# Chapter 1

# Introduction

"In his famous survey of mathematical ideas, F. Klein championed 'the fusion of arithmetic with geometry' as a major unifying principle of mathematics. Klein's seminal analysis of the structure and history of mathematics brings to light two major processes by which mathematics grows and becomes organized. They may be aptly referred to as the algebraic and the geometric. The one emphasizes algebraic structure while the other emphasizes geometric interpretation. Klein's analysis shows one process alternatively dominating the other in the historical development of mathematics. But there is no necessary reason that the two processes should operate in mutual exclusion. Indeed, each process is undoubtedly grounded in one of two great capacities of the human mind: the capacity for language and the capacity for spatial perception. From the psychological point of view, then, the fusion of algebra with geometry is so fundamental that one could well say, 'Geometry without algebra is dumb! Algebra without geometry is blind!' "

— D. Hestenes, 1984.

## 1.1 Leibniz's dream

The algebraization of geometry started with R. Descartes' introduction of coordinates into geometry. This is one of the greatest achievements in human history, in that it is a key step from qualitative description to quantitative analysis. However, coordinates are sequences of numbers, they have no geometric meaning by themselves.

Co-inventor of calculus, the great mathematician G. Leibniz, once dreamed of having a geometric calculus dealing directly with geometric objects rather than with sequences of numbers. His dream is to have an algebra that is so close to geometry that every expression in it has a clear geometric meaning of being either a geometric object or a geometric relation between geometric objects, that the algebraic manipulations among the expressions, such as addition, subtraction, multiplication and division, correspond to geometric transformations. Such an algebra, if exists, is rightly called *geometric algebra*, and its elements called *geometric numbers*.

Then what is a geometry? In his classic book *Three-dimensional Geometry and Topology* [180], Fields Medalist W. Thurston wrote: " Do we think of a geometry as a space equipped with such notions as lines and planes, or as a space equipped with a notion of congruence, or as a space equipped with either a metric or a Riemannian metric? There are deficiencies in all of these approaches. The best way to think of a geometry, really, is to keep in mind these different points of view all at the same time."

Thurston defined a geometry as a space $X$ equipped with a group $G$ of congruences. Technically, $X$ is a manifold that is connected and simply connected, and $G$ is a Lie group of diffeomorphisms of $X$, whose action on $X$ is transitive and whose point stabilizers are compact. For a classical geometry, the geometric space $X$ is embedded in a real vector space $\mathcal{V}^n$, and the transformation group $G$ of $X$ is a subgroup of the general linear group $GL(\mathcal{V}^n)$.

To search for a geometric algebra dreamed of by Leibniz, we start with the most fundamental geometry, Euclidean plane geometry. The geometric space is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Points are represented by vectors starting from the origin of $\mathbb{R}^2$, so they can be added, and be multiplied with a scalar of $\mathbb{R}$. The transformation group is the *Euclidean group*, where each element can be decomposed into two parts $(\mathbf{R}, \mathbf{t})$, such that

$$\mathbf{x} \mapsto \mathbf{R}\mathbf{x} + \mathbf{t}, \ \forall \mathbf{x} \in \mathbb{R}^2 \tag{1.1.1}$$

is the group action on the geometric space, with $\mathbf{R}$ being a 2D rotation matrix, and $\mathbf{t} \in \mathbb{R}^2$ being a vector of translation.

Two vectors can be multiplied using the complex numbers product, if $(x_1, x_2) \in \mathbb{R}^2$ is identified with $x_1 + ix_2 \in \mathbb{C}$:

$$\mathbf{x}\mathbf{y} = (x_1, x_2)(y_1, y_2) = (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1). \tag{1.1.2}$$

Although the result is still a vector (complex number), it lacks invariance under the Euclidean group. In other words, the geometric information encoded in the product is unable to be separated from the interference of the reference coordinate frame. Regarding this aspect, we can say that the complex numbers product is *geometrically meaningless*, because its geometric interpretation is always related to the real axis of the specific complex numbers coordinate system of the 2D plane.

If we change the product of $\mathbf{x}, \mathbf{y}$ to

$$\overline{\mathbf{x}}\mathbf{y} := (x_1, -x_2)(y_1, y_2) = (x_1y_1 + x_2y_2, x_1y_2 - x_2y_1), \tag{1.1.3}$$

then under any 2D rotation $f : \mathbf{x} \mapsto \mathbf{x}e^{i\theta}$ centered at the origin, $(\overline{\mathbf{x}e^{i\theta}})(\mathbf{y}e^{i\theta}) = \overline{\mathbf{x}}\mathbf{y}$ is invariant, in the sense that

$$\overline{f(\mathbf{x})}f(\mathbf{y}) = \overline{\mathbf{x}}\mathbf{y}. \tag{1.1.4}$$

Since any reflection in the plane with respect to a line passing through the origin is the composition of a rotation and the complex conjugate $\mathbf{z} \mapsto \overline{\mathbf{z}}$ for $\mathbf{z} \in \mathbb{C}$, by $\overline{\overline{\mathbf{x}}e^{i\theta}}\,(\overline{\mathbf{y}}e^{i\theta}) = \mathbf{x}\overline{\mathbf{y}} = \overline{\overline{\mathbf{x}}\mathbf{y}}$, we get that the product (1.1.3) is conjugate-invariant under any reflection $g$:

$$\overline{\overline{g(\mathbf{x})}g(\mathbf{y})} = \overline{\overline{\mathbf{x}}\mathbf{y}}. \tag{1.1.5}$$

So the product (1.1.3) can be called a *geometric product* of 2D *orthogonal geometry*, where the geometric space is still $\mathbb{R}^2$ but the transformation group is the 2D orthogonal group $O(2)$. For the geometric interpretation, let $\mathbf{x} = |\mathbf{x}|e^{i\theta_{\mathbf{x}}}$ and $\mathbf{y} = |\mathbf{y}|e^{i\theta_{\mathbf{y}}}$, then

$$\overline{\mathbf{x}}\mathbf{y} = |\mathbf{x}||\mathbf{y}|e^{i(\theta_{\mathbf{y}}-\theta_{\mathbf{x}})} = |\mathbf{x}||\mathbf{y}|e^{i\angle(\mathbf{x},\mathbf{y})}, \tag{1.1.6}$$

where $\angle(\mathbf{x}, \mathbf{y})$ is the angle of rotation from vector $\mathbf{x}$ to vector $\mathbf{y}$.

The two components of the geometric product (1.1.3) are also invariant by $O(2)$. They are both geometrically meaningful, and can also be termed as being "*geometric*". To distinguish among the three products, the real part of the geometric product $\overline{\mathbf{x}}\mathbf{y}$ is called the *inner product* between $\mathbf{x}$ and $\mathbf{y}$, denoted by $\mathbf{x}\cdot\mathbf{y}$; the pure imaginary part of the geometric product is called the *outer product* between $\mathbf{x}$ and $\mathbf{y}$, denoted by $\mathbf{x}\wedge\mathbf{y}$:

$$\begin{aligned} \mathbf{x}\cdot\mathbf{y} &= \frac{1}{2}(\mathbf{x}\overline{\mathbf{y}} + \overline{\mathbf{x}}\mathbf{y}), \\ \mathbf{x}\wedge\mathbf{y} &= \frac{1}{2}(\overline{\mathbf{x}}\mathbf{y} - \mathbf{x}\overline{\mathbf{y}}). \end{aligned} \tag{1.1.7}$$

One can immediately recognize that $\mathbf{x}\cdot\mathbf{y}$ is exactly the inner product of vectors $\mathbf{x}, \mathbf{y}$ in vector algebra, and $\mathbf{x}\times\mathbf{y} = (x_1y_2 - x_2y_1)\mathbf{n} = -i(\mathbf{x}\wedge\mathbf{y})\mathbf{n}$ is the cross product of the two vectors in space, where $\mathbf{n}$ is the unit normal direction of the plane described by complex numbers. In trigonometric form,

$$\begin{aligned} \mathbf{x}\cdot\mathbf{y} &= \ |\mathbf{x}||\mathbf{y}|\cos\angle(\mathbf{x},\mathbf{y}), \\ \mathbf{x}\wedge\mathbf{y} &= i|\mathbf{x}||\mathbf{y}|\sin\angle(\mathbf{x},\mathbf{y}). \end{aligned} \tag{1.1.8}$$

