$, we see the length spectrum is smooth on this moduli space.

For about the moduli space of helices of proper order 3 on $\mathbb{R}M^n(c)$ $(n \geq 3)$, the feature depends on sectional curvature c. All helices of proper order 3 on a Euclidean space $\mathbb{R}^n$ are unbounded. For a standard sphere $S^n(c)$ of constant sectional curvature c, we have a canonical foliation $\{\mathcal{G}_\alpha\}_{\alpha \in (1,\infty)}$ on $\mathcal{K}_3\big(S^n(c)\big)$ which is related with the length spectrum and is given as

$$\mathcal{G}_\alpha = \left\{ [\gamma_{\kappa_1,\kappa_2}] \;\middle|\; \kappa_1^2 + (\kappa_2 - \alpha\sqrt{c}/2)^2 = c(\alpha^2 - 1)/4 \right\},$$

where $[\gamma_{\kappa_1,\kappa_2}]$ denotes the congruence class of helices of proper order 3 on $S^n(c)$ with geodesic curvatures κ_1, κ_2 (see Figure 1).

Theorem 3.1. *The length spectrum $\mathcal{L} : \mathcal{K}_3\big(S^n(c)\big) \to (0,\infty]$ is constant on each leaf. Each leaf is set theoretically maximal with respect to this property. A leaf $\mathcal{G}_\alpha$ consists of congruence classes of closed helices if and only if α and $\sqrt{\alpha^2-1}$ are rational.*

For a real hyperbolic space $H^n(c)$ of constant sectional curvature c, we have a canonical foliation $\{\mathcal{G}_\alpha\}_{\alpha\in(-\infty,\infty)}$ on $\mathcal{K}_3\big(H^n(c)\big)$ which is given as

$$\mathcal{G}_\alpha = \big\{ [\gamma_{\kappa_1,\kappa_2}] \bigm| \kappa_1^2 + (\kappa_2 - \alpha\sqrt{|c|}/2)^2 = -c(\alpha^2+1)/4 \big\}.$$

We should note that the moduli space $\mathcal{BK}_3\big(H^n(c)\big)$ of bounded helices of proper order 3 on $H^n(c)$ is given as $\big\{ [\gamma_{\kappa_1,\kappa_2}] \bigm| \kappa_1^2+(\kappa_2-\sqrt{|c|}/2)^2 > -c/2 \big\}$. On this space the foliation $\{\mathcal{G}_\alpha\}_{\alpha\in(1,\infty)}$ satisfies the same property as of the foliation on $\mathcal{K}_3\big(S^n\big)$. In both cases of a standard sphere and of a real hyperbolic space, these foliations can be naturally extend to a foliation or a lamination on $\mathcal{K}_2\big(\mathbb{R}M^n(c)\big) \cup \mathcal{K}_3\big(\mathbb{R}M^n(c)\big)$.

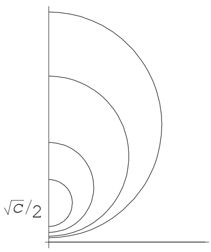

Fig. 1. Foliation on $\mathcal{K}_3\big(S^n(c)\big)$

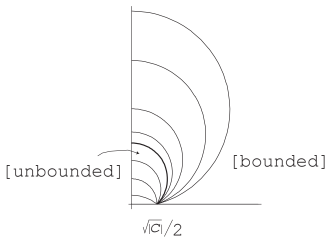

Fig. 2. Foliation on $\mathcal{K}_3\big(H^n(c)\big)$

4. Moduli space of circles on a complex space form

We now study the moduli space of helices on a non-flat complex space form. On a Kähler manifold, the complex torsion τ_{12} of each circle γ is always constant along γ, because

$$\tau_{12}' = \langle \nabla_{\dot\gamma}\dot\gamma, JY_2\rangle + \langle \dot\gamma, J\nabla_{\dot\gamma}Y_2\rangle = \kappa_1\big(\langle Y_2, JY_2\rangle - \langle \dot\gamma, J\dot\gamma\rangle\big) = 0.$$

Therefore we see the moduli space $\mathcal{K}_2\big(\mathbb{C}M^n\big)$ of circles of positive geodesic curvature on a non-flat complex space form is set theoretically bijective to the product $(0,\infty)\times[0,1]$ when $n \geq 2$. In this section we suppose $n \geq 2$ and we shall denote by $[\gamma_{\kappa,\tau}]$ the congruence class of circles with geodesic curvature κ and complex torsion $\tau_{12} = \tau$ on a complex space form $\mathbb{C}M^n(c)$ of constant holomorphic sectional curvature c.

