Chapter 1

Introduction

In order to understand complex phenomena in nature and social sciences, people desire to achieve the most accurate description and modeling of such complex phenomena and to understand the corresponding physics. Nonlinear dynamics, as an engine of modern science, drives investigations on nonlinear science in interdisciplinary fields, and dynamics is the most fascinating field in the modern science. The modern science began in the 17^{th} century, and since then, one has paid a great deal of attentions on dynamics. The work of Galileo, Newton, Laplace, Clausius, Raylaigh lay on developing the fundamental physical principles and applications of dynamics. The work of Lagrange, Hamilton and Jacobi provided the further and formal developments of dynamics theory. In the 19^{th} century, Hill and Poincare provided the qualitative analysis of dynamical systems to realize the remarkable significance of dynamics for scientific thought. The three-body problem in celestial mechanics is one of the origins for one to keep interests and to develop the fundamental theories and methodologies in dynamics. The ordinary differential equation theory in the real domain is a core to play an important role in dynamics. In the 20^{th} century, one followed the Poincaré's idea to develop and apply the qualitative theory to understand the complexity in dynamical systems. Birkhoff was a key person to push the further development and applications of the Poincare's qualitative theory. The Taylor series expansion and perturbation analyses play a central role in qualitative and quantitative analyses. However, the Taylor series expansion analysis is valid in the finite domain under certain convergent conditions, and the perturbation analysis based on the small parameters, as an approximate estimate, is only acceptable for a very small domain during a short time period. Therefore, the qualitative theory based on the Taylor series and perturbation analysis cannot provide enough useful tools to understand the global complexity in nonlinear systems in the infinite time interval. The author would like to present a different view to look into the fundamental theory in dynamics. The ideas presented in this book are less formal and rigorous in an informal and lively manner. The author hopes the new ideas can give some inspirations in the field of nonlinear dynamics. In the first chapter of this book, the brief development history of the modern theory of dynamics will be presented. Further, chaotic dynamics in nonlinear Hamiltonian systems will be discussed and the corresponding existing methodologies for approximate predictions of the onset, growth and destruction of chaotic layers will be reviewed. The current status of researches on dissipative, nonlinear dynamical systems will be addressed and the crucial, unsolved problems in nonlinear dynamical systems will be briefly discussed. Finally, the summary and layout of contents in this book will be given.

1.1. A brief history of dynamics

The modern theory of dynamics originates from the Poincare's qualitative analysis. Poincare (1892) discovered that the motion of nonlinear coupled oscillator is sensitive to the initial condition, and qualitatively presented that the inherent characteristics of the motion in the vicinity of unstable fixed points of nonlinear oscillation systems may be *stochastic* under regular applied forces. In addition, Poincare developed the perturbation theory for periodic motions in planar dynamical systems. Birkhoff (1913) continued the Poincaré's work, and provided a proof of Poincare's geometric theorem. Birkhooff (1927) showed that both stable and unstable fixed points of nonlinear oscillation systems with two degrees of freedom must exist whenever their frequency ratio (or called *resonance*) is rational. The sub-resonances in periodic motions of such systems change the topological structures of phase trajectories, and the island chains are obtained when the dynamical systems renormalized with fine scales are used. The work of Poincaré and Birkhoff implies that the complexity of topological structures in phase space exists for nonlinear dynamic systems. The question is whether the complicated trajectory can fill the entire phase space or not. The formal and normal forms in the vicinity of equilibrium are developed through the Taylor series to investigate the complexity of trajectory in the neighborhood of the equilibrium. Since the trajectory complexity exists in the vicinity of hyperbolic points, one focused on investigating the dynamics in such vicinity of hyperbolic points.

