Chapter 1

A Role of the Melnikov-Type Methods in Applied Sciences

In this chapter an important role of the classical Melnikov method and its extension to study engineering systems is briefly described. Advantages and disadvantages of the Melnikov-type approaches are discussed. A state of the art development and application of analytical homo- and heteroclinic intersections is presented and various applications of the mentioned techniques to analyze and control dynamical systems are reviewed.

1.1 Introduction

The celebrated for many years classical Melnikov method is used in many cases to predict the occurrence of chaotic orbits in non-autonomous smooth one-degree-of-freedom nonlinear systems. It is applied construct the Melnikov function, and hence to predict either regular or chaotic behaviour of a studied simple dynamical system. On the other hand, it belongs to one of the widely studied perturbative approaches, and hence it has many advantages associated with application of these methods described in numerous monographs, including the frequently cited reference [58]. This group of methods contains the contribution of Igor V. Melnikov [53], who proposed the method, further referred to as MM. In its classical form it concerns two-dimensional dynamical systems, perturbed by relatively small periodic external excitations. It is possible to determine a set of the system parameters with perturbation, assuming a knowledge of a homoclinic bifurcation of a system without perturbation, which has a singular saddle point along with a homoclinic orbit governed by a parametric type equation.

The MM is one of the so-called *small (perturbation) parameter* methods. The above statement follows from the fact that the original system is perturbed by a relatively small excitation. This implies that the Melnikov function determines a measure of distance between stable and unstable manifolds in the Poincaré map using a linear term in the Taylor expansion. A question arises in this situation if such an approach is adequate to the considered problem. This problem was studied in [62]. The authors, analyzing the Duffing oscillator as a discretization of the system of masses distributed along spatial coordinates continuously, included in the analysis the terms of an order higher than the first one. In the conclusion to this work one can find a statement that taking into account only relatively small perturbations is sufficient for a qualitative description in the parameter space of the criteria of chaos occurrence, and taking into consideration higher order terms only small quantitative improvements are implied.

Although the Melnikov method is merely approximative, it is one of a few methods allowing analytical prediction of chaos occurrence. Moreover, it can be applied to a relatively large class of dynamical systems.

It is needless to emphasize that both the classical Melnikov method and its generalization in the form of the Melnikov-Gruendler method (MGM) (see [33-35]) found their application in the analysis of dynamical systems. The main advantages of both methods cover:

(i) possibility of obtaining analytical results;
(ii) possibility of applying the method in dynamical systems characterized by arbitrary but integrable characteristics (including discontinuities which occur in a finite number of points like e.g. friction characteristics);
(iii) high efficiency of the verification of numerically generated results;
(iv) possibility of examination of strongly nonlinear systems.

Both mentioned methods (MM and MGM) are not ideal, since they exhibit the following drawbacks:

(i) they are applicable to systems characterized by a specific phase-portrait, namely homoclinic orbits of a critical saddle point;
(ii) they are not exact but approximative methods which use a small parameter;
(iii) non-perturbed system should be integrable;
(iv) they enable prediction of values of the parameters associated only with the so-called homoclinic and chaos ;
(v) they are associated with rather complicated algebraic computations.

1.2 Application of the Melnikov-type methods

Despite the aforementioned imperfections, on the current level of nonlinear dynamical systems development, the Melnikov method has been accepted by the scientific community and it is a useful tool to perform advanced investigations in nonlinear dynamics, including deterministic chaos. In order to illustrate the importance of the MM and its impact on applied sciences, we briefly describe a direct use of this method together with its slight modifications to solve many important problems for both applied mathematicians and engineers.

Bulsara *et al.* [18] applied the MM to a system with stochastic excitation of probability distribution of finite mean value and variation. A formula, defining the Melnikov function, was derived for a considered system in a form of probability distribution (random variable) and its mean value. The obtained analytical results were compared with numerical simulations, performed for a system of the Duffing oscillator, mathematical pendulum and an abstract system, while high compliance of predictions and numerical experiments was achieved.

Formulas derived in reference [9] enable determination of the Melnikov function for oscillators (in particular mechanical systems with stick-slip behavior are addressed) in a simple form. The proposed approach simplifies the calculations needed to define conditions for homoclinic bifurcation exhibited by oscillating dynamical systems.