All complex numbers form a field, so any nonzero vector in $\mathbb{R}^2$ is invertible. The *geometric division* of $\mathbf{x}$ by $\mathbf{y}$ is just the geometric product of $\mathbf{x}$ and $\mathbf{y}^{-1}$, where the inverse is with respect to the geometric product (1.1.3) instead of the usual complex numbers product:

$$\mathbf{y}^{-1} = \frac{\mathbf{y}}{\mathbf{y}\overline{\mathbf{y}}}, \qquad \overline{\mathbf{x}}\mathbf{y}^{-1} = \frac{\overline{\mathbf{x}}\mathbf{y}}{\mathbf{y}\overline{\mathbf{y}}} = |\mathbf{x}||\mathbf{y}|^{-1}e^{i\angle(\mathbf{x},\mathbf{y})}. \tag{1.1.9}$$

The above analysis leads to the following theorem: The complex numbers equipped with the scalar multiplication, addition, subtraction, geometric product (1.1.3) and geometric division (1.1.9), are a geometric algebra of 2D orthogonal geometry.

For Euclidean plane geometry, Leibniz's dream is partially realized by complex numbers when the transformation group is restricted to the orthogonal group. His dream cannot be fully realized by complex numbers, because neither (1.1.3) nor

(1.1.9) is invariant under translation. With the increase of the dimension of the geometric space and the generalization of the transformation group, realizing Leibniz's dream becomes more and more difficult.

*Can Leibniz's dream of geometric algebras be realized for $n$D classical geometries?* The answer is affirmative:

- For $n$D projective geometry, the geometric algebra is *Grassmann-Cayley algebra*.
- For $n$D affine geometry, the geometric algebra is *affine Grassmann-Cayley algebra*.
- For $n$D orthogonal geometry, the geometric algebra is *Clifford algebra*, also called *Geometric Algebra* by Clifford himself.
- For $n$D Euclidean geometry, $n$D similarity geometry, $n$D conformal geometry, $n$D spherical geometry, $n$D hyperbolic geometry, and $n$D elliptic geometry, the geometric algebras are the same. It is *conformal geometric algebra*, which is composed of conformal Grassmann-Cayley algebra and conformal Clifford algebra.

## 1.2 Development of geometric algebras

In 1844, H. Grassmann published his book *Linear Extension Theory* [68], where he proposed the very original concepts in linear algebra such as linear independence, $n$D linear space, and linear extension of linear subspaces. Grassmann and A. Cayley are the two co-founders of linear algebra. While Grassmann established the concept of linear space, Cayley set up the matrix representation of linear maps. However, the algebra now bearing both their names, *Grassmann-Cayley algebra*, is not an algebra of matrices. It is an integration of *Grassmann algebra*, also called *exterior algebra*, and the dual of Grassmann algebra called *Cayley algebra* [57].

In Grassmann's vision, a point is zero dimensional, and can be represented by a vector, or a 1D direction. A line is generated by two points on it, and should be represented by a *product* of the two vectors representing the two points. The value of the product of two vectors should be a 2D direction. Two vectors in the same direction cannot span a 2D direction, so their product should be zero. Likewise, a plane is generated by three points on it, and should be represented by the product of the three vectors representing the points. The value of the product of three vectors should be a 3D direction. Three linearly dependent vectors cannot span a 3D direction, so their product should be zero. The product should be associative, as the plane spanned by line **12** and point **3** is identical to the plane spanned by point **1** and line **23**.

For any vector space $\mathcal{V}^n$, there exists a unique associative product, denoted by "$\wedge$", satisfying the requirements that the product of any vector $\mathbf{a} \in \mathcal{V}^n$ with itself is zero, and the product is linear with respect to every participating vector. This is Grassmann's product, nowadays called the *outer product*, or the *exterior product*. Vector space $\mathcal{V}^n$ supplemented with the linear combinations of all kinds of outer product results, becomes a *$\mathbb{Z}$-graded vector space* of dimension $2^n$, called the *Grassmann space* over base space $\mathcal{V}^n$.

The $\mathbb{Z}$-grading in a Grassmann space is induced by the outer product. The outer product of $r$ vectors represents an $r$D direction, and its *grade* is $r$. The operator extracting the $r$-graded part is called an *$r$-grading operator*.

In modern terms, Grassmann's point is a projective point, or a 1D linear subspace. His lines and planes are projective lines and planes respectively, or in other words, 2D and 3D linear subspaces respectively. His product is the direct sum extension of linear subspaces: if the intersection of two linear subspaces is just the zero vector, or in other words, they have *zero intersection*, then their outer product is their *direct sum*; if the two linear subspaces have nonzero intersection, their outer product is zero.

For example, in $\mathbb{R}^2$, the outer product of two vectors $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ is, by (1.1.3) and (1.1.7),

$$\mathbf{x} \wedge \mathbf{y} = i(x_1 y_2 - x_2 y_1). \tag{1.2.1}$$

Grassmann took $i$ as a unit 2D direction representing the complex numbers plane, which is also a real projective line, so that $x_1 y_2 - x_2 y_1$ is the scale, or coefficient, of the outer product $\mathbf{x} \wedge \mathbf{y}$ with respect to the unit $i$, the latter serving as a basis vector of the one-dimensional linear space of 2D directions.

Since $i(\mathbf{x} \wedge \mathbf{y})$ equals the signed area of the parallelogram spanned by vectors $\mathbf{y}, \mathbf{x}$, it is invariant under any affine transformation of the plane. Under any general linear transformation $T$ of the plane, the outer product changes by a scale independent of $\mathbf{x}, \mathbf{y}$:

$$T(\mathbf{x}) \wedge T(\mathbf{y}) = \det(T)\, \mathbf{x} \wedge \mathbf{y}. \tag{1.2.2}$$

It is called a *relative invariant* in classical invariant theory, and is meaningful in projective geometry.

However, there is no way for one vector to divide another in Grassmann algebra, because no vector in $\mathbb{R}^2$ is invertible with respect to the outer product: for any two vectors $\mathbf{a}, \mathbf{b}$, their outer product is never a scalar.

In $(n-1)$D *projective geometry*, the geometric space is composed of all 1D linear subspaces of an $n$D vector space $\mathcal{V}^n$, the transformation group is the general linear group $GL(\mathcal{V}^n)$. Addition and subtraction of two generic vectors in $\mathcal{V}^n$ are purely algebraic operations without any meaning in projective geometry, because by arbitrarily rescaling one vector while fixing the other, the sum of the two vectors can be in any 1D subspace of the 2D space spanned by the two vectors.

Grassmann algebra is a geometric algebra of projective geometry where the product of vectors represents the extension of linear subspaces. However, this geometric algebra is incomplete both algebraically and geometrically: algebraically, it lacks the division operation; geometrically, it does not have any operation that represents the intersection of linear subspaces.

To provide the Grassmann algebra with an algebraic operation representing the geometric intersection, Cayley proposed a second product called the *meet product*, which is algebraically dual to the outer product. Geometrically, the meet product of two linear subspaces of $\mathcal{V}^n$, whose dimensions when summed up are not less than $n$, represents the intersection of the two linear subspaces. A Grassmann space equipped with both the outer product and the meet product is an algebra called *Grassmann-Cayley algebra*.

*Grassmann-Cayley algebra is a geometric algebra for nD projective geometry whose two products represent the extension and intersection of linear subspaces.*

For example, in $\mathbb{R}^2$, the meet product of two vectors $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ is, using the complex numbers representation,

$$\mathbf{x} \vee \mathbf{y} := x_1 y_2 - x_2 y_1 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} := [\mathbf{xy}]. \tag{1.2.3}$$

The *bracket* $[\mathbf{xy}]$ is the short-hand notation of the determinant formed by the two vectors $\mathbf{x}, \mathbf{y}$. It is just the coefficient of the 2D direction $\mathbf{x} \wedge \mathbf{y}$ with respect to the unit $i$.