From a topological point of view, the Smale's horseshoe was presented in Smale (1967). Further, a differentiable dynamical system theory was developed. Such a theory has been extensively used to interpret the homoclinic tangle phenomenon in nonlinear dynamics. Smale found the infinite, many periodic motions, and a perfect minimal Cantor set near a homoclinic motion can be formed. However, Smale's results cannot apply to Hamiltonian systems with more than two-degrees of freedom. Because the differentiable dynamical system theory is based on the linearization of dynamical systems at hyperbolic points, it may not be adequate to explain the complexity of chaotic motions in nonlinear dynamical systems. To continue the Birkhoff' formal stability, Glimm (1963) investigated the formal stability of an equilibrium (or a periodic solution) of Hamiltonian systems through the rational functions instead of the power series expansion. Such an investigation just gave another kind of approximation. Though those theories are extensively applied in nonlinear dynamical systems, such analyses based on the formal and normal forms are still the local analyses in the vicinity of equilibrium. Those theories cannot be applied for the global behaviors of nonlinear dynamical systems.

To understand the complexity of motion in nonlinear Hamiltonian systems, based on the non-rigorous theory of perturbation, Kolmogorov (1954) postulated the KAM theorem. In the KAM theorem, Kolmogorov suggested a procedure which ultimately led to the stability proof of the periodic solutions of the Hamiltonian systems with two-degrees of freedom. This problem is intimately connected with the difficulty of small divisors. The aforementioned theorem was proved under different restrictions in Arnold (1963) and Moser (1962). Further, Arnold (1964) investigated the instability of dynamical systems with several degrees of freedom, and the diffusion of motion along the generic separatrix was discussed. The results of Arnold (1964) extended the Kolmogorov's results to the Hamiltonian system with several degrees of freedom system. The stability in the sense of Lyapunov cannot be inferred. The KAM theory is based on the separable oscillators with weak interactions. In fact, once the perturbation exists, the dynamics of the perturbed Hamiltonian systems may not be well-behaved to the separable dynamical systems. In physical systems, the interaction between two oscillators in a nonlinear dynamical system cannot be very small. The KAM theorem may provide an acceptable prediction only when the interaction perturbation is very weak. The KAM theory is based on separable, integrable Hamiltonian systems. In fact, the complexity of motions in non-integrable, nonlinear Hamiltonian systems is much beyond what the KAM theory stated.

The instability zone (or *stochastic layer*) of Hamiltonian systems, as investigated in Arnold (1964), is a domain of chaotic motion in the vicinity of the generic separatrix. Even if the width of the separatrix splitting was estimated, the dynamics of the separatrix splitting was not developed. Henon and Heiles (1964) gave a numerical investigation on the nonlinear Hamiltonian system with two-degrees of freedom in order to determine whether or not a well-behaved constant of the motion exists for such a Hamiltonian. Izrailev and Chirikov (1964) first pointed out that the periodically forced, nonlinear Hamiltonian system with one degree of freedom exhibits a KAM instability leading to the stochastic behavior (or stochastic and resonance layers). Walker and Ford (1969) investigated the amplitude instability and ergodic behavior for nonlinear Hamiltonian systems with two degrees of freedom to develop the verifiable scheme for prediction of the onset of the amplitude instability. Isolated resonance and double resonance were investigated and the resonance was determined through the transformed coordinates. Such ergodic behavior in nonlinear Hamiltonian system originates from Birkhoff (1927). In other words, to investigate the enormous complexity of non-special motions in dynamical systems from geodesic flows, Birkhoff (1927) presented that the set of non-special motions (or chaotic motions) is measurable in the sense of Lebesgue, and the set of the special motions (or regular motion) is of zero measure. Furthermore, the ergodic theory had been developed in the 20$^{\text{th}}$ century and it is as a fundamental base for fractal theory. The thorough study of the geodesic flows in the ergodic theory can be found in Hopf's book (1937). Those ideas were generalized by Anosov (1962) to study a class of differential equations, which can be also referenced to Sinai (1976). Even though the ergodic theory is a foundation for fractality of chaotic motions in nonlinear dynamical systems, such a theory still cannot provide enough hopes to understand the complexity of chaotic motions in nonlinear dynamics.