Taki [65] applied the Melnikov method to a bistable optical system which is characterized by two homoclinic orbits connected with the same saddle point, while one of them is contained in the other. A criterion of the homoclinic chaos for one of the orbits was numerically determined on the basis of the formula describing the Melnikov function. A criterion corresponding to the second orbit was not considered, since it predicted chaos occurrence for relatively large values of the parameter of the considered dynamical system which should be relatively small on the assumption. The obtained results were compared with the numerical simulations. High compliance was achieved between the simulations and predictions of chaotic motions obtained by the Melnikov method.

Holmes and Marsden [41] applied the method to a periodically driven buckled beam. By applying the Galerkin method they obtained an equivalent mathematical model in a form of the Duffing oscillator.

Mielke and Holmes [54] used the MM to examine a problem of buckling of a strongly curved rod. In the work, a special attention was paid to chaos which can occur in systems with at least two heteroclinic orbits and where the Melnikov function has no single roots. This happens when stable and unstable differentiable manifolds of the same saddle point (which belongs to different homoclinic orbits when there is no perturbation) approach each other and form a chaotic attractor represented by infinite, countable number of intersections of stable and unstable manifolds but they will not intersect simultaneously with other manifolds of the second equilibrium point. In this case, the Melnikov function evaluated along the heteroclinic orbit, being a measure of distance between non-intersecting manifolds, will not have roots. Thus, it was proved that in some dynamical systems the chaos could occur although the Melnikov method showed regular motions.

Moon and Li [57] considered a dependence of the fractal structure of a basin of attraction on the occurrence of homoclinic bifurcation taking as an example the Duffing oscillator. The threshold parameters predicted by MM imply the presence of a chaotic attractor. Next, basins of attractions for several selected values of the parameters were reported. It was shown, among the others, that the structure of the basin of attraction boundary was related to homoclinic bifurcation and chaos.

Moon [55], and Moon and Holmes [56] examined the system of an elastic and harmonically excited pendulum of uniformly distributed mass in the form of a steel flat bar of small thickness which was located in a specific magnetic field. By applying the Galerkin method to a system governed by partial differential equations a discrete mathematical model of the examined system in a form of the Duffing equation was derived. Results obtained using the classical MM were compared with laboratory investigations. A quantitative divergence was observed between predictions, which followed from the Melnikov method and experimental investigations at qualitative compliance. Generally, a larger amplitude of external excitation turned out to be necessary to make deterministic chaos arise in the examined system than it was predicted by the method. Although the approximation obtained by Holmes [40] was better than this given by the Melnikov method, the mentioned results should be approached critically, since instead of using the Lyapunov exponents the Fourier transform was applied. It seems that methodological imperfections could have been a reason of incorrect identification of weak chaos as regular motions for smaller values of the external excitation.

Guckenheimer and Holmes [35] applied the Melnikov method to determine limiting parameters of a harmonically driven and damped Duffing oscillator at which the homoclinic bifurcation occurred. Conditions of occurrence of subharmonic vibrations in the parameters space by example of the mentioned oscillator were detected. Thus, the Melnikov method was applied to a system with heteroclinic orbits in the phase plane.

Koch and Leven [44] applied the Melnikov method to examine a parametrically driven mathematical pendulum. The Melnikov function and criterion of homoclinic and subharmonic bifurcations were determined for the considered system. The numerical investigations confirmed the occurrence of tangency of stable and unstable orbits for critical parameters.

The Melnikov method was used as a detector of global homoclinic structures in [17]. As a result of the performed investigations the averaged frequency spectra were obtained analytically (the Fourier integral transformation applied) for several arbitrarily selected values of the parameters.

Salam [63] used the Melnikov method to study dissipative systems by example of a strongly damped mathematical pendulum at relatively small time-periodic external excitation and constant driving torque. He showed numerically that the system without excitation had a heteroclinic orbit and gave intersection of stable and unstable differentiable manifolds for relatively small excitation. Predictions based on the Melnikov method were numerically confirmed for several selected parameters of the given system.

Some self-excited systems with dry friction are included in a group of discontinuous systems and have been studied extensively for a long time. The Melnikov method was applied to non-smooth systems by example of a self-excited, relatively weakly driven (quasi-autonomous) Duffing oscillator of one-degree-of-freedom with polynomial-type friction [7] for the first time derived in [9]. The analytically defined Melnikov function for one degree-of-freedom oscillators, simultaneously described both types of motions: stick and slip. Taking into account only one criterion of chaos was a disadvantage of this method, though two such criteria could have been expected. In reference [7], due to the application of the Melnikov method, one proved analytically that chaotic attractors could occur in autonomous systems with almost zero initial excitation and with dry friction. Hence, with a help of the Melnikov method one verified the phenomenon of deterministic chaos, known from numerical and laboratory investigations [11, 29, 61], in self-excited systems with dry friction without external excitation. The results, obtained by means of the Melnikov method, were numerically confirmed in reference [3].