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- Proceedings of the 8th International Workshop on Complex Structures and Vector Fields

For a complex projective space $\mathbb{C}P^n(c)$, we have a lamination structure $\{\mathcal{F}_\mu\}_{\mu\in[0,1)\cup\{\star\}}$ on $\mathcal{K}_2\big(\mathbb{C}P^n(c)\big)$ defined by

$$\mathcal{F}_\mu = \begin{cases} \big\{[\gamma_{\kappa,0}] \bigm| \kappa>0\big\}, & \text{if } \mu=0,\\ \big\{[\gamma_{\kappa,\tau}] \bigm| 3\sqrt{3}c\kappa\tau(4\kappa^2+c)^{-3/2}=\mu\big\}, & \text{if } 0<\mu<1,\\ \big\{[\gamma_{\kappa,1}] \bigm| \kappa>0\big\}, & \text{if } \mu=\star. \end{cases}$$

Theorem 4.1. *The length spectrum $\mathcal{L}:\mathcal{K}_2\big(\mathbb{C}P^n(c)\big)\to(0,\infty]$ is smooth on each leaf with respect to the canonical induced Euclidean differential structure. Each leaf is maximal with respect to this property.*

1) *The leaf $\mathcal{F}_\star$ consists of congruence classes of closed circles satisfying $\mathcal{L}([\gamma_{\kappa,1}])=2\pi/\sqrt{\kappa^2+c}$.*
2) *The leaf $\mathcal{F}_0$ also consists of congruence classes of closed circles satisfying $\mathcal{L}([\gamma_{\kappa,0}])=4\pi/\sqrt{4\kappa^2+c}$.*
3) *The leaf $\mathcal{F}_\mu$ $(0<\mu<1)$ consists of congruence classes of closed circles if and only if $\mu=q(9p^2-q^2)(3p^2+q^2)^{-3/2}$ with some relatively prime positive integers p,q satisfying $p>q$. On this leaf $\mathcal{L}([\gamma_{\kappa,\tau}])=2\delta(p,q)\pi\sqrt{(3p^2+q^2)/\{3(4\kappa^2+c)\}}$, where $\delta(p,q)=1$ when the product pq is odd and $\delta(p,q)=2$ when pq is even.*

For a complex hyperbolic space $\mathbb{C}H^n(c)$, we also have a lamination structure $\{\mathcal{F}_\mu\}_{\mu\in[0,\infty]\cup\{\star\}}$ on $\mathcal{K}_2\big(\mathbb{C}H^n(c)\big)$ defined by

$$\mathcal{F}_\mu = \begin{cases} \big\{[\gamma_{\kappa,0}] \bigm| \kappa>0\big\}, & \text{if } \mu=0,\\ \big\{[\gamma_{\kappa,\tau}] \bigm| 3\sqrt{3}|c|\kappa\tau|4\kappa^2+c|^{-3/2}=\mu\big\}, & \text{if } 0<\mu<\infty,\\ \big\{\big[\gamma_{\sqrt{|c|}/2,\tau}\big] \bigm| 0<\tau<1\big\}, & \text{if } \mu=\infty,\\ \big\{[\gamma_{\kappa,1}] \bigm| \kappa>0\big\}, & \text{if } \mu=\star. \end{cases}$$

This lamination has the same properties as of the lamination on $\mathcal{K}_2\big(\mathbb{C}P^n\big)$ if we restrict ourselves on the moduli space

$$\mathcal{BK}_2\big(\mathbb{C}H^n(c)\big)=\big\{\,[\gamma_{\kappa,\tau}] \bigm| 0\le\tau<\nu(\kappa)\,\big\}\cup\big\{\,[\gamma_{\kappa,1}] \bigm| \kappa>\sqrt{|c|}\,\big\}$$

of bounded circles on $\mathbb{C}H^n(c)$ (see [2]). Here $\nu:(0,\infty)\to[0,1]$ is given by

$$\nu(\kappa)=\begin{cases} 0, & \text{if } 0<\kappa\le\sqrt{|c|}/2,\\ (4\kappa^2+c)^{3/2}/(3\sqrt{3}\,|c|\kappa), & \text{if } \sqrt{|c|}/2<\kappa<\sqrt{|c|},\\ 1, & \text{if } \kappa\ge\sqrt{|c|}. \end{cases}$$

In view of the features of these laminations on the moduli spaces of circles, we find the set $\{\,[\gamma_{\kappa,1}] \mid \kappa>0\,\}$ of congruence classes of *trajectories*