For a nonlinear Hamiltonian system with n-degrees of freedom, it is very difficult to understand the mechanism of chaotic motions. To date, such a problem is unsolved. Around 1960, one considered extremely simple, nonlinear Hamiltonian systems to investigate such a mechanism. Melnikov (1962) used the concept of Poincaré (1892) to investigate the behavior of trajectories of perturbed systems near autonomous Hamiltonian systems. Melnikov (1963) further investigated the behavior of trajectories of perturbed Hamiltonian systems and the width of the separatrix splitting were approximately estimated. The width gives the domain of the chaotic motion in the vicinity of the generic separatrix. Even if the width of the separatix splitting was approximately estimated, the dynamics of the separatrix splitting was not developed. From a physical point of view, Chirikov (1960) investigated the resonance processes in magnetic traps, and the resonance overlap was presented initially. Zaslavsky and Chirikov (1964) discussed the mechanism of one-dimensional Fermi acceleration and determined the stochastic property of such a system. Rosenblut et al (1966) investigated the appearance of a stochastic instability (or chaotic motion) of trapped particles in the magnetic field of a traveling wave under a perturbation. Filonenko et al (1967) further discussed the destruction of magnetic surface generated by the resonance harmonics of perturbation. The destruction of such a magnetic surface demonstrates the formation and destruction of the resonant surface. Zaslavsky and Filonenko (1968) gave a systematic investigation of the stochastic instability of trapped particles through the separatrix map (or whisker motion in Arnold (1964)), and the fractional equation for diffusion was developed. Zaslavsky and Chirikov (1972) further presented the stochastic instability of nonlinear oscillations. Chirikov (1979) refined the resonance overlap criterion to predict the onset of chaos in stochastic layers. In addition, the most important achievements for prediction of the appearance of chaotic motions were summarized. Escande and Doveil (1981) used the resonance overlap concept and gave a criterion through a renormalization group method (also see, Escande, 1985). The details for the resonance overlap theory and renormalization group scheme can be referred to references (e.g., Lichtenberg and Lieberman, 1992; Reichl, 1992). Though the resonant overlap criterion can provide a rough prediction of the onset of chaotic motion in the stochastic layers, the mechanism of the chaotic motion in the stochastic layers still cannot be fully understood until now.

Luo (1995) proposed the resonance theory for chaotic motions in the vicinity of generic separatrix in nonlinear Hamiltonian system (also see, Luo and Han, 2001), and it was asserted that chaotic motions in nonlinear Hamiltonian systems are caused by the resonant interaction. Furthermore, the mechanism for the formation, growth and destruction of stochastic layers in nonlinear Hamiltonian systems was discussed in Luo and Han (2001). In Luo et al (1999), the resonant webs formed in the stochastic layer were presented, and it was observed that the webs are similar to the stochastic layer of the parametrically forced pendulum system. The recent investigations (e.g., Han and Luo, 1998; Luo, 2001b, 2001c, 2002) discovered that the resonance interaction generates the resonant separatrix, and the chaotic motion forms in vicinity of such a resonant separatrix. The corresponding criteria were presented for analytical predictions of chaotic motions in 1-DOF nonlinear Hamiltonian systems with periodic perturbations. The maximum and minimum energy spectrum methods were developed for numerical predictions of chaotic motions in nonlinear Hamiltonian systems (Luo et al, 1999; and Luo, 2002). The energy spectrum approach is applicable not only for small perturbations but for the large perturbation. The recent achievements for stochastic layers in periodically forced Hamiltonians with one-degree of freedom were summarized in Luo (2004a). Luo (2006a) investigated quasi-periodic and chaotic motions in n-dimensional nonlinear Hamiltonian systems. The energy spectrum method was systematically presented for arbitrary interactions of the integrable nonlinear Hamiltonian systems. The internal resonance was discussed analytically for weak interactions, and the chaotic and quasi-periodic motions can be predicted. From a theory for discontinuous dynamical system in Luo(2006b), Luo (2007a) presented a general theory for n-dimensional nonlinear dynamical systems. The global tangency and trans-versality to the separatrix were discussed from the first integral quantity. The first integral quantity increment was introduced to investigate the periodic and chaotic flows. In the following section, the basic theory and methodology for nonlinear Hamiltonian system will be addressed.