On the other hand, on the basis of references [1] and [7], an interesting application of the Melnikov method to a rotary cylinder-bush system was presented in [12]. One considered a physical-mathematical model governing the influence of heat emitted during the dry friction process on the friction force magnitude. However, only a numerical analysis of the Melnikov function was performed due to its rather complex form.

Litak *et al.* [51] examined a classical Froude pendulum numerically. The numerical investigations showed the occurrence of deterministic chaos, despite numerous simplifications of restitutive characteristics and friction in the form of Taylor expansions up to the third order. Authors of [39] applied the Melnikov method to analyze the Froude pendulum with dry friction, however analytical calculations were not verified by numerical investigations.

Systems with nonlinearities of dry friction-type are typical of mechanical engineering, but a similar behavior can be found in majority of electric circuits containing diodes, transistors, logical systems etc., with elements of *piecewise differentiable* or *discontinuous* (jump variable) characteristics.

Endo and Chua [28] applied the Melnikov method to an electronic system, governed by equations equivalent to the equations of motion of a mathematical pendulum and attempted to apply the described method of homoclinic chaos prediction to a system characterized by linear restitutive characteristics (commonly known as the Chua system [2, 10]). Two different criteria of chaos for each of the examined systems, i.e. one for each of the homoclinic orbits, were derived. Chaotic dynamics of a rotated pendulum using the Melnikov method was predicted in reference [38]. It was shown, among the others, that the one degree-of-freedom system could exhibit either two or four homoclinic orbits and one or two Melnikov criteria could be applied, respectively.

A large rotating nonlinear multibody system (drag line) with energy dissipation exhibiting chaotic instabilities was studied in reference [52]. The sufficient analytical criterion for critical parameters set responsible for occurrence of chaotic motion was derived using the Melnikov method.

Asymmetric spacecraft dynamics perturbed by small aerodynamic drag torque with periodic in time moments of inertia was investigated in reference [42]. Transientchaotic behaviour of the system was predicted by means of the Melnikov method. Chaotic critical thresholds of a nonlinear elastic beam large deflections were formulated with a help of the Melnikov function method in reference [36].

Existence, stability and bifurcation theorems for subharmonics of the planar Hamiltonian systems were reported in reference [68] using the modified Melnikov theory.

Both Galerkin and Melnikov approaches were used to study chaotic motion of an elastic cylindrical shell in reference [37].

In reference [43] global bifurcation behavior of 2-DOF nonlinear oscillator was analyzed using the Melnikov method. A Smale horseshoe type of chaos was illustrated, among the others.

The noise-induced chaos predicted by the Melnikov method (MM) was investigated in the softening Duffing oscillator in reference [30]. In reference [27] MM was extended also to study an inverted pendulum externally driven and impacting rigid walls. The Melnikov function up to the n-th order was computed and a critical set of parameters for the persistence of homoclinic impact cycles were estimated.

Yagasaki [69] studied periodic and homoclinic motions in periodically forced and weakly coupled oscillators with perturbation of two independent planar Hamiltonian systems. The relationship between the subharmonic and homoclinic Melnikov theories was illustrated, and the modified homoclinic MM was directly applied to study two types of periodic orbits.

Chen [22] extended MM to perturbed planar non-Hamiltonian integrable systems with slowly-varying angle variables, giving the condition of transversely homoclinic intersection.

Lamarque and Bastien [45] studied a forced pendulum with viscous damping and Coulomb friction. Lyapunov exponents were computed and a Melnikov relation was obtained as a limit of regularised Coulomb friction.

Chaotic attitude motion and its control of a magnetic rigid spacecraft in an arbitrary circular orbit round the Earth with a help of MM was investigated by Chen and Liu [23]. The feedback control and its local linearization were used to control chaotic attitude motions to the given either fixed point or periodic motion.

Yagasaki [70] analyzed a homoclinic behavior in resonance cases of non-conservative forced oscillators. He applied MM to get a simple condition under which separatrix splittings with exponentially small upper bounds might appear. He also used MM to study codimension-two Bogdanov-Takens bifurcation for subharmonics in periodic perturbations of planar Hamiltonian systems [71]. Criteria of the Bogdanov-Takens bifurcations, approximate relations for saddle-node, Hopf and homoclinic bifurcation sets and the Bogdanov-Takens bifurcation points were derived.