To define a division operation in the Grassmann-Cayley algebra over $\mathbb{R}^2$, we need to find the inverse of vector $\mathbf{y} \in \mathbb{R}^2$. This is possible only for the meet product. For vector

$$\mathbf{z} = (-\frac{y_2}{y_1^2 + y_2^2},\ \frac{y_1}{y_1^2 + y_2^2}), \tag{1.2.4}$$

since $\mathbf{y} \vee \mathbf{z} = 1$, $\mathbf{z}$ is a *right inverse* of $\mathbf{y}$ with respect to the meet product. It is not a left inverse because $\mathbf{z} \vee \mathbf{y} = -1$. Such a right inverse is not unique, because any $\mathbf{z} + \lambda \mathbf{y}$ for $\lambda \in \mathbb{R}$ is also a right inverse of $\mathbf{y}$. Nevertheless, the right inverse is *unique modulo* $\mathbf{y}$, *i.e.*, any right inverse of $\mathbf{y}$ is of the form $\mathbf{z} + \lambda \mathbf{y}$.

If we require that the right inverse of $\mathbf{y}$ is orthogonal to $\mathbf{y}$, *i.e.*, its inner product with $\mathbf{y}$ equals zero, then (1.2.4) is the unique right inverse of $\mathbf{y}$ with respect to the meet product, called the *orthogonal right inverse* of $\mathbf{y}$, and denoted by $*\mathbf{y}^{-1}$. The *division* of vector $\mathbf{x}$ by vector $\mathbf{y}$ from the right, can be defined by

$$\mathbf{x}/\mathbf{y} := \mathbf{x} \wedge (*\mathbf{y}^{-1}) = \mathbf{x} \wedge \mathbf{z} = i \frac{x_1 y_1 + x_2 y_2}{y_1^2 + y_2^2}, \tag{1.2.5}$$

which congrues with $i$ times the inner product between $\mathbf{x}$ and $\mathbf{y}^{-1} = \mathbf{y}/(\mathbf{y}\overline{\mathbf{y}})$ in complex numbers notation, according to (1.1.3) and (1.1.9).

Similarly, if we define the division of $\mathbf{x}$ by $\mathbf{y}$ from the left as the outer product of the left inverse of $\mathbf{y}$ with $\mathbf{x}$, the result is still (1.2.5). So (1.2.5) can be uniformly called the division of $\mathbf{x}$ by $\mathbf{y}$. This division operation is elegant, but lacks invariance under $GL(\mathbb{R}^2)$, so it is meaningless in projective geometry. However, it is meaningful in orthogonal geometry, because it is invariant under $O(2)$.

*Projective geometry does not have a geometric algebra in which the division operation is geometrically meaningful.*

The above construction of the $O(n)$-invariant division by the outer product and the meet product, indicates a new way of constructing the geometric algebra $\mathbb{C}$ of 2D orthogonal geometry: by denoting

$$*\mathbf{y} := *(\mathbf{y}^{-1})^{-1} = *(\frac{\mathbf{y}}{|\mathbf{y}|^2})^{-1} = (-y_2, y_1), \tag{1.2.6}$$

we have

$$\begin{aligned}\overline{\mathbf{x}}\mathbf{y} &= (x_1y_1 + x_2y_2, x_1y_2 - x_2y_1)\\ &= (x_1y_1 + x_2y_2) + i(x_1y_2 - x_2y_1)\\ &= -i\mathbf{x}/\mathbf{y}^{-1} + \mathbf{x} \wedge \mathbf{y}\\ &= \mathbf{x} \vee *\mathbf{y} + \mathbf{x} \wedge \mathbf{y}\\ &= \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \wedge \mathbf{y}.\end{aligned} \tag{1.2.7}$$

The last expression of (1.2.7) as a geometric product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ can be extended to more than two dimensions. Historically, in 1843, W.R. Hamilton established the theory of *quaternions*, a geometric algebra of 3D orthogonal geometry as the extension of the complex numbers to space. In 1873, W.K. Clifford extended quaternions to *dual quaternions*, a geometric algebra of 3D Euclidean geometry. Both can be taken as extensions of the left side of (1.2.7) to some geometric products of 3D geometry.

In 1878, Clifford established the general theory of *Clifford algebra*, which he originally called geometric algebra, as "an application of Grassmann's extensive algebra". It is an extension of the last expression of (1.2.7) to a geometric product of $n$D orthogonal geometry. Quaternions and dual quaternions are both Clifford algebras.

*Clifford algebra is a geometric algebra of $n$D orthogonal geometry in which the division operation is geometrically meaningful. Its product after decomposition also gives the extension and intersection of linear subspaces.*

From now on, we only call the product in Clifford algebra the *geometric product*. A Grassmann space equipped with the geometric product is called a *Clifford algebra*. Formally, for any vector space $\mathcal{V}^n$ equipped with an inner product structure, the *Clifford algebra* over base space $\mathcal{V}^n$ is the universal associative algebra generated by the relation $\mathbf{aa} = \mathbf{a} \cdot \mathbf{a}$ for any vector $\mathbf{a} \in \mathcal{V}^n$, if the product, denoted by juxtaposition, is linear with respect to every participating vector.

**Example 1.1.** Clifford algebra $\mathcal{CL}(\mathbb{R}^2)$ over the Euclidean plane $\mathbb{R}^2$.

Is it just the algebra $\mathbb{C}$ of complex numbers? Certainly not. The dimension of $\mathcal{CL}(\mathbb{R}^2)$ as a real vector space is $2^2 = 4$, while $\mathbb{C}$ as a real vector space is of dimension 2. Let $\mathbf{e}_1, \mathbf{e}_2$ be an orthonormal basis of $\mathbb{R}^2$. Their geometric product is

$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{e}_2 + \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2. \tag{1.2.8}$$

It represents a 2D direction, and so is called a *2-vector*. In complex numbers notation, $\mathbf{e}_1\mathbf{e}_2 = (1, 0)(0, 1) = i$ is the pure imaginary unit.

As a vector space, $\mathcal{CL}(\mathbb{R}^2)$ is the direct sum of three vector subspaces:

- the subspace $\mathbb{R}$ of scalars, or 0D directions (0-vectors);
- the subspace $\mathbb{R}^2$ of vectors, or 1D directions (1-vectors);
- the subspace of 2-vectors, or 2D directions, spanned by $i = \mathbf{e}_1\mathbf{e}_2$.

The direct sum of the first and the third subspaces is composed of elements of the form $x + y\mathbf{e}_1\mathbf{e}_2 = x + iy$, where $x, y \in \mathbb{R}$. It is a subalgebra of $\mathcal{CL}(\mathbb{R}^2)$ isomorphic to $\mathbb{C}$.

So $\mathcal{CL}(\mathbb{R}^2)$ is composed of both the real representation $\mathbb{R}^2$ and the complex representation $\{x + y\mathbf{e}_1\mathbf{e}_2 \,|\, x, y \in \mathbb{R}\}$ of the Euclidean plane. As an algebra it is isomorphic to the algebra $M_{2\times 2}(\mathbb{R})$ of 2 by 2 real matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a+d}{2} + \frac{a-d}{2}\mathbf{e}_1 + \frac{b+c}{2}\mathbf{e}_2 + \frac{b-c}{2}\mathbf{e}_1\mathbf{e}_2. \tag{1.2.9}$$

The algebraic isomorphism between $\mathcal{CL}(\mathbb{R}^2)$ and $M_{2\times 2}(\mathbb{R})$ is not a coincidence. By a theorem of É. Cartan, any real Clifford algebra is isomorphic to a matrix algebra over either the real numbers, or the complex numbers, or the quaternions.

Although created as a geometric algebra for orthogonal geometry, Clifford algebra did not attract more attention than quaternions in the 19th century. At that time, it was generally taken as a mathematical curiosity of being an algebra of "hypercomplex numbers", meaning that its elements, called *Clifford numbers*, are just complex numbers in high dimensional space.