1.2. Nonlinear Hamiltonian systems

The *stochastic layer* in nonlinear Hamiltonian systems is a domain of chaotic motion in the vicinity of separatrices (e.g., Chirikov, 1979; Lichtenberg and Lieberman, 1992). The chaotic motion is generated through resonance interaction in nonlinear Hamiltonian systems. The separatrices of a Hamiltonian system divide the system phase space into different domains, and from numerical simulations, resonance interactions in all the domains are distinguishing themselves (e.g., Han and Luo, 1998; Luo and Han, 1999). Therefore, the resonance interaction in stochastic layers is a key for a better understanding of the mechanism of such a chaotic motion in the vicinity of separatrix. Poincare (1890) described qualitatively such a chaotic motion formed through the separatrix splitting of Hamiltonian systems for the first time (also see, Poincare, 1892), and Melnikov (1963) continued the Poincaré's investigation and computed the width of the separatrix splitting. Since then, the separatrix splitting has been further investigated quantitatively (e.g., Melnikov, 1963; Holmes et al, 1988; Lazutkin et al, 1989; Gelfreich et al, 1991; Gelfreich et al, 1994; Treschev, 1995, 1998), and the separatrix and standard map approaches have been developed for chaotic motions in the stochastic layer (e.g., Filonenko et al, 1967; Zaslavsky and Filonenko, 1968; Luo and Han, 1999; Luo, 1995). After the KAM torus in vicinity of separatrix is destroyed, the resonance overlap, generated by the interaction of resonance between the unperturbed and perturbed orbits, occurs in the stochastic layer in Chirikov (1979). However, such a resonant overlap mechanism of the stochastic layer needs to be further investigated. Therefore, the resonant characterization of stochastic layers in nonlinear systems was extensively investigated in recent years.

Consider a nonlinear Hamiltonian system in Chirikov (1979) as

$$H(I,\varphi)=H_0(I)+\mu H_1(I,\varphi,\Omega t,) \tag{1.1}$$

where $(I,\varphi)=(I_1,\cdots,I_n,\varphi_1,\cdots,\varphi_n)$ ($n>1$) are canonical variables, and t is time. The unperturbed integrable Hamiltonian $H_0(I)$ and the $2\pi/\Omega$-periodic perturbation $H_1(I,\varphi,\Omega t)$ are real analytic function (i.e., C^k ($k\geq 2$) continuous function) and the small parameter $\mu>0$ is the magnitude of perturbation, and the parameter Ω is perturbation frequency. As in Chirikov (1979), the Fourier series expansion of Eq.(1.1) for given I is

$$H(I,\varphi)=H_0(I)+\mu\sum_{m,n}^{\infty}H_{mn}(I)\exp(\mathrm{i}[n\varphi+m\Omega(t+t_0)]) \tag{1.2}$$

where m,n are integer and $\mathrm{i}=\sqrt{-1}$. $H_{mn}(I)$ is the Fourier series coefficient of $H_1(I,\varphi,\Omega t)$. From Eq.(1.2) the resonant condition is

$$n\omega(I)+m\Omega=0. \tag{1.3}$$

Due to different Ω, the following forcing will be discussed: (a) fast forcing ($\Omega=1/\varepsilon$ and $0<\varepsilon<<1$), (b) slow forcing ($\Omega=\varepsilon$ and $0<\varepsilon<<1$), (c) regular forcing (μ and Ω not small). Before discussing the nonlinear dynamics of stochastic layers in nonlinear Hamiltonian systems with at least a separatrix, the separatrix splitting should be discussed first.

