Chapter 1

Introduction

In the last two decades or so, there has been a resurgence in the analysis of the behavior in complex networks of interacting systems. During this time, two parallel branches of research activities have emerged. On the one hand, starting with the 1983 paper by Fujisaka and Yamada on synchronization in coupled systems [Fujisaka and Yamada (1983)] and subsequently the 1990 paper by Pecora and Carroll on synchronization in chaotic systems [Pecora and Carroll (1990)], there has been a plethora of activity on the synchronization of coupled chaotic systems. In this form of synchronization, called *complete* synchronization[1], the state variables of individual systems converge towards each other. Complete synchronization is more restrictive than phase synchronization that was studied as early as the 17th century by Huygens [Bennett *et al.* (2002)], and can be easier to analyze. In the last few years, agreement and consensus protocols of interconnected autonomous agents have been actively studied in the control systems community. In these problems the goal is to have the states of the agents agree to each other. For instance, the state can be the heading of a mobile agent in flocking problems where the goal is to have all agents move in the same direction. These problems, whose state equations form a linear system, can also be considered as a synchronization problem.

On the other hand, novel models of random graphs have been proposed to study the complex networks that we observe around us. In 1998 Watts and Strogatz proposed a model of a small-world network [Watts and Strogatz (1998)] and in 1999 Albert and Barabási proposed a model of a scale-free network based on preferential attachment [Barabási and Albert (1999)]. These graph models mimic complex networks in natural and man-made systems more accurately than the classical random graph mod-

[1] This is sometimes also referred to as identical synchronization

els studied by Rapoport [Rapoport (1957)] and by Erdös and Renyi [Erdös and Renyi (1959)] in the late 1950's. Examples of such networks include communication networks, transportation networks, neural networks and social interaction networks. Although features of these networks have been studied in the past, see for example Milgram's letter passing experiments [Milgram (1967)] and Price's citation network model [de Solla Price (1965, 1976)], it was only recently that massive amount of available data and computer processing power allow us to more easily analyze these networks in great detail and verify the applicability of various models.

The present book studies the intersection of these two very active research areas. In particular, the main object of study is synchronization phenomena in networks of coupled dynamical systems where the coupling topology can be expressed as a complex network. We attempt to combine recent results in these two interdisciplinary areas to obtain a view of such synchronization phenomena.

The focus of this book is on complex interacting systems that can be modelled as an interconnected network of identical systems. In particular, we are interested in the relationship between the coupling topology and the ability to achieve coherent behavior in the network. A typical interconnected system is illustrated in Fig. 1.1. We characterize the coupling topology by means of a directed graph, called the *interaction graph* . If system i influences system j, then there is a directed edge (i, j) starting from system i and ending in system j. In this case, the circles in Fig. 1.1 are vertices and the arrows are edges of this graph. We consider weighted directed graphs, where the weight of an edge indicates the coupling strength of that connection. By definition, the interaction graph of a network is *simple*, i.e. it does not contain self-loops from a vertex to itself and there is at most one edge between vertices.

One of the main results in this text is the following intuitive conclusion. If there is a system (called the *root system*) which influences directly or indirectly all other systems, then coherent behavior is possible for sufficiently strong coupling. The root system can change with time and the ability of the root system to influence all other systems can occur at each moment in time, or through a number of time steps. To reach this conclusion we use tools from dynamical systems, graph theory, and linear algebra.

This text is organized as follows. Chapter 2 summarizes basic graph theory, properties of graphs and linear algebra that we need. Detailed proofs of some of the results in Chapter 2 are found in Appendix A.

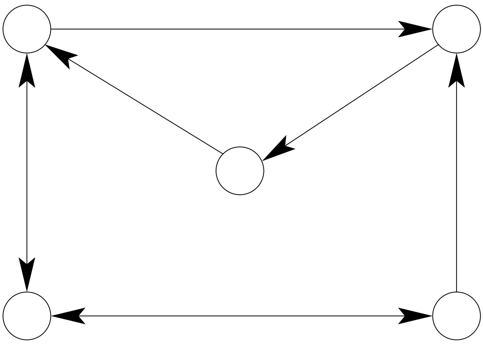

Fig. 1.1 Network of interconnected systems. Circles indicate individual systems and arrows indicate the coupling between them.

In Chapter 3 we study various models of graphs that have been proposed to model man-made and naturally occurring networks.[2]

In Chapter 4 we study synchronization in a network of nonlinear continuous time dynamical systems. The case of discrete-time systems will be addressed in Chapter 5. In these 2 chapters we establish a relationship between synchronizability and properties of the interaction graph.

In Chapter 6 we consider the special case where the coupling and the individual systems are linear. In particular, we show how the discrete time case is related to ergodicity of inhomogeneous Markov chains.

Finally, in Chapter 7 we study several consensus and agreement problems of autonomous agents that have been formulated as a network of coupled nonautonomous linear systems.

[2]We use the words *graph* and *network* interchangeably.