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for Kähler magnetic fields is quite different from other part of $\mathcal{K}_2\bigl(\mathbb{C}M^n(c)\bigr)$. Since each trajectory lies on some totally geodesic $\mathbb{C}M^1$ and other circles do not lie on $\mathbb{C}M^1$, we shall classify helices by this property. We call a helix on $\mathbb{C}M^n$ of proper order $2k-1$ or $2k$ *essential* if it lies on some totally geodesic $\mathbb{C}M^k$. We denote by $\mathcal{EK}_d\bigl(\mathbb{C}M^n(c)\bigr)$ the set of all congruence classes of essential Killing helices of proper order d on $\mathbb{C}M^n(c)$.

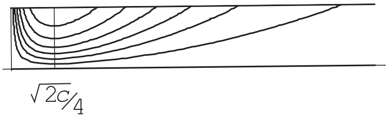

Fig. 3. Lamination on $\mathcal{K}_2\bigl(\mathbb{C}P^n(c)\bigr)$

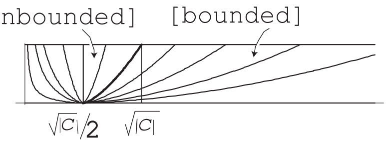

Fig. 4. Lamination on $\mathcal{K}_2\bigl(\mathbb{C}H^n(c)\bigr)$

5. Moduli spaces of Killing helices of orders less than 5 on a complex space form

Though all circles on a non-flat complex space form are Killing, helices of proper order greater than 2 are not necessarily Killing. Computing τ'_{ij} by using (1.1) we see a helix of proper order d on $\mathbb{C}M^n$ is Killing if and only if

$$-\kappa_{i-1}\tau_{i-1j}+\kappa_i\tau_{i+1j}-\kappa_{j-1}\tau_{ij-1}+\kappa_j\tau_{ij+1}=0,\quad 1\le i<j\le d, \tag{5.1}$$

where we set $\tau_{0k}=\tau_{kk}=\tau_{kd+1}=0$ ([7]). Applying these relations to a helix of proper order 3 on $\mathbb{C}M^n$ $(n\ge 2)$, we find it is Killing if and only if its geodesic curvatures and complex torsions satisfy $\tau_{13}=0$ and $\kappa_1\tau_{23}=\kappa_2\tau_{12}$. If we consider the initial frame we find the following.

1) A helix is essential Killing if and only if $\tau_{12}=\pm\kappa_1\big/\sqrt{\kappa_1^2+\kappa_2^2}$, $\tau_{13}=0$, $\tau_{23}=\pm\kappa_2\big/\sqrt{\kappa_1^2+\kappa_2^2}$, where the double signs take the same signature.
2) When $n\ge 3$, a helix is Killing if and only if its complex torsions satisfy $\tau_{12}=\kappa_1\tau$, $\tau_{13}=0$, $\tau_{23}=\kappa_2\tau$ with some τ satisfying $|\tau|\le 1\big/\sqrt{\kappa_1^2+\kappa_2^2}$.

Thus we see the moduli space $\mathcal{EK}_3\bigl(\mathbb{C}M^n\bigr)$ of essential Killing helices of proper order 3 is bijective to a quater of a plane $(0,\infty)^2$ and the moduli space $\mathcal{K}_3\bigl(\mathbb{C}M^n\bigr)$ is bijective to the set $(0,\infty)^2\times[0,1]$.

When we consider Killing helices of proper order 4, the relations (5.1) turn to $\kappa_1\tau_{23}+\kappa_3\tau_{14}=\kappa_2\tau_{12}$, $\kappa_3\tau_{23}+\kappa_1\tau_{14}=\kappa_2\tau_{34}$ and $\tau_{13}=\tau_{24}=0$. Considering the initial frame we find a helix of proper order 4 on $\mathbb{C}M^n$ $(n\ge 2)$ is essential Killing if and only if its complex torsions satisfy one of the following conditions:

i) $\tau_{12} = \tau_{34} = \pm(\kappa_1 + \kappa_3)\Big/\sqrt{\kappa_2^2 + (\kappa_1 + \kappa_3)^2}, \quad \tau_{13} = \tau_{24} = 0,$
$\tau_{23} = \tau_{14} = \pm\kappa_2\Big/\sqrt{\kappa_2^2 + (\kappa_1 + \kappa_3)^2},$

ii) $\tau_{12} = -\tau_{34} = \pm(\kappa_1 - \kappa_3)\Big/\sqrt{\kappa_2^2 + (\kappa_1 - \kappa_3)^2}, \quad \tau_{13} = \tau_{24} = 0,$
$\tau_{23} = -\tau_{14} = \pm\kappa_2\Big/\sqrt{\kappa_2^2 + (\kappa_1 - \kappa_3)^2}.$