1.2.1. Separatrix splitting

Poincare (1892) gave the measure of the splitting being exponentially small in μ (i.e., $\exp(-\frac{\pi}{\sqrt{2\mu}})$). The series solutions for limiting trajectories of Eq.(1.1) include the terms of $\exp(-\frac{\alpha}{\sqrt{\mu}})$ and $\alpha>0$ (e.g.,Poincare,1892; Melnikov,1962,1963). Arnold (1964) investigated the diffusion in the stochastic layer (or instability zone) through the separatrix splitting, and the average velocity for the shift of action variables is of the order of $\exp(-\frac{\alpha}{\sqrt{\mu}})$. Some explicit upper bounds for the speed of the Arnold diffusion in steep unperturbed Hamiltonian were presented (see also, Nekhoroshev, 1977; Neishtadt, 1984). For small μ, the stochastic layer is a very narrow domain in vicinity of the separatrix, and both the width and angle are of the order $\exp(-\frac{\alpha}{\sqrt{\mu}})$ in Neishtadt (1984). Holmes et al (1988) gave both upper and lower estimates of the exponentially small splitting of separatrices for rapidly forced systems (e.g., rapid forcing $\mu\sin(t/\varepsilon)$ and $0<\varepsilon<<1$) through the splitting distance computed by the Melnikov function. As in Treschev (1995), the Melnikov method gives the correct asymptotics of the splitting only when the parameter μ is sufficiently small. The other exponentially small effects were investigated in Slutskin (1964). Such effects are generated by a very slow perturbation (or slow forcing $\mu\sin(\varepsilon t)$ and $0<\varepsilon<<1$) rather than a very rapid perturbation of an integrable system.

Consider a 2-dimensional dynamic system possessing at least a homoclinic or heteroclinic orbit and such a system is perturbed by a periodic excitation, which is given by

$$\left.\begin{aligned}\dot{x}&=f_1(x,y)+\mu g_1(x,y,t),\\ \dot{y}&=f_2(x,y)+\mu g_2(x,y,t);\end{aligned}\right\}\tag{1.4}$$

where $f_1(x,y)=\partial H_0/\partial y$, $f_2(x,y)=-\partial H_0/\partial x$. The unperturbed Hamiltonian $H_0=H_0(x,y)$, $g_1(x,y,t)$ and $g_2(x,y,t)$ are analytic. In Melnikov (1962), the limiting trajectories of Eq.(1.4) were expressed by a series in the power of $\sqrt{\mu}$ and the trajectories converge for all the $\sqrt{\mu}$ -terms. Consider an oscillatory frequency $\omega=\frac{1}{\sqrt{\mu}}$, and then the terms of $\exp(-\frac{\alpha}{\sqrt{\mu}})$ with ($\alpha>0$) should exist in the series solution of the foregoing equation. To investigate the separatrix splitting of the Hamiltonian system in Eq.(1.4), from Melnikov (1963), the asymptotic solution to Eq.(1.4) is determined by

$$\left.\begin{aligned}x_\varepsilon(t,\omega,t_0)&=x_0(t-t_0)+\sum_{n=1}^{\infty}x_n(t,\omega,t_0)\mu^n,\\ y_\varepsilon(t,\omega,t_0)&=y_0(t-t_0)+\sum_{n=1}^{\infty}y_n(t,\omega,t_0)\mu^n;\end{aligned}\right\}\tag{1.5}$$

where $\dot{x}_0=f_1(x_0,y_0)$ and $\dot{y}_0=f_2(x_0,y_0)$ gives a solution of the unperturbed system (i.e., $x_0(t-t_0)$ and $y_0(t-t_0)$). Through the perturbation analysis, equations for variables (x_n,y_n) can be obtained from the order of ε^n, and the corresponding perturbed solution (i.e., $x_n(t,\omega,t_0)$ and $y_n(t,\omega,t_0)$) is determined by

$$\begin{Bmatrix} \dot{x}_n \\ \dot{y}_n \end{Bmatrix} = \begin{bmatrix} \frac{\partial f_1(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_2(x,y)}{\partial y} \end{bmatrix}_{(x_0,y_0)} \begin{Bmatrix} x_n \\ y_n \end{Bmatrix} + \begin{Bmatrix} P_n\left(x_0,\cdots,x_{n-1},y_0,\cdots,y_{n-1}\right) \\ Q_n\left(x_0,\cdots,x_{n-1},y_0,\cdots,y_{n-1}\right) \end{Bmatrix}. \quad (1.6)$$