MM regarding generalized Hamiltonian systems was used to explain the control rules of directing chaotic motion towards low-periodic motion in the Lorenz equations in reference [19]. Numerical simulation verified the effectiveness of the applied Melnikov's technique.

Casasayas et al. [21] showed that the conditions for the persistence under perturbation of the invariant manifolds also ensured the convergence of the Melnikov integral in the case when the unperturbed system had a parabolic orbit with a homoclinic loop.

MM was applied to analyze the intermittency transition between order and in the chaotic Duffing-type system [66]. Weak signals were detected and estimated.

The application and extension of Melnikov's idea by considering Poincaré sections non-orthogonal to the flux and by applying both the so called "one-half" and "full" Melnikov functions are proposed by Lenci and Rega [46–49]. In the mentioned papers it is shown (among the others) that classical Melnikov's technique is practically inaccurate for both small and large excitation frequencies with respect to degenerated homo/heteroclinic bifurcations and in the case of generic periodic excitations.

A sufficient condition for controlling chaos using a weak resonant excitation is given in reference [20]. In addition, it is shown how the MM illustrates a vital role of the initial phase difference in suppressing or inducing either chaotic or quasi-periodic dynamics.

Some new methods to suppress chaos via slightly modified Melnikov function are proposed by Leung and Zengrong [50].

The dynamics of both stochastic and resonant layers for rotated pendulum equations with a use of MM were studied in reference [24]. A simultaneous occurrence of chaotic and subharmonic dynamics was proven.

MM is applied to derive criteria for chaos of a cylindrical shell single model approximation in reference [25]. A discussion of one and double modes approximation is supplemented.

Zhu and Liu [72] derived the random Melnikov process, and they proposed a mean square criterion to determine the threshold amplitude of the bounded noise for the onset of chaos in coupled pendulums and harmonic oscillator under bounded noise excitation.

Control and Melnikov chaos of an harmonically excited particle from a catastrophic single well potential are studied in reference [59]. Both energy and Melnikov methods are used to obtain critical external forcing amplitudes for catastrophe and chaos.

Dynamics of a nonlinear elastic shallow shell of large deflection subjected to constant boundary loading and harmonic lateral excitation is studied in [26]. Three different dynamic equations of the shell were derived and the associated Melnikov functions were formulated. Critical condition for chaos occurrence were provided, and then regular and chaotic vibrations of the shell were studied using history plots, phase diagrams and Poincaré maps.

The Melnikov method, despite its numerous applications in the classical form, has an essential disadvantage: the method can be applied to mechanical systems of one degree-of-freedom with excitation. This disadvantage does not occur in the Melnikov-Gruendler method, as it will be shown in this book.

Guckenheimer and Holmes [35] were the first who based on the KAM theorem, determined a set of parameters of relatively weak perturbed Hamiltonian systems of two-degrees-of-freedom, at which homoclinic bifurcation occurred.

Another possibility is the approach based on the assumption that if a mechanical system of a finite number of degrees-of-freedom is an integrable system, then it is described by the Routh equation [16]. If the number of first integrals allows the Routh equation to satisfy the assumptions of the Melnikov method, then it is possible to apply the classical Melnikov method to a system of a larger number of degrees-of-freedom. In the case when the number of first integrals is unknown, one needs to apply the Melnikov-Gruendler method [34], which is an extension of the Melnikov method to non-Hamiltonian dynamical systems of arbitrary, finite number of degrees-of-freedom. For some types of systems (symmetry of variational equations on a homoclinic orbit, no coupling, etc.) the formulas determining chaos criteria can be simplified.

In contrast to the classical Melnikov method, the Melnikov-Gruendler method is not widely known. It was applied in [34] to consider small vibrations of a spherical pendulum subjected to magnetic field with a relatively small and then large viscous damping (a non-Hamiltonian system). Moreover, the Melnikov-Gruendler method was used in both physical interpretation and abstract dynamical systems [32, 33], in order to detect deterministic chaos and analyze structural changes of homoclinic orbits.

Reference [14] played a key role in the attempt to apply the Melnikov-Gruendler method to self-excited systems with polynomial-type friction of two degrees-of-freedom. However, the obtained results did not reflect complexity of the multi-dimensional problem entirely, by making use of only 8% of the method capabilities. For this reason, one can say that the problem of homoclinic chaos occurrence in two degrees-of-freedom systems with dry friction has not been examined by means of the Melnikov-Gruendler method so far. Mechanical systems with friction of three degrees-of-freedom need investigation with a help of the Melnikov-Gruendler method. Other and these types of problems are addressed in this monograph.