Here the double signs take the same signatures in each case. Thus the moduli space $\mathcal{EK}_4(\mathbb{C}M^n)$ of essential Killing helices of proper order 4 is bijective to the set $(0,\infty)^2 \times \big(\mathbb{R}\setminus\{0\}\big)$. When $\kappa_3 > 0$, a point $(\kappa_1, \kappa_2, \kappa_3)$ corresponds to the congruence class of Killing helices with geodesic curvatures κ_1, κ_2, κ_3 and complex torsions in the condition i), and when $\kappa_3 < 0$, it corresponds to the congruence class of Killing helices with geodesic curvatures κ_1, κ_2, $-\kappa_3$ and complex torsions in the condition ii). Thus we find moduli spaces of essential Killing helices of proper order less than 5 on $\mathbb{C}M^n$, which are

$$\mathcal{EK}_1(\mathbb{C}M^n) = \{0\}, \qquad \mathcal{EK}_2(\mathbb{C}M^n) = (0,\infty),$$
$$\mathcal{EK}_3(\mathbb{C}M^n) = (0,\infty)^2, \qquad \mathcal{EK}_4(\mathbb{C}M^n) = (0,\infty)^2 \times \big(\mathbb{R}\setminus\{0\}\big),$$

set theoretically form a “building structure” like $\mathcal{K}(\mathbb{R}M^n)$. Since the moduli spaces $\mathcal{K}_2(\mathbb{C}M^n)$, $\mathcal{K}_3(\mathbb{C}M^n)$ do not form such structure in canonical sense, we restrict ourselves on moduli spaces of essential Killing helices.

6. Lamination on moduli space of essential Killing helices

We here consider laminations on the moduli spaces of essential Killing helices of proper orders 3 and 4. For $\mathbb{C}P^n(c)$ we have a foliation $\{\mathcal{G}_\mu\}_{\mu\in(-1,1)}$ corresponding to the length spectrum on $\mathcal{EK}_3\big(\mathbb{C}M^n(c)\big)$: It is given as

$$\mathcal{G}_\mu = \left\{ [\gamma(\kappa_1,\kappa_2)] \;\middle|\; \frac{8(\kappa_1^2+\kappa_2^2)^2 + c(9\kappa_1^2 - 18\kappa_2^2)}{(\kappa_1^2+\kappa_2^2)^{1/2}(4\kappa_1^2+4\kappa_2^2+3c)^{3/2}} = \mu \right\},$$

where $[\gamma(\kappa_1,\kappa_2)]$ denotes the congruence class containing essential Killing helices of proper order 3 with geodesic curvatures κ_1, κ_2 (see Figure 5).

Theorem 6.1. *The length spectrum $\mathcal{L} : \mathcal{EK}_3\big(\mathbb{C}P^n(c)\big) \to (0,\infty]$ is smooth on each leaf $\mathcal{G}_\mu$ with respect to the canonical induced Euclidean differential structure. Each leaf is maximal with respect to this property.*

1) *The leaf $\mathcal{G}_0$ consists of congruence classes of closed helices of proper order 3 satisfying $\mathcal{L}([\gamma(\kappa_1,\kappa_2)]) = 2\sqrt{3c}\,\pi\big/\sqrt{4\kappa_1^2+4\kappa_2^2+3c}$.*
2) *The leaf $\mathcal{G}_\mu$ $(\mu \neq 0)$ consists of closed helices of proper order 3 if and only if $\mu = \pm q(9p^2-q^2)(3p^2+q^2)^{-3/2}$ with some relatively prime positive integers p, q satisfying $p > q$.*

By this theorem we find there are embeddings of $\mathcal{K}_2\big(\mathbb{C}P^n\big) \setminus \mathcal{EK}_2\big(\mathbb{C}P^n\big)$ into $\mathcal{EK}_3\big(\mathbb{C}P^n\big)$ with respect to the induced Euclidean differential structures which preserve the foliation structure. In other words, there is a two-to-one continuous map $\mathcal{EK}_3\big(\mathbb{C}P^n\big) \to \mathcal{K}_2\big(\mathbb{C}P^n\big) \setminus \mathcal{EK}_2\big(\mathbb{C}P^n\big)$ which preserves the foliation structure. The reader should compare Figures 1 and 5. He can find difference between their features near the half line $\big\{(\kappa, 0) \bigm| \kappa > 0\big\}$.