The relation between a Clifford algebra and the corresponding Grassmann-Cayley algebra over the same base space is as follows: first, they are isomorphic as linear spaces, or more accurately, as $\mathbb{Z}$-graded linear spaces; second, both the meet product and the outer product can be expressed as polynomials of the geometric product in Clifford algebra, called the *ungrading* of the two products; third, the geometric product can always be decomposed into a polynomial of the two products in Grassmann-Cayley algebra, called the *grading* of the geometric product. Because of the latter decomposition, Grassmann-Cayley algebra provides a natural representation for Clifford algebra, and can be taken as a version of Clifford algebra if its base space has an inner product structure.

In the 20th century, with the development of spinors and their representations in Clifford algebras in the first half of the century, by great mathematicians É. Cartan, H. Weyl, C. Chevalley, M. Riesz, *et al.*, and by great physicists W. Pauli, P. Dirac, *et al.*, with the applications in index theorems and gauge theory in the second half of the century, by M. Atiyah, I. Singer, N. Seiberg, E. Witten, *et al.*, with the extension of complex analysis to Clifford numbers to form an alternative theory of several complex variables called *Clifford analysis*, by A.C. Dixon, F. Klein, R. Delanghe, L. Ahlfors, *et al.*, with the development into a universal geometric algebra by D. Hestenes and his school, and with many other accomplishments and applications by mathematicians, physicists and computer scientists, Clifford algebra has become a conflux of algebra, analysis and geometry, with wide range of applications in mathematics, theoretical physics, computer science and engineering.

It is hard to imagine that the reason behind so many successful applications of Clifford algebra is other than the intrinsic property that it is a geometric algebra describing the incidence and metric properties of linear subspaces. Just quote one comment from differential geometers:

> "It is a striking (and not commonplace) fact that Clifford algebras and their representations play an important role in many fundamental aspects of differential geometry. These algebras emerge repeatedly at the very core of an astonishing variety of problems in geometry and topology.
>
> Even in discussing Riemannian geometry, the formalism of Clifford multiplication will be used in place of the more conventional exterior tensor calculus. The Clifford multiplication is strictly richer than exterior multiplication; it reflects the inner symmetries and basic identities of the Riemannian structure. The effort invested in becoming comfortable with this algebraic formalism is well worthwhile."
>
> — H. Lawson Jr. and M.-L. Michelson, 1989.

Following the line of developing a universal geometric language out of Clifford algebra, D. Hestenes launched a new approach to Clifford algebra, and formulated a version of this algebra that is more "geometric" than any other version. He called this version *Geometric Algebra*, where the two initial capitals distinguish this specific geometric algebra from other geometric algebras [77], [78], [82], [83].

Hestenes regarded Clifford algebra as a geometric extension of the real number system to provide a complete algebraic representation of the geometric notions of high dimensional directions and magnitudes. The building blocks of *Geometric Algebra* are the outer products of vectors, called *blades*, or *decomposable extensors* in Grassmann algebra. The geometric product of blades describes relations among the spaces they represent. In classical geometries, the primitive elements are points, and geometric objects are point sets with properties. The properties are of two main types: structural and transformational. Geometric objects are characterized by structural relations and compared by transformations. Geometric Algebra provides a unified algebraic framework for both kinds of properties.

There are two main differences between Geometric Algebra and other versions of Clifford algebra. The first difference is "structural", or representational. On one hand, in the versions of hypercomplex numbers and matrix algebra, any expression representing a geometric object or property is a linear combination of a fixed set of basis expressions. In the Grassmann-Cayley algebra version of Clifford algebra, an expression representing an orthogonal transformation of a geometric object is in the form of a polynomial of outer products and meet products.

On the other hand, in Geometric Algebra, any expression representing a geometric object or property is a *graded Clifford monomial*, *i.e.*, a monomial generated by vectors using the geometric product and $\mathbb{Z}$-grading operators. The inner product, outer product and meet product are secondary operations. They can all be expressed as compositions of $\mathbb{Z}$-grading operators and the geometric product.

Structurally, Geometric Algebra is more "geometric" than other versions of Clifford algebra, because geometric objects or properties are expressed more multiplicatively than additively in this algebra. In symbolic form, more addition leads to more algebra, and more multiplication preserves more geometry. While the geometric product and many of the $\mathbb{Z}$-grading operators are geometrically meaningful, the decomposition into a sum of expressions is a way of getting more "algebraic", and often breaks up the high dimensional relations among geometric objects.

To be more specific and exemplary, coordinatization is a typical way of algebraization by decomposing (or "discretizing") a high dimensional geometric object into a sequence of one-dimensional representations. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be a basis of $\mathcal{V}^n$, then the coordinate representation of any vector $\mathbf{x} \in \mathcal{V}^n$, $(x_1, x_2, \ldots, x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$, expresses $\mathbf{x}$ by scalars $x_i$ each measuring the affinity of $\mathbf{x}$ with a basis vector $\mathbf{e}_i$. For two vectors $\mathbf{x}, \mathbf{y} \in \mathcal{V}^n$, their relation in $\mathcal{V}^n$ is decomposed into the sum of 1D relations between the $x_i$ and $y_i$, with the basis vectors $\mathbf{e}_i$ as external agencies.

The second difference between Geometric Algebra and other versions of Clifford algebra is "transformational", or computational. Other versions emphasize the multilinear nature of Clifford algebra, so expansions based on multilinearity are common and frequent in manipulating expressions. The idea behind such manipulations is to normalize the expressions into canonical forms, just like the normalization of the multiplication of two polynomials by expanding it into a sum of monomials. Multilinear expansion is a way of decomposing a high dimensional multiplication into the sum of lower dimensional multiplications. It inevitably decreases the "geometric" feature of Clifford algebra.

Multiplication and division operations are the gist of Geometric Algebra. They are the only two geometrically meaningful operations upon algebraic expressions having geometric meaning, and are always preferred to addition and subtraction. Consequently, as manipulations inverse to expansions and normalizations, factorizations for the multiplicatively decomposed form and contractions to reduce the number of terms are common and frequent in Geometric Algebra. Symbolic computations of geometric problems based on factorizations and contractions prove to be more efficient and effective than those based on expansions and normalizations.

## 1.3 Conformal geometric algebra

Having surveyed the history of Clifford algebra from the viewpoint of developing a geometric algebra for $n$D orthogonal geometry, we return to the original problem of Leibniz's dream, *i.e.*, developing a geometric algebra for $n$D Euclidean geometry.

The Euclidean group $E(n)$ is the semi-direct product of the orthogonal group

$O(n)$ and the translational group $\mathbb{R}^n$. To find a homogeneous space embedded in a real vector space $\mathcal{V}^m$ for $E(n)$ to act on as a transitive transformation group, the orthogonal group $O(\mathcal{V}^m)$ of the surrounding space $\mathcal{V}^m$ should contain $E(n)$ as a subgroup. The only $(n+1)$D real vector space having this property is the space $\mathbb{R}^{n,0,1}$ spanned by $\mathbb{R}^n$ and a *null vector* $\mathbf{e}$ orthogonal to $\mathbb{R}^n$. A nonzero vector is said to be *null* if its inner product with itself is zero.

The orthogonal group of $\mathbb{R}^{n,0,1}$ is isomorphic to the similarity group of $\mathbb{R}^n$, which is generated by the Euclidean group $E(n)$ and dilations

$$D_\lambda : \ \mathbf{x} \longmapsto \lambda \mathbf{x}, \ \forall \mathbf{x} \in \mathbb{R}^n, \tag{1.3.1}$$

for all $\lambda \in \mathbb{R} - \{0\}$. When $n = 3$, the algebra of even-graded elements in $\mathcal{CL}(\mathbb{R}^{n,0,1})$ is isomorphic to the algebra of *dual quaternions*.

The product in dual quaternions is invariant under $E(3)$, so it is a Euclidean geometric product. However, the dual quaternion representations of primitive geometric objects such as points, lines and planes in space are not *covariant*. More accurately, the representations are not *tensors*, they depend upon the position of the origin of the coordinate system irregularly. As a consequence, the compositions of $\mathbb{Z}$-grading operators and the geometric product in $\mathcal{CL}(\mathbb{R}^{3,0,1})$ have rather poor geometric meaning, because they are all related to the origin in use. The algebraic properties of $\mathcal{CL}(\mathbb{R}^{3,0,1})$ are also poor. Because the inner product in $\mathbb{R}^{3,0,1}$ is degenerate, many important invertibilities in non-degenerate Clifford algebras are lost.