From Eq.(1.6), Melnikov (1963) developed an analytical method to predict the gap (or width, splitting distance) between two branches of separatrix splitting, and the gap was computed by

$$\Delta_\mu\left(t_0,\omega\right) = \sum_{n=1}^{\infty} \Delta_n\left(t_0,\omega\right)\mu^n, \quad (1.7)$$

where

$$\begin{aligned} \Delta_n\left(t_0,\omega\right) = & \int_{-\infty}^{\infty}\left[Q_n\left(x_0,\cdots,y_{n-1}\right)f_1\left(x_0,y_0\right) - P_n\left(x_0,\cdots,y_{n-1}\right)f_2\left(x_0,y_0\right)\right] \\ & \times\exp\left(-\int_{t_0}^{t-t_0}\left[\frac{\partial f_1(x,y)}{\partial x} + \frac{\partial g_1(x,y)}{\partial y}\right]_{(x_0(\xi),y_0(\xi))} d\xi\right)dt. \end{aligned} \quad (1.8)$$

For $P_1(x_0,y_0) = f_2(x_0,y_0)$ and $Q_1(x_0,y_0) = g_2(x_0,y_0)$ in Eq.(1.8), the quantity $\Delta_n(t_0,\omega)$ becomes $\Delta_1(t_0,\omega)$ which is called the Melnikov function. Equation (1.7) gives the asymptotic behavior of the function $\Delta_\mu(t_0,\omega)$ as $\omega \to \pm\infty$.

In recent years, analytic estimates of the exponentially small effect of separatrix splitting were very active through the standard map (e.g., Lazutkin et al, 1989; Gelfreich et al, 1994; Treschev,1998) and for continuous Hamiltonian systems (e.g., Gelfreich et al, 1991; Treschev, 1995). When the homoclinic orbit is splitting in perturbed Hamiltonian systems with a small parameter μ, its stable and unstable manifolds intersect each other, and the tangent vectors of the two manifolds are nonzero intersection angles. The splitting of homoclinic orbit is characterized by the maximum angle of intersection and the width of the stochastic layer. The angle and width are of the order $\exp(-\frac{\alpha}{\sqrt{\mu}})$ $(\alpha > 0)$ in Poincaré (1890) (also see, Melnikov, 1962). For the exponentially small splitting of a homoclinic orbit γ, Gelfreich et al (1991) introduced the homoclinic invariant instead of the maximum angle of intersection since this angle is not symplectically invariant. When the stable and unstable manifolds $\varphi^{s,u}(t)$ related to the hyperbolic fixed point z_0 are used, the homoclinic invariant is

$$\omega\left(\gamma\right) = \| \tfrac{d\varphi^u(t)}{dt}|_{z_h} \| \times \| \tfrac{d\varphi^s(t)}{dt}|_{z_h} \| \sin\phi, \quad (1.9)$$

where ϕ is the intersection angle of two manifolds $\varphi^{s,u}(t)$ at z_h and $\|\cdot\|$ is a Riemanian metric. A generalized standard map in Gelfreich et al (1991) is

$$x_{n+1} = x_n + y_{n+1}, \text{ and } y_{n+1} = y_n + \mu\sum_{k=1}^{\infty} a_k \sin\left(kx_n + b_k\right). \quad (1.10)$$