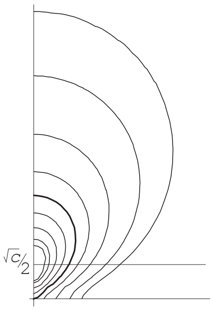

Fig. 5. Folation on $\mathcal{EK}_3\big(\mathbb{C}P^n\big)$

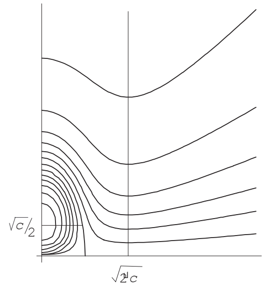

Fig. 6. Foliation on $\mathcal{MK}_4\big(\mathbb{C}P^n\big)$

On $\mathcal{EK}_4\big(\mathbb{C}P^n\big)$ we also have a foliation $\big\{\mathcal{H}_\mu\big\}_{\mu\in(-1,1)}$ corresponding to the length spectrum. It is an extension of the foliation on $\mathcal{EK}_3\big(\mathbb{C}P^n\big)$ and we have a projection

$$\mathcal{EK}_3\big(\mathbb{C}P^n\big) \cup \mathcal{EK}_4\big(\mathbb{C}P^n\big) \cong (0,\infty)^2 \times \mathbb{R} \to \mathcal{K}_2\big(\mathbb{C}P^n\big) \setminus \mathcal{EK}_2\big(\mathbb{C}P^n\big)$$

which preserves the foliation structure. In order to see this foliation visually we take a plane $\mathcal{MK}_4\big(\mathbb{C}P^n(c)\big) = \big\{\, [\gamma(\kappa_1, \kappa_2, -\kappa_1)] \bigm| \kappa_1, \kappa_2 > 0 \,\big\} \subset \mathcal{EK}_4\big(\mathbb{C}P^n(c)\big)$ which consists of congruence classes of Killing helices of proper order 4 with complex torsions $\tau_{12} = -\tau_{34} = 0$, $\tau_{23} = \tau_{14} = \pm 1$. We note that except on this plane absolute values of complex torsions of essential Killing helices of proper order 4 are less than 1. Leaves of the foliation $\big\{\mathcal{H}\big\}_\mu$ are transversal to this plane (see Figure 6).

For $\mathbb{C}H^n$ we have a lamination $\big\{\mathcal{G}_\mu\big\}_{\mu\in(-\infty,\infty)}$ on $\mathcal{EK}_3\big(\mathbb{C}H^n\big)$ corresponding to the length spectrum which is given by just the same manner as on $\mathcal{EK}_3\big(\mathbb{C}P^n\big)$. It has the same property as of the foliation on $\mathcal{EK}_3\big(\mathbb{C}P^n\big)$ if we restrict ourselves on the moduli space $\mathcal{BEK}_3\big(\mathbb{C}H^n(c)\big)$ of bounded essential Killing helices of proper order 3 which is given by

$$\left\{ [\gamma(\kappa_1,\kappa_2)] \;\middle|\; \begin{array}{l} 4\kappa_1^2 + 4\kappa_2^2 + 3c > 0, \\[6pt] \dfrac{|8(\kappa_1^2+\kappa_2^2)^2 + c(9\kappa_1^2 - 18\kappa_2^2)|}{(\kappa_1^2+\kappa_2^2)^{1/2}(4\kappa_1^2+4\kappa_2^2+3c)^{3/2}} < 1 \end{array} \right\}.$$

This moduli space has a cusp at a point $(\sqrt{6c}/3, \sqrt{3c}/6)$ and the lamination on $\mathcal{EK}_3\big(\mathbb{C}H^n(c)\big)$ has a singularity at this point. We can extend this lamination onto the moduli space $\mathcal{EK}_4\big(\mathbb{C}H^n(c)\big)$. We should note that some essential Killing helices of proper order 4 are obtained as trajectories for canonical magnetic fields on real hypersurfaces of type A_1 in $\mathbb{C}M^n$ (see [3]).

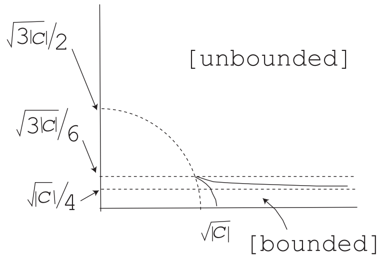

Fig. 7. $\mathcal{K}_3\big(\mathbb{C}H^n(c)\big)$

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