To find covariant representations of geometric objects, we need to go up one more dimension. In the $(n+2)$D Minkowski space $\mathbb{R}^{n+1,1}$, there is a well-known result [22], [35], [130], [147], saying that the orthogonal group of $\mathbb{R}^{n+1,1}$ is a double covering of the *conformal group* $M(n)$ of $\mathbb{R}^n$, the latter being composed of angle-preserving diffeomorphisms called *Möbius transformations* in $\mathbb{R}^n$. Since $E(n)$ is a subgroup of $M(n)$, it can be represented by orthogonal transformations in $\mathbb{R}^{n+1,1}$.

The geometric space on which $E(n)$ acts is neither a linear subspace of $\mathbb{R}^{n+1,1}$, nor an affine subspace. Instead, it is a quadric surface of dimension $n$. This surface, denoted by $\mathcal{N}_{\mathbf{e}}$, is isometric to $\mathbb{R}^n$ as distance space. This model of $n$D Euclidean geometry has its origin in the work of F. Wachter (1792-1817), a student of Gauss [70]. A revised version of this model, called the *Lie model*, was developed by S. Lie in his Ph. D. dissertation (1872) to study *contact geometry*, also known as *Lie sphere geometry*. In applications, this model is often called the *conformal model* [18], [56], [151], [168].

The Geometric Algebra $\mathcal{CL}(\mathbb{R}^{n+1,1})$ established upon the conformal model is called *conformal geometric algebra*. It is an integration of the previously developed conformal Clifford algebra [5], [44], [72], [80], [130], [136], [184], with the recently developed conformal Grassmann-Cayley algebra [110].

Conformal Clifford algebra studies the conformal transformations in $\mathbb{R}^n$ using various versions of Clifford algebra, ranging from hypercomplex numbers and Clifford matrices to Geometric Algebra. It provides a "transformational" geometric algebra for $n$D conformal geometry.

Conformal Grassmann-Cayley algebra, on the other hand, provides primitive Euclidean geometric objects such as points, lines, circles, planes and spheres, with covariant representations in $\mathcal{CL}(\mathbb{R}^{n+1,1})$. It represents and computes geometric constructions such as the intersection, extension, contact, and knotting of primitive geometric objects, by combining the meet product, outer product, inner product, and geometric product. It provides a "structural" geometric algebra for $n$D conformal geometry.

*As an integration of conformal Grassmann-Cayley algebra and conformal Clifford algebra, conformal geometric algebra provides the first geometric algebra for $n$D conformal geometry, including Euclidean geometry as a special case.*

Another classical result on conformal groups is that the conformal groups of $n$D Euclidean space, $n$D spherical space, and $n$D hyperbolic space respectively, are isomorphic to each other [35], [147]. Besides the conformal model $\mathcal{N}_{\mathbf{e}}$ of Euclidean geometry, the conformal models $\mathcal{N}_{\mathbf{p}}$ and $\mathcal{N}_{\mathbf{a}}$ of $n$D spherical geometry and hyperbolic geometry respectively also can be established as $n$D quadric surfaces in Minkowski space $\mathbb{R}^{n+1,1}$ [111], [113].

To unify the three conformal models, the whole set $\mathcal{N}$ of null vectors in $\mathbb{R}^{n+1,1}$ must be put into service. This space when equipped with a conformal distance function, provides a *universal homogeneous model* for Euclidean, spherical, and hyperbolic geometries [112]. By "homogeneous" we mean that two vectors in $\mathcal{N}$ represent the same point if and only if they differ by scale. Because of the homogeneity, the model is conformal instead of isometric, hence can represent various classical geometries of different metrics.

*The Geometric Algebra established upon the universal homogeneous model, still called conformal geometric algebra, is a unified geometric algebra for classical geometries.*

## 1.4 Geometric computing with invariant algebras

The purpose of developing geometric algebras is to use them to compute geometric problems and prove geometric theorems. Now that Leibniz's dream has been completely realized, using geometric algebras to solve difficult unsolved problems in geometry seems highly prospective. However, to change the prospect into reality, a lot of developments are needed.

What is geometric computing? Computing is an algebraic thing. In a suitable algebraic framework, a geometric problem can be translated into an algebraic one, and an algebraic result can be obtained by either symbolic or numerical computing. If the result can be translated into geometric terms, then it is geometrically meaningful. *Geometric computing* refers to a procedure of algebraic manipulations generating a geometrically meaningful result from an input of geometric data.

In the ancient times before algebra was invented, geometric terms such as lengths, areas, ratios and angles were used directly in geometric computing. This synthetic approach to geometry has rather limited capabilities both in representing and in manipulating geometric objects and properties.

By decomposing high dimensional geometry into a sequence of one dimensional geometries, Descartes' introduction of coordinates into geometry greatly facilitates the representation and manipulation of geometric objects. A side-effect is that this factitious decomposition induces two hard-to-solve problems:

(1) The results from algebraic computations based on coordinates are either difficult to explain geometrically, or geometrically meaningless because they are neither invariant nor covariant under coordinate transformations. In other words, their dependencies upon the specific coordinate systems are either difficult or impossible to be separated from the geometric properties they represent.

(2) Even in the most favorable coordinate system, there is the following "middle expression swell" phenomenon: both the input expression and the output expression are small in size, but the expressions in the middle computing steps are huge. Because of this, some computations are possible only theoretically.

We shall see that geometric algebras can alleviate the above difficulties faced by coordinate methods. A geometric algebra can be represented either by coordinates or in a coordinate-free manner. In coordinate form, no result from algebraic computing by a geometric algebra is geometrically meaningless, owning to the invariant nature of the operations in such an algebra. Still the obtaining of a geometric interpretation may be possible only theoretically, because of the difficulty in translating the result from coordinate form to geometric terms.

On one hand, the coordinate-free versions of geometric algebras have obvious representational advantage over coordinates. For example, for two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, by (1.2.3) and (1.2.2), bracket $[\mathbf{xy}]$ is a *projective geometric invariant*. By Laplace expansion, the determinant in (1.2.3) becomes a polynomial of two terms. By the same expansion, a single bracket monomial $[\mathbf{a}_1\mathbf{a}_2][\mathbf{a}_3\mathbf{a}_4]\cdots[\mathbf{a}_{2k-1}\mathbf{a}_{2k}]$, where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{2k} \in \mathbb{R}^2$, becomes a coordinate polynomial of as many as $2^k$ terms.

Recall that the geometric algebra of projective geometry is Grassmann-Cayley algebra. In this algebra, all brackets form an algebraic system by multiplication, addition and subtraction, called a *bracket algebra*. Bracket algebra is the coordinate-free version of the algebra of determinants. A theorem in classical invariant theory says that all projective geometric invariants are generated algebraically by such brackets. Bracket algebra is in fact the algebra of all projective geometric invariants.

On the other hand, the representational advantage of brackets does not necessarily lead to any manipulational advantage. In bracket algebra, all brackets are indeterminates. Contrary to the coordinates of generic vectors, which are independent of each other, brackets composed of generic vectors are algebraically dependent, and their algebraic relations are called *syzygies*.

