

## Chapter 1

# Introduction

The classical Calculus of Variations deals with finding minima of functionals  $\Phi : X \rightarrow \mathbb{R}$  that are bounded below. The basic idea of the direct method is to consider a minimizing sequence  $\Phi(u_n) \rightarrow \inf \Phi$ , to find a convergent subsequence  $u_{n_k} \rightarrow u$ , and to show that  $\Phi(u) = \inf \Phi$ . In order to make this work the space  $X$  should have a topology which is rather weak for the existence of a convergent subsequence, and rather strong so that  $\Phi$  is lower semicontinuous. In many applications the functional is not bounded below and instead of a minimizer one is interested in critical points. This is the concern of the Calculus of Variations in the Large or Critical Point Theory, which has undergone an enormous development in the last century due to the work of mathematicians like Morse, Lusternik, Schnirelman, Palais, Smale, Rabinowitz, Ambrosetti, Lions, Struwe, Witten, Floer and many others, with applications to problems from analysis, geometry and mathematical physics. Here one usually requires  $X$  to be a Banach manifold and  $\Phi$  to be differentiable. An essential ingredient is the construction of a flow  $\varphi$  on  $X$  so that  $\Phi(\varphi(t, u))$  is decreasing in  $t$ . This flow is used in the spirit of Morse theory, to construct deformations of sublevel sets  $\Phi^c = \{u \in X : \Phi(u) \leq c\}$ , and to find Palais-Smale sequences  $(u_n)_n$ , that is:  $\Phi(u_n)$  is bounded and  $\Phi'(u_n) \rightarrow 0$ , replacing the minimizing sequences. Typical results are the mountain pass theorem of Ambrosetti and Rabinowitz or various linking theorems. The proofs use in an essential way topological concepts based on the Brouwer or Leray-Schauder degree. The theory has also been extended to deal with (semi-)continuous functions on metric spaces, forced by problems from nonlinear elasticity (see [Degiovanni and Schuricht (1998)]). Another generalization concerns variational methods for functionals on closed convex subsets of Banach spaces developed by Struwe [Struwe (1989)] for Plateau's problem. Such functionals appear also in variational inequalities.

Motivated by several applications, for instance to finite- and infinite-dimensional Hamiltonian systems, nonlinear Schrödinger equations and nonlinear Dirac equations, we were led to consider  $C^1$ -functionals  $\Phi : E = E^- \oplus E^+ \rightarrow \mathbb{R}$  defined on the product  $E = E^- \oplus E^+$  of Banach spaces  $E^\pm$  with  $\dim E^\pm = \infty$  but where one needs to work with the weak topology on  $E^-$  in order to gain compactness. The

functionals typically have the form

$$\Phi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \Psi(u) \quad \text{for } u = u^- + u^+ \in E^- \oplus E^+. \quad (1.1)$$

Since  $\dim E^\pm = \infty$  the functional is strongly indefinite. Thus all of its critical points have infinite Morse index. Moreover,  $\Psi' : E \rightarrow E^*$  is not completely continuous and the Palais-Smale condition does not hold in our applications. This makes applications of Leray-Schauder degree type arguments rather subtle. On the other hand the functional  $\Psi : E \rightarrow \mathbb{R}$  is weakly sequentially lower semicontinuous and  $\Psi' : E \rightarrow E^*$  is weakly sequentially continuous. It turns out that the product topology

$$\mathcal{T} = (\text{weak topology on } E^-) \times (\text{norm topology on } E^+)$$

is well suited for certain arguments because  $\Phi : (E, \mathcal{T}) \rightarrow \mathbb{R}$  is sequentially upper semicontinuous, and  $\Phi' : (E, \mathcal{T}) \rightarrow (E^*, \text{weak}^* \text{ topology})$  is continuous. Given a finite-dimensional subspace  $F \subset E^+$  the unit ball of  $E^- \oplus F$  is  $\mathcal{T}$ -compact, and given a bounded sequence  $(u_n)_n$  the negative part  $(u_n^-)_n$   $\mathcal{T}$ -converges (up to a subsequence). When one wants to develop critical point theory with this topology on  $E$  one needs to construct deformations on  $E$  which are  $\mathcal{T}$ -continuous. Deformations are usually obtained by integrating vector fields which in turn are constructed with the help of partitions of unity. So one needs to construct these in a  $\mathcal{T}$ -Lipschitz continuous way. A more difficult situation occurs when one is interested in “normalized solutions”, that is critical points of  $\Phi$  constrained to the unit sphere  $SE = \{u \in E : \|u\| = 1\}$  or to other finite-codimensional submanifolds  $X$  of  $E$ .

