Chapter 1

Introduction

1.1 What is Phase-Locked Loop?

The phase-locked loop (PLL) is an electronic system which has numerous important applications.

It consists of three elements forming a feedback loop: voltage controlled oscillator (VCO), phase detector (PD) and low-pass filter.

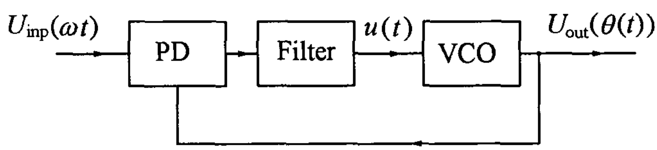

Fig. 1.1 Scheme of the phase-locked loop.

Generator VCO produces electrical oscillations $U_{out}\big(\theta(t)\big)$ periodic with respect to θ. The waveforms of these oscillations may be different, for example sinusoidal, saw-tooth, rectangular or other. The angular frequency $\frac{d\theta}{dt}$ depends on slowly varying signal $u(t)$ controlling the oscillator. Usually, if the controlling voltage increases, the generator frequency also increases.

The aim of the phase detector PD is to produce a signal which controls the generator. This signal depends on the phase difference $\theta(t) - \omega t$ of the generator signal and the input signal U_{inp}. Usually PD is a system which multiplies both these signals. If, for example, $U_{inp} = U \sin\big(\omega t + \varphi_1(t)\big)$ and $U_{out} = \cos\big(\omega t + \varphi_2(t)\big)$ then at the output of the phase detector we get

$$U_{inp} \cdot U_{out} = -\frac{1}{2} U \sin\big(\varphi_2(t) - \varphi_1(t)\big) + \frac{1}{2} U \sin\big(2\omega t + \varphi_1(t) + \varphi_2(t)\big). \quad (1.1)$$

If $\frac{d\varphi_1}{dt} \ll \omega$ and $\frac{d\varphi_2}{dt} \ll \omega$, the first term of the right-hand side of (1.1) is a slowly varying signal used to tune the VCO, whereas the second high--frequency component is harmful and should be suppressed by a low-pass filter.

The feedback system tends to minimize the error signal $\frac{1}{2}U\sin(\varphi_2 - \varphi_1)$ and, in consequence, the output signal phase tracks the changes of the input signal phase.

The basic property of the PLL, useful in practice, is its ability to synchronize (to lock in). It means that when the input signal is periodic then, in a steady-state, the mean frequency of output signal generated by the VCO is equal to the input signal frequency, and the phase difference of these signals is small and almost constant.

There are two basic kinds of phase detectors. The first of them compares the phases continuously, and the second does it only in discrete moments of time determined by the output (or input) signal phase. The phase detector is then a sampling-and-holding system and the controlling voltage $u(t)$ changes (jumps) at the moments when the output (input) signal phase equals an integer multiple of 2π. The magnitude of this change depends on the input (resp. output) signal phase. Such systems are used, among other purposes, to multiply and divide the frequency.

In technical literature on the PLL systems (see Refs. [9], [10], [19], [22], [28], [33], [34], [35], [53], [58], [59]) one can find a long list of various applications. The most important of them are:

1. Locked oscillators and frequency synthesizers,
2. Modulators, demodulators and converters (AM, FM, PM),
3. Recovery of the clock signal,
4. The tracking filters,
5. Frequency multipliers and dividers, coherent transponders,
6. Synchronization of digital transmission.

The PLL systems are produced also in integrated version and find use in many technical devices of electronics and telecommunication.

1.2 PLL and differential or recurrence equations

The phase-locked loop with continuous time is most frequently described by the system of ordinary differential equations. The order of these equations is greater by one then the order of the filter which connects the phase detector and the voltage controlled oscillator. In the state vector of the system the first coordinate is the phase $\theta(t)$ of the output signal $U_{\text{out}}(\theta(t))$, the second is the voltage $u(t)$ controlling the VCO, and the remaining coordinates characterize the inner structure of the filter.

In particular, if the filter is of the first order then the equations of the PLL system are of the second order

$$\frac{d\theta}{d\tau} = F_1(\theta, u, \tau), \qquad \frac{du}{d\tau} = F_2(\theta, u, \tau), \tag{1.2}$$

where $\tau = \omega t$.