In classical invariant theory, a *syzygy* refers to a polynomial of invariants that equals zero when expanded into coordinate form. For example, in $\mathbb{R}^2$, brackets $[\mathbf{a}_i\mathbf{a}_j]$ for $1 \leq i < j \leq 4$ satisfy

$$[\mathbf{a}_1\mathbf{a}_2][\mathbf{a}_3\mathbf{a}_4] - [\mathbf{a}_1\mathbf{a}_3][\mathbf{a}_2\mathbf{a}_4] + [\mathbf{a}_1\mathbf{a}_4][\mathbf{a}_2\mathbf{a}_3] = 0. \tag{1.4.1}$$

The left side is a syzygy among the six brackets. It becomes zero automatically when expanded into coordinate form using $\mathbf{a}_i = (x_i, y_i)^T$:

$$(x_1y_2 - x_2y_1)(x_3y_4 - x_4y_3) - (x_1y_3 - x_3y_1)(x_2y_4 - x_4y_2) \\ + (x_1y_4 - x_4y_1)(x_2y_3 - x_3y_2) = 0.$$

In the special case where $\mathbf{a}_1 = \mathbf{e}_1$, $\mathbf{a}_2 = \mathbf{e}_2$ are an orthonormal basis of $\mathbb{R}^2$, and $\mathbf{a}_3 = \mathbf{x}$, $\mathbf{a}_4 = \mathbf{y}$ are two generic vectors, this sygyzy is just the Laplace expansion of bracket $[\mathbf{xy}]$, *i.e.*,

$$[\mathbf{e}_1\mathbf{e}_2][\mathbf{xy}] - [\mathbf{e}_1\mathbf{x}][\mathbf{e}_2\mathbf{y}] + [\mathbf{e}_1\mathbf{y}][\mathbf{e}_2\mathbf{x}] = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} - x_2(-y_1) + y_2(-x_1).$$

The algebraic dependencies among bracket indeterminates pose two long-lasting problems to the invariant theory community: factorization of a bracket polynomial without resorting to coordinates, and reduction of a bracket polynomial to the least number of terms. Having been open for about a century, there is no sign that the two problems can be solved in the near future.

In classical invariant theory, a bracket polynomial is changed into normal form by Young's *straightening algorithm* proposed in 1928. In the straightening procedure, a bracket monomial is transformed many times into bracket polynomials of many terms. This "expansion" procedure does not have any control of the middle expression swell.

Geometric interpretation is also a problem for bracket algebra. Although each bracket, as a determinant of homogeneous coordinates of the constituent points, can be interpreted in affine geometry as the *signed volume* of the simplex spanned by the points as vertices, a bracket polynomial is by no means easily interpretable with geometric terms. If the bracket polynomial can be written as a rational monomial in Grassmann-Cayley algebra, then it can be given a projective geometric interpretation based on the latter form. This translation from bracket algebra to Grassmann-Cayley algebra is called *Cayley factorization.* Except for some special cases [192], [117], this problem is still open.

So in classical invariant theory, the two major problems faced by the coordinate approach are still alive, although in some cases the algebraic manipulations can be simplified because of the simplicity in algebraic representations. Due to the algebraic dependencies among brackets, new difficulties arise, which are by no means easy to handle. In the invariant-theoretic approach, people do not get rid of algebraic dependencies, otherwise it becomes a traditional coordinate approach.

Bracket algebra is the algebra of invariants in projective geometry, and after some revision, also the algebra of invariants in affine geometry. For Euclidean geometry, bracket algebra needs to be supplemented with inner products of vectors.

The resulting algebra, called *inner-product bracket algebra*, is the algebra of invariants in Euclidean geometry. The algebraic relations among inner products of vectors are much more complicated than those among brackets. The task to control middle expression swell is much heavier.

The phenomenon that in classical geometries, the results computed using algebras generated by basic invariants such as determinants and inner products of vectors, are often big rational polynomials of the basic invariants, without any clear geometric meaning, indicates that the generating elements of such algebras are too low-level, *both algebraically and geometrically.* To control the expression size effectively in middle computing steps, at the same time to alleviate the difficulty of translating algebraic results into geometric terms, more advanced invariant algebras are needed.

## 1.5 From basic invariants to advanced invariants

*Advanced invariants* are polynomials of basic invariants. By putting them into the algebra of basic invariants as new indeterminates, and treating their polynomial expressions by basic invariants as their defining syzygies, an *algebra of advanced invariants* is obtained.

The purpose of proposing advanced invariants is to simplify algebraic computation and keep geometric meaning. The criteria used in singling out advanced invariants from the polynomials of basic invariants, are the following:

- an advanced invariant should have clear geometric meaning;
- the system of advanced invariants should be hierarchical;
- an advanced invariant should have relatively nice symmetry with respect to its vector constituents.

(1) To keep geometric meaning, we can resort to the geometric algebra of the corresponding geometry. Since the product in such an algebra is always geometrically meaningful, a scalar-valued monomial in such an algebra is naturally an advanced invariant with immediate geometric interpretation. Such monomials occur naturally in representing geometric constructions. If not expanded into polynomials of basic brackets, they can keep the geometric nature within their algebraic structures.

(2) The structure of an algebra of advanced invariants has to be hierarchical: the bottom level is the basic invariants, and each higher level invariant is a polynomial of the lower level ones. The *level* of an advanced invariant is determined by the level of composition of geometric product operations that are used to construct the advanced invariant.

The connections between high-level invariants and low-level ones are through expansions and factorizations. An *expansion* is the transformation of a high-level invariant to a polynomial of low-level ones, by eliminating at least one geometric product operation from the high-level invariant. The reverse procedure to produce a high-level invariant in monomial form is called *factorization*.

To distinguish the expansion and factorization here from those based on multilinearity properties, the terminology usually bears the name of the associated geometric algebra. For example, in projective geometry – *Cayley expansion* and *Cayley factorization*, in orthogonal geometry – *Clifford expansion* and *Clifford factorization*, and in Euclidean geometry – *null expansion* and *null factorization*.

(3) Advanced invariants when put in the form of monomials of the associated geometric algebra, are easy to be given geometric explanations, but are more difficult to manipulate than low-level ones if they do not have nice symmetries within their monomial structures.

Symmetries within advanced invariants, if expressed by basic invariants, are generally complicated syzygies. They provide the most economical way of reining syzygies. Employing the structural symmetries within advanced invariants can drastically, and in some cases, even magically, simplify the manipulation of syzygies.

**Example 1.2.** Advanced invariants in projective geometry.

All scalar-valued monomials in Grassmann-Cayley algebra generate an advanced invariant algebra, called *Cayley bracket algebra*. The monomials are graded by the number of meet products involved, and each monomial has clear geometric meaning by explaining the outer product as the geometric extension, and the meet product as the geometric intersection.

The role played by symmetries within advanced invariants in simplifying algebraic manipulations can be seen from the following typical example. In $\mathcal{V}^3$, bracket

$$[\mathbf{abc}] := \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{1.5.1}$$

is a projective geometric invariant. Vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ represent projective points in the plane. If $\mathbf{a} = \mathbf{x}_1\mathbf{x}_2 \cap \mathbf{x}_3\mathbf{x}_4$, $\mathbf{b} = \mathbf{y}_1\mathbf{y}_2 \cap \mathbf{y}_3\mathbf{y}_4$, $\mathbf{c} = \mathbf{z}_1\mathbf{z}_2 \cap \mathbf{z}_3\mathbf{z}_4$ are intersections of lines, then in the Grassmann-Cayley algebra over $\mathcal{V}^3$, the three points have expressions

$$\begin{aligned} \mathbf{a} &= (\mathbf{x}_1 \wedge \mathbf{x}_2) \vee (\mathbf{x}_3 \wedge \mathbf{x}_4), \\ \mathbf{b} &= (\mathbf{y}_1 \wedge \mathbf{y}_2) \vee (\mathbf{y}_3 \wedge \mathbf{y}_4), \\ \mathbf{c} &= (\mathbf{z}_1 \wedge \mathbf{z}_2) \vee (\mathbf{z}_3 \wedge \mathbf{z}_4). \end{aligned} \tag{1.5.2}$$

Substituting them into bracket $[\mathbf{abc}]$, we get, in the notation of Cayley bracket algebra, the following advanced projective geometric invariant:

$$((\mathbf{x}_1\mathbf{x}_2)(\mathbf{x}_3\mathbf{x}_4))\,((\mathbf{y}_1\mathbf{y}_2)(\mathbf{y}_3\mathbf{y}_4))\,((\mathbf{z}_1\mathbf{z}_2)(\mathbf{z}_3\mathbf{z}_4)). \tag{1.5.3}$$

It can be expanded into 16847 different bracket polynomials [117]!