The  $\mathcal{T}$ -topology on  $X$  is not metrizable, therefore the by now well developed critical point theory for (semi-)continuous functions on metric spaces cannot be applied. Instead the  $\mathcal{T}$ -topology is generated by a family  $\mathcal{D}$  of semi-metrics. A pair  $(X, \mathcal{D})$  consisting of a set  $X$  and a family of semi-metrics is called a *gage space*; see [Kelley (1995)]. The paper [Bartsch and Ding (2006I)] is a first step to develop critical point theory on gage spaces. We begin by settling some basic topological questions. We introduce the concept of a Lipschitz map  $(X, \mathcal{D}) \rightarrow \mathbb{R}$  and of a Lipschitz normal gage space (disjoint closed sets can be separated by Lipschitz maps). We find conditions on  $(X, \mathcal{D})$  so that  $X$  is Lipschitz normal and so that Lipschitz partitions of unity (subordinated to a given open cover) exist. In particular, we show that given a Banach space  $B$ , an arbitrary subset  $B_0 \subset B$ , and letting  $\mathcal{D}$  be the family of semi-metrics on  $X = B^*$  given by  $d_b(x, y) := |\langle b, x - y \rangle_{B, B^*}|$ ,  $b \in B_0$ , the gage space  $(B^*, \mathcal{D})$  is Lipschitz normal. More generally, if  $(Y, d_Y)$  is a metric space then the product gage space  $(B^*, \mathcal{D}) \times (Y, d_Y)$  is Lipschitz normal and has Lipschitz partitions of unity. In addition, if  $B$  is separable and  $B_0 \subset B$  is dense then also every locally closed subset (that is, an intersection of an open and a closed subset) of this product gage space is Lipschitz normal and has Lipschitz partitions of unity subordinated to an arbitrary open cover.

We then present some nonlinear problems where the abstract theory developed here can be applied. These problems arise in mechanics, physics, control theory and

other topics, which are variational in nature with the feature that their solutions correspond to critical points of certain strongly indefinite functionals of the form (1.1). We are interested in the existence and multiplicity of solutions to these problems. The details are arranged in the last four chapters. In Chapter 5 we study the homoclinic orbits in the classical Hamiltonian systems

$$\begin{cases} \mathcal{J} \frac{d}{dt} z + L(t)z = R_z(t, z) & \text{for } t \in \mathbb{R} \\ z(t) \rightarrow 0 & \text{as } |t| \rightarrow \infty \end{cases}$$

with periodic or non-periodic (with respect to the time  $t$ ) Hamiltonians. Chapter 6 is devoted to the standing waves of the nonlinear Schrödinger equations

$$\begin{cases} -\Delta u + V(x)u = g(x, u) & \text{for } x \in \mathbb{R}^N \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty \end{cases}$$

with  $V$  and  $g$  being periodic in  $x$ . We also treat here semiclassical states of a Hamiltonian system of perturbed Schrödinger equations:

$$\begin{cases} -\varepsilon^2 \Delta \varphi + \alpha(x)\varphi = \beta(x)\psi + F_\psi(x, \varphi, \psi) \\ -\varepsilon^2 \Delta \psi + \alpha(x)\psi = \beta(x)\varphi + F_\varphi(x, \varphi, \psi) \\ (\varphi, \psi) \in H^1(\mathbb{R}^N, \mathbb{R}^2) \end{cases}$$

without any periodicity assumption. Chapter 7 deals with localized solutions of the nonlinear Dirac equations with external fields

$$\begin{cases} -i\hbar \sum_{k=1}^3 \alpha_k \partial_k u + \beta m u + M(x)u = G_u(x, u) & \text{for } x \in \mathbb{R}^N \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty \end{cases}$$

with either scale potentials (i.e.,  $M(x) = \beta V(x)$ ), or vector potentials (say, the Coulomb-type potentials). We also study semiclassical solutions (as  $\hbar \rightarrow 0$ ). Finally, in Chapter 8 we handle solutions of homoclinic type to the systems of diffusion equations

$$\begin{cases} \partial_t u - \Delta_x u + \mathbf{b}(t, x) \cdot \nabla_x u + V(x)u = H_v(t, x, u, v) \\ -\partial_t v - \Delta_x v - \mathbf{b}(t, x) \cdot \nabla_x v + V(x)v = H_u(t, x, u, v) \end{cases}$$

for  $(t, x) \in \mathbb{R} \times \mathbb{R}^N$  with  $u(t, x), v(t, x) \rightarrow 0$  as  $|t| + |x| \rightarrow \infty$ . In all these problems the nonlinear terms are assumed to be either asymptotically linear or super linear. In the arguments certain analytical estimates which are needed to check the assumptions of the abstract results require different techniques. We prove new results extending the previous relative works in the literature.