For the phase-locked loop without filter the equation is of the first order

$$\frac{d\theta}{d\tau} = F_0(\theta, \tau). \tag{1.3}$$

The input signal $U_{\mathrm{inp}}(\omega t)$ is most frequently a periodic function of time of the period $\frac{2\pi}{\omega}$. Then the functions F_0, F_1, F_2 are 2π-periodic with respect to the variables θ and τ. This specific property of equations enables us to treat the variables θ and τ as cyclic variables determined up to a multiple of 2π. To visualize the solutions of equations we then choose a space with suitable geometric properties (suitable system of coordinates). For the equation (1.3) it will be a surface of torus instead of plane, and for equations (1.2) torus or cylinder (solid).

Equations of each physical system depend on a number of parameters. In PLL systems two parameters are especially important: the amplitude and the frequency of a periodic input signal. So, it is necessary to investigate how the whole family of differential equation solutions corresponding to all admissible initial conditions depends on these parameters. Further investigations concern the dependence of solutions on constructional parameters of the system: the cut-off frequency of the filter, the coefficient of amplification and the quiescent frequency of VCO. It is not a simple task because these equations are strongly nonlinear and their solving demands using the advanced theory of differential equations.

The number of essentially different physical phenomena observed in the second order system (1.2) is greater then that occurring in the first order system (1.3). However, in some ranges of parameters the high order system behaves "just the same" as the second order system (and similarly, the system of the second order may behave like a system of the first order). This happens when there exists a globally attracting stable integral manifold of a suitably low dimension. The conditions for the existence of such a manifold have substantial practical significance and this is why we devote them a lot of place (Secs. 3.5, 3.9, 5.3).

In technical literature one usually admits many simplifications. The most important of them is neglecting the fast-varying part of the input signal of VCO. For the phase difference $\varphi(\omega t) = \theta(t) - \omega t$ (but not for $\theta(t)$) we get the autonomous system of equations

$$\frac{d\varphi}{d\tau} = F_{a1}(\varphi, u; \omega), \quad \frac{du}{d\tau} = F_{a2}(\varphi, u; \omega), \quad \text{where } \tau = \omega t, \tag{1.4}$$

instead of equations (1.2). The frequency ω occurs here as a parameter. This procedure of averaging will be justified in the next section. Of course, the equations (1.4) are easier to analyze then the equations (1.2), (compare Secs. 2.2, 3.2).

The second simplification is a linearization of the system in a neighborhood of the stable steady-state (compare Sec. 3.9.4). For the autonomous system the linearized equations are most frequently solved using the Laplace transformation and the so called transfer function. Transfer function has simple physical interpretation and provides electrical engineers with valuable information about the behavior of the linearized system. However, the linearization of the system permits to investigate only its local properties in a small neighborhood of the steady-state, but it gives no answer to such an important question as the range of initial conditions and parameters for which the system remains in synchronization (pull-in and hold-in ranges).

For discrete-time phase-locked loops the output signal phase $\theta(t)$ is compared with input signal phase $\tau = \omega t$ only in discrete moments t_n, for example, when $\theta(t_n) = 2\pi n$, for $n = 0, 1, 2, \ldots$ The behavior of the second order system is completely determined by the properties of the point sequence

$$(\tau_0, u_0),\ (\tau_1, u_1),\ (\tau_2, u_2),\ (\tau_3, u_3), \ldots \tag{1.5}$$

where $\tau_n = \omega t_n$ and $u_n = u(t_n)$. Instead of the differential equation the system is described by the recurrence equation (map), which relates a position of the point (τ_{n+1}, u_{n+1}) with position of the point (τ_n, u_n) in a plane or on a cylindrical surface if we treat the variable τ as cyclic variable determined up to a multiplicity of 2π.

For more simplified models of phase-locked loops instead of (1.5) it is sufficient to investigate the sequence of points

$$\tau_0, \quad \tau_1, \quad \tau_2, \quad \tau_3, \ldots \tag{1.6}$$

on a straight line or on a circle (with perimeter normalized to 2π).

From the behavior of sequence (1.5) or (1.6) one can deduce whether the PLL system synchronizes, when there is multiplication or division of the input signal frequency, what the sensitivity of the system is with respect to small perturbations, how fast a synchronization state is reached, and also, when chaotic oscillations appear.