The equality between any two of the 16847 bracket polynomials is a nontrivial syzygy in bracket algebra. In the monomial form (1.5.3) of this advanced invariant, all these equalities are embodied in the following simple symmetries:

(i) the product of each pair of vectors, *e.g.* $\mathbf{x}_1\mathbf{x}_2$ of $\mathbf{x}_1$ and $\mathbf{x}_2$, is antisymmetric;

(ii) the product of each pair of 2-vectors, *e.g.* $(\mathbf{x}_1\mathbf{x}_2)(\mathbf{x}_3\mathbf{x}_4)$ of $\mathbf{x}_1\mathbf{x}_2$ and $\mathbf{x}_3\mathbf{x}_4$, is antisymmetric;

(iii) the product of $(\mathbf{x}_1\mathbf{x}_2)(\mathbf{x}_3\mathbf{x}_4)$, $(\mathbf{y}_1\mathbf{y}_2)(\mathbf{y}_3\mathbf{y}_4)$, $(\mathbf{z}_1\mathbf{z}_2)(\mathbf{z}_3\mathbf{z}_4)$, is both antisymmetric and associative;

(iv) the product is invariant under any duality between vectors and 2-vectors: if

$$\mathbf{x}_1\mathbf{x}_2 \mapsto \mathbf{u}_1, \ \ \mathbf{x}_3\mathbf{x}_4 \mapsto \mathbf{u}_2, \ \ \mathbf{y}_1\mathbf{y}_2 \mapsto \mathbf{v}_1, \ \ \mathbf{y}_3\mathbf{y}_4 \mapsto \mathbf{v}_2, \ \ \mathbf{z}_1\mathbf{z}_2 \mapsto \mathbf{w}_1, \ \ \mathbf{z}_3\mathbf{z}_4 \mapsto \mathbf{w}_2$$

is a special linear map from the space of 2-vectors to $\mathcal{V}^3$, then (1.5.3) equals $(\mathbf{u}_1\mathbf{u}_2)(\mathbf{v}_1\mathbf{v}_2)(\mathbf{w}_1\mathbf{w}_2)$.

In general, a representation of a geometric property by advanced invariants can be transformed fairly easily into a representation by basic invariants or coordinates. The converse transformation is usually much more difficult to be practically possible. As to computation, a computing procedure based on basic invariants or coordinates generally cannot be translated into one based on advanced invariants. The converse translation is generally also impossible, because basic invariants and coordinates have their own rules of computation, by either straightening or expanding into their own normal forms.

The idea of *advanced invariant computing* is to manipulate advanced invariants by their own mechanism, without resorting to low-level invariants or coordinates. The following *non-commutative diagram* describes the relation among computations based on advanced invariants, basic invariants, and coordinates respectively.

$$\begin{array}{ccccc}
\text{Representation} & & \text{Middle steps} & & \text{Result} \\[1em]
\text{advanced invariants} & \xrightarrow{compute} & \text{advanced invariants} & \xrightarrow{compute} & \text{advanced invariants} \\
\Big\downarrow{\scriptstyle translate} & & \Big\downarrow{\scriptstyle translate} & & \Big\downarrow{\scriptstyle translate} \\
\text{basic invariants} & \xrightarrow{compute} & \text{basic invariants} & \xrightarrow{compute} & \text{basic invariants} \\
\Big\downarrow{\scriptstyle translate} & & \Big\downarrow{\scriptstyle translate} & & \Big\downarrow{\scriptstyle translate} \\
\text{coordinates} & \xrightarrow{compute} & \text{coordinates} & \xrightarrow{compute} & \text{coordinates}
\end{array}$$

**Example 1.3.** Advanced invariants in orthogonal geometry.

The algebra of basic invariants in orthogonal geometry is inner-product bracket algebra. Geometrically, the inner product of two vectors represents the cosine of the angle between them. For the cosine of the angle formed by two planes or higher dimensional objects, its representation in inner-product bracket algebra is usually a complicated polynomial of inner products of vectors.

Bracket algebra when supplemented with inner products of high dimensional directions, becomes an algebra of advanced invariants in orthogonal geometry, called *graded inner-product bracket algebra* [50], [51], [136]. In this algebra, the inner products of $r$D directions are naturally graded by the dimension $r$. Still this algebra cannot provide simple representations for the cosines of sums of angles. To find monomial representations of such angle objects, we need the geometry of angles, which is the geometric algebra of orthogonal geometry – the Clifford algebra (or Geometric Algebra) over inner-product space $\mathcal{V}^n$.

*Clifford bracket algebra* [115] is generated by two scalar-valued $\mathbb{Z}$-grading operators acting on the geometric product results of vectors in Clifford algebra, called *angular bracket operator* and *square bracket operator* respectively, together with the graded inner products of vectors. The two brackets are naturally graded by the number of vectors in them.

Algebraically, the angular bracket is the prolongation of the inner product of two vectors to a "hyper-inner-product" of any $2l$ vectors of $\mathcal{V}^n$, for any $l \geq 1$; the square bracket is the prolongation of the determinant of $n$ vectors to a "hyper-determinant" of any $n+2l$ vectors of $\mathcal{V}^n$. They are often referred to as *long brackets*. Geometrically, the two brackets are trigonometric functions of the compositions of directed angles.

Clifford bracket algebra is an advanced invariant algebra of orthogonal geometry. It is not an algebra of invariants in Euclidean geometry, where the transformation group is generated by orthogonal transformations and translations. While a translation is numerically almost trivial, it leads symbolically to the exponential growth of the expression size in the traditional normalization approach. For a single monomial $x_1x_2\cdots x_m$ in indeterminates $x_1, x_2, \ldots, x_m$, the translation $x_i \mapsto x_i + t$ changes the monomial into a polynomial of as many as $2^m$ terms after expansion:

$$(x_1 + t)(x_2 + t)\cdots(x_m + t) = x_1x_2\cdots x_m + tx_2\cdots x_m + \cdots + x_1t^{m-1} + t^m.$$

To construct advanced invariants in Euclidean geometry, we need the universal geometric algebra of classical geometries – conformal geometric algebra. In this algebra, all vectors representing points in Euclidean space are null vectors. The Clifford bracket algebra generated by null vectors is called *null bracket algebra* (NBA). Being nilpotent, null vectors provide great benefits to algebraic manipulations, and as a result, long brackets composed of null vectors have much richer symmetries than those composed of other vectors.

Null bracket algebra is a universal algebra of advanced invariants for classical metric geometries. It is also an algebra of advanced invariants for a class of non-classical geometry – contact geometry, where all points, hyperplanes and spheres are represented by null vectors.

## 1.6 Geometric reasoning with advanced invariant algebras

Geometric reasoning is a common task in mathematics education, computer-aided design, computer vision and robot navigation. Traditional geometric reasoning follows either a logical programming approach in artificial intelligence, or a coordinate approach in computer algebra [96], [201], or a hybrid approach based on both basic geometric invariants such as areas, volumes and distances, and dynamic geometric database management [40], [41].

In algebraic approaches to geometric reasoning and theorem proving, the input of a geometric problem is formulated by a set of symbols and their algebraic relations, and the algebraic computing of the conclusion expression, if geometrically meaningful, is called "symbolic geometric computing".

It has been widely known that algebraic methods such as the Shaw Prize winner W.-T. Wu's method of characteristic sets, the Gröbner base method, and the resultant method, are much more efficient than the logical programming approach to mechanical geometric theorem proving. In these methods, coordinates are introduced to formulate the geometric theorems in question. Because of the difficulty in providing geometric interpretations for algebraic expressions in coordinate form, much of the recent research on automated geometric deduction has focused on developing high-level coordinate-free techniques.

Using geometric invariants in symbolic geometric computing has been an active research subject since 1980's. Apart from the benefit of better geometric interpretability when compared with coordinates, geometric invariants have a salient feature of reducing the size of symbolic manipulations.

In the 1990's, several invariant algebraic methods were proposed, including the bracket polynomial straightening method [175], the distance method [70], the area method [40], the biquadratic final polynomial method [148], and the vectorial equation solving method [103]. From the viewpoint of advanced invariants, these methods can all be classified as methods of basic invariants.

The first successful method of advanced invariants for automated theorem proving in projective geometry was proposed in 2003 [117]. Advanced invariants from Cayley bracket algebra are used together with basic invariants in an alternating manner, by eliminations of geometric constructions to get advanced invariants from basic invariants, and by Cayley expansions to get polynomials of basic invariants from advanced invariants. In projective incidence geometry, all the theorems tested by the method of advanced invariants are given *binomial proofs*, which means that at every step of the proving procedure, the conclusion expression as a polynomial of basic and advanced invariants, is always two-termed.

In projective conic geometry, however, many theorems cannot be given binomial proofs. The reason is that the Grassmann-Cayley algebra over $\mathcal{V}^n$ is on intersections and extensions of *linear geometric objects* such as points, lines and planes. For *quadratic geometric objects* such as conics and quadrics, the base space $\mathcal{V}^n$ is too small. Recently (see Chapter 4), the Grassmann-Cayley algebra of $n$D projective quadric geometry, called *quadratic Grassmann-Cayley algebra*, is established. In the case $n = 3$, this is a geometric algebra of projective conic geometry, by which the algebra of basic conic invariants and the algebra of advanced conic invariants can be set up.

For automated theorem proving in Euclidean geometry, a breakthrough was made by the author in 2007 [124], during his preparation of this book. The author proposed a recipe for symbolic computing with conformal geometric algebra and null bracket algebra. The recipe is composed of three parts:

- use long geometric product to represent and compute multiplicatively;
- use "**breefs**" to control the expression size locally;
- use Clifford factorization for term reduction and translation back to geometry.

(1) Long geometric product:

This strategy has two aspects. In representation, it refers to representing geometric constructions and relations *multiplicatively*, or more accurately, with as few terms as possible. In computation, it refers to eliminating most of the $\mathbb{Z}$-grading operators from a graded Clifford monomial by generating a polynomial of long brackets with as few terms as possible, called *least ungrading*.

Eliminating $\mathbb{Z}$-grading operators is a way of prolonging the geometric product by breaking up the barriers to the associativity of the geometric product. The purpose is to replace complicated syzygy manipulations by simple symmetry manipulations of the long geometric product, with the cost of generating a minimal number of terms out of a monomial in Geometric Algebra.

(2) **Breefs**:

In most cases, the goal of algebraic computing is to make *simplification* to algebraic expressions, *i.e.*, to shorten the expressions by reducing the number of terms, and to make the expressions more decomposed by producing more factors. If at every middle step of the computing, a factored and shortest result is reached, then not only the goal of computing can be realized more easily, but the middle expression swell can be effectively controlled.

The new guideline in invariant computing, called "**breefs**" – **b**racket-oriented **r**epresentation, **e**limination, **e**xpansion for **f**actored and **s**hortest result, is contrary to the traditional guideline of computing by normalization, in that it does not transform an expression into normal form, which usually incurs middle expression swell, instead it seeks to squeeze the expression locally to make it more compact. The new guideline leads to a lot of unique techniques. For example, if a vector indeterminate representing a geometric construction occurs several times in an expression, it can be given different algebraic representations at different parts of the same expression.

(3) Clifford factorization:

It includes two new algebras grown out of the interplay of null bracket algebra, Grassmann-Cayley algebra and conformal geometric algebra. They are *null Geometric Algebra* (NGA) and *null Grassmann-Cayley algebra* (NGC). The former is used to make factorizations and term reductions in null bracket algebra, while the latter is used to represent geometric constructions multiplicatively and to make Cayley factorizations.

About one hundred theorems in Euclidean geometry are tested, among which about four-fifth are given monomial or binomial proofs. In particular, more than one third of the theorems are given *monomial proofs*, *i.e.*, the conclusion of each theorem is represented by a single monomial, and keeps to be in monomial form till the end of the proof. This feature makes it impossible to find any analytic proof that is more elegant.

We do not have enough space left in this book to further explore this recipe and its magic effect in simplifying invariant geometric computing, so we advertize the forthcoming sequel [125] of this book for reference.

## 1.7 Highlights of the chapters

Chapter 2 is an introduction of bracket algebra, Grassmann-Cayley algebra and coalgebra, and the advanced invariant algebra for projective geometry – Cayley bracket algebra. The highlights of this chapter are

- the two new concepts *total meet product* and *reduced meet product*;
- Cayley expansion theory, of which most of the contents are moved to Appendix A to ease reading difficulty;
- Cayley bracket algebra.

Chapter 3 is on simplification techniques in bracket algebra, and applications to automated theorem proving in projective incidence geometry. Highlights:

- rational invariants, which are as important as classical invariants but seem to have been overlooked all the time;
- bracket polynomial divisions, with the discovery that divisions of invariants generate covariants;
- factorization and contraction algorithms.

Chapter 4 is on invariant representations of projective conics, the **breefs** principle, simplification techniques in conic computation, and applications to automated theorem proving. This chapter may be skipped by those not particularly interested in projective conics and quadrics. Highlights:

- conic representations and computations based on Cayley bracket algebra;
- quadratic Grassmann-Cayley algebra and quadratic bracket algebra, proposed by the author when he was an AvH Fellow at Christian-Albrechts Universität zu Kiel in 1998, but never formally published;
- **breefs**.

Chapter 5 is on inner-product bracket algebra, Clifford algebra, and Clifford expansion theory. The highlight is the *Clifford expansion theory* of expanding a monomial of Clifford algebra into a polynomial of inner-product Grassmann algebra. This theory is the foundation of Clifford bracket algebra. The study of Clifford expansions was initiated in the second half of the 20th century, first by physicists

E. Caianiello [32], A. Lotze [129], and later by mathematicians G.-C. Rota and J. Stein [158], A. Brini [29], *et al.*

Driven by the need to find factored and shortest expansions of the geometric product of a sequence of elements into a polynomial of inner products and outer products, and totally ignorant of any existing work on this subject, the author re-derived the whole theory of Clifford expansions in 2001 [115]. In comparison, the author's results contain not only all the formulas discovered earlier by other people, but also many new discoveries that can be taken as simplifications and extensions of the existing results. The author thanks B. Fauser and A. Brini for getting him to know of the earlier work on Clifford expansions.

Chapter 6 is on Geometric Algebra, its main computing techniques, Clifford coalgebra, and Clifford bracket algebra. Among the computing techniques unique to Geometric Algebra, there are three prominent ones that prove to be very powerful: (a) symmetry and commutation, (b) ungrading, (c) monomial simplification. Below we present a short introduction to the latter technique.

Monomial simplification is to reduce the redundancy of participating vectors in the geometric product. It includes *monomial factorization* to generate scalar-valued factors, and *monomial compression* to reduce the number of effective vectors. While monomial factorization techniques are to be developed in [125], monomial compression is the highlight of this chapter. This problem seems rather interesting, and inspires a sequence of conjectures on the algebraic structures of Clifford algebras, *e.g.*, the *maximal grade conjectures*. The author thanks D. Fontijne for putting up the invertible monomial (versor) compression problem to him.

Chapter 7 is on algebraic models of affine, Euclidean, and contact geometries, and on the corresponding geometric algebras of describing their incidence constructions – affine Grassmann-Cayley algebra, conformal Grassmann-Cayley algebra, and the Grassmann-Cayley algebra upon the Lie model. There are two attractive discoveries:

(1) The geometric exploration of the meet products of Minkowski blades in the conformal model, leads to an elegant characterization of the *knotting relation* between two objects.

(2) Positive vectors and negative vectors in the Lie model have the geometric interpretations of representing pencils of *intersecting Lie spheres* and *separating Lie spheres* respectively. So besides the contact problem among points, hyperplanes and spheres, the intersection and separation problems can also be dealt with in the Grassmann-Cayley algebra upon the Lie model.

Chapter 8 is on conformal Clifford algebra, dual Clifford algebra, and the universal homogeneous model of classical geometries. Four kinds of representations of 3D conformal transformations in conformal Clifford algebra are developed in full detail, for the purpose of applying them immediately to engineering problems:

- the geometric product of Minkowski blades [110] is a representation of the conformal transformation by direct geometric constructions;

- the Cayley transform [130] provides a large-scale representation of the conformal transformation by a polynomial of degree four;
- the outer exponential [130] provides a local representation of the conformal transformation by a polynomial of degree two;
- the fractional linear form of $n$D conformal transformations [184] is a natural extension of the classical complex fractional linear function representation of 2D conformal transformations.

In each chapter, those sections and subsections marked with asterisks at the end of their titles, are suggested to be skipped by inexperienced beginners for their first-time reading.