1.3 Averaging method

In technical literature the high-frequency component of the output signal of PD (the second term on right-hand side of (1.1)) is usually neglected. The motivation of this fact is purely physical: the high frequencies are damped by the low-pass filter and, therefore, have inessential influence on the behavior of the system.

There is also a formal mathematical reasoning (called the averaging method) for such simplification of equations. Below we present the averaging method and its range of applications.

Let us consider the differential equation

$$\frac{dz}{dt} = F(z, \omega t), \quad \text{where } z = \{z_1, z_2, \dots, z_n\}, \tag{1.7}$$

with positive parameter ω which takes a large value. Let

$$F_a(c) = \lim_{T\to\infty} \frac{1}{T} \int_0^T F(c,\tau) d\tau \tag{1.8}$$

denote the mean value of the function $F(c,\tau)$ with respect to τ.

The autonomous differential equation

$$\frac{dz_a}{dt} = F_a(z_a) \tag{1.9}$$

is called the *averaged equation* with respect to the original (1.7).

The solutions of the averaged equation (1.9) approximate solutions of the original equation in the following sense.

Theorem 1.1. *Let the following assumptions be satisfied:*

1) *$F(z,\tau)$ is uniformly continuous and bounded function for all z and $\tau > 0$,*

2) *the limit* (1.8) *is uniform with respect to c,*

3) *the equation* (1.9) *with an initial value $z_a(0) = z^*$ has exactly one solution $z_a(t)$ for $t \in [0,\infty)$.*

Under these assumptions, for arbitrary numbers δ (small) and t^* (large), there exists ω^* such that for all $\omega \in (\omega^*, \infty)$ the solution $z(t;\omega)$ of equation (1.7) *with the initial value z^* satisfies the inequality*

$$\max_{0<t<t^*} ||z(t;\omega) - z_a(t)|| < \delta, \tag{1.10}$$

where $||z||$ denotes the norm of vector z.

The proof is omitted. Theorem 1.1 is one of many specialized theorems about solutions of a differential equation depending on a parameter. One can find a rich collection of similar theorems in Ref. [42]. A substantial part of the proof (see also Ref. [31]) uses Ascoli's theorem on compactness of any set of uniformly bounded functions equicontinuous on a bounded closed domain.

From the physical point of view the parameter ω is a frequency of periodic signal perturbing the physical system. According to the above theorem, if the frequency is sufficiently large then in any finite time interval the system behaves almost the same as the autonomous system in which the time-varying perturbation is replaced by its mean value. However, replacement of the finite time interval $[0, t^*]$ by infinite interval $[0, \infty)$ is generally not possible (if δ decreases or if t^* increases then ω^* increases). A small distance between the solutions of both equations, i.e. (1.7) and (1.9) in an infinite time interval (in a steady-state) occurs rather exceptionally and under much stronger assumptions.

One of the theorems on the averaging method related to steady-state signals is formulated below.

Theorem 1.2. *Let the assumptions* 1) *and* 2) *of the theorem 1.1 be satisfied, and moreover:*

3) $F(z, \tau)$ *is a periodic function of* τ,

4) *there exists* z_0 *such that* $F_a(z_0) = 0$,

5) *the Jacobian matrix* F'_a *is continuous in a neighborhood of the point* z_0 *and it is non-singular at the point* z_0 (*i.e.* $\det F'_a(z_0) \neq 0$).

Under these assumptions there exists ω^* *such that for all* $\omega \in (\omega^*, \infty)$ *the equation* (1.7) *has in a neighborhood of* z_0 *exactly one periodic solution* $z_{\text{per}}(\omega t)$ (*of the same period as* $F(z, \omega t)$) *and*

$$\sup_{-\infty < t < +\infty} ||z_{\text{per}}(\omega t) - z_0|| \to 0 \quad \text{as} \quad \omega \to \infty. \tag{1.11}$$

Moreover, if all eigenvalues of the Jacobian matrix F'_a *have negative real parts at the point* z_0 *then the periodic solution* $z_{\text{per}}(\omega t)$ *attracts neighboring solutions* (*it is asymptotically stable*).

The proof is omitted (the implicit function theorem is used in the proof).

It is worth a mention that in mathematical literature the theorems on averaging are usually formulated for equation of the form:

$$\frac{dz}{d\tau} = \varepsilon F(z, \tau), \quad \text{where } \varepsilon \text{ is a small parameter.} \tag{1.12}$$

This equation becomes identical with equation (1.7) after time rescaling $\tau = \omega t$ and changing the parameter $\varepsilon = \frac{1}{\omega}$.

Averaged equations of phase-locked loops are solved in Secs. 2.2, 3.2. However, conclusions which follow from these solutions have limited applications. The behavior of solutions of averaged equations in a long time interval (of the order of several hundreds of input signal periods) may be qualitatively different from the behavior of solutions of original PLL equations, where the frequency ω is a fixed parameter (and is not a value which one can freely increase). An extensive part of this book deals with comparing solutions of original equations with solutions of averaged equations. In particular we investigate how the high-frequency component at the input of VCO changes such solutions of averaged equations as, for example, equilibrium points, stable periodic trajectories and separatrices of saddle points (the borders of attractive domains).

1.4 Organization of the book

The purpose of this book is an investigation of nonlinear deterministic models of PLL systems when the high-frequency term of the signal at the input of VCO is not neglected, but to the contrary, its essential influence on the system's behavior is emphasized. Subsequent chapters concern continuous-time systems of the first and second orders as well as discrete-time systems also of the first and second orders. In this way we pass from easier problems to more difficult ones accepting a few repetitions of some small parts of the material. Each chapter can be read almost independently. The sections which present concrete conclusions concerning the behavior of PLL systems are preceded by sections which include the necessary mathematical notions and theorems with the proofs. Some proofs are omitted (when they can be found in most textbooks, or when they concern less important theorems).

Chapter 2 presents a nonlinear model of phase-locked loop without filter, described by differential equation on a torus. The solutions of averaged Adler's equation are investigated. The notions of rotation number, devil's staircase and Arnold's tongues are introduced as basic characteristics of synchronized oscillators which depend on parameters. This material shows the mechanizm of multiplication and dividing the frequency (fractional synchronization) in the most simple phase-locked loop systems. The cases of sinusoidal and rectangular waveform signals are discussed in the first place.

In Chapter 3 we are dealing with the second order phase-locked loop system with a low-pass filter. The phase-plane portrait of an averaged system on a cylinder is examined, and conclusions concerning hold-in range and pull-in range are presented. The considered system is periodically perturbed by high-frequency component of input signal. We investigate what happens with some selected trajectories (fixed points, stable periodic orbits and separatrices which connect saddle points) after the perturbation. The conditions for the existence of a stable one-dimensional integral manifold are given. Under these conditions the dynamics of the second order system is reduced to dynamics of the first order system. The Melnikov theorem is applied to examine different types of homoclinic trajectories, and their influence on the transient chaos and on the form of attractive domain borders of stable fixed points. The last Sec. 3.9 is devoted to systems with higher order filters. A theorem on the existence of a globally stable two-dimensional integral manifold for the equations of a phase-locked loop of higher order is formulated and proved. The conditions are given for which a higher order system is reducible to the second order.

In Chapter 4 a discrete-time phase-locked loop is investigated. A continuous two-modal mapping of a circle is accepted as the mathematical model of this system. The properties of periodic points, rotation intervals, frequency locking regions and attractive domains of stable periodic orbits of different types are discussed. The emphasis is on bifurcations of periodic orbits (saddle-node and period doubling bifurcations), Feigenbaum's cascade, the skeleton of superstable orbits and bifurcation of rotation interval on the border of a frequency locking region. Some characteristics of chaotic dynamics, especially the invariant measures and Liapunov's exponent are also discussed.

Chapter 5 deals with a more realistic model of discrete-time phase-locked loop, described by a two-dimensional continuous map of the cylinder. Stable periodic points and hold-in regions in the plane of parameters are examined. The conditions for the existence and decay of a one-dimensional stable invariant manifold are investigated. Different types of attractors are discussed, e.g.: stable periodic orbits, invariant curves and strange attractors. In particular, the relations between homoclinic trajectories and bifurcations of strange attractors (crisis bifurcation) are given. Transitions from stable chaotic oscillations to a transient chaos are illustrated by numerical experiments.