The corresponding homoclinic invariant is

$$\omega(\gamma)=\tfrac{4\pi|\Theta|}{n^2\mu}\exp(-\tfrac{2\pi\rho}{\sqrt{\mu}})\sum_{k=1}^{m}c_k\cos\tfrac{2\pi\beta_k}{\sqrt{\mu}}+o\left(\mu^{-1+\delta}\exp(\tfrac{2\pi\rho}{\sqrt{\mu}})\right). \tag{1.11}$$

The constant in the o-estimate depends on the choice of parameters. $\delta\in(0,\frac{1}{2(n+1)})$ and $z_k=\beta_k$ ($k=0,1,\cdots,m$). $c_k=1$ if $\beta_k=0$ and $c_k=2$ if $\beta_k\neq 0$. $|\Theta|=$ 1118.827 705 95$\cdots$. If $k=1$, we have $\omega(\gamma)=4\phi$ and Eq.(1.11) reduces to the maximum intersection angle in Lazutkin et al (1989)

$$\phi=\tfrac{\pi|\Theta|}{\mu}\exp(-\tfrac{\pi^2}{\sqrt{\mu}})\left[1+o\left(\mu^{1/8-\delta}\right)\right]. \tag{1.12}$$

The other exponentially small effects, generated by a very slow perturbation (or slow forcing $\mu\sin(\varepsilon t)$ and $0<\varepsilon<<1$) rather than the very rapid forcing of an integrable system, were investigated in Slutskin (1964). Analytical estimates for the exponentially small splitting of separatrices in the standard Chirikov map were presented (e.g., Lazutkin et al, 1989; Gelfreich et al, 1991, 1994; Treschev, 1995). Treschev(1995) investigated the separatrix splitting in nonlinear Hamiltonian systems, which is exponentially close to integrable Hamiltonian systems through the averaging method in Slutskin (1964), and such a separatrix splitting includes all the aforementioned separatrix splitting. Treschev (1998) estimated the width of stochastic layers in near-integrable two-dimensional symplectic maps, and the upper and lower bounds for both the width and area of the stochastic layers were also estimated through the separatrix splitting.

The separatrix splitting for slow forcing ($\mu>1$) and usual forcing is still not estimated because the aforementioned method is based on the perturbation analysis. In addition, only the separatrix is used for estimates of the separatrix splitting without the periodic solutions of an unperturbed Hamiltonian system in the vicinity of the separatrix. The exponentially small separatrix splitting is crucial to understand the behavior and characteristics of trajectories in the stochastic layer, and the widths and/or intersection angle of the separatrix splitting in the stochastic layer have been discussed. However, no any criteria associated with the widths for transition to global stochasticity is provided as in the standard map (e.g., Chirikov, 1979). Therefore, the amplitudes (or strengths) of external excitation cannot be determined when a primary resonance is absorbed in the stochastic layer.

1.2.2. Standard and whisker maps

Zaslavsky and Filonenko (1968) first introduced the separatrix map to investigate the one-dimensional motion of a charged particle in the field of a traveling wave nder a perturbation. Such a whisker map is based on the following three assumptions:

(i) The approximation of the first complete elliptic integral gives an expression $K(k) \approx \log(4/\sqrt{1-k^2})$ when the periodic orbits of an unperturbed system are close to the separatrix (e.g., Chirikov, 1979).
(ii) Only the period of libration is used for the estimate of phase change without the rotation.
(iii) The energy change (ΔH_0) for one period of a periodic orbit of an unperturbed Hamiltonian system in vicinity of separatrix in Eq.(1.1) is approximated by the one (ΔH_0^{sx}) along the separatrix (e.g., Chirikov, 1979; Gelfreich et al, 1994; Arnold, 1964; Reichl, 1992), i.e.,

$$\begin{aligned}\Delta H_0 \approx \Delta H_0^{sx} &= \mu \int_{-\infty}^{\infty} \frac{dH_0}{dt}\bigg|_{(I^{sx},\varphi^{sx})} dt \\ &= \mu \int_{-\infty}^{\infty} \left[H_0\left(I^{sx}\right), H_1\left(I^{sx},\varphi^{sx},\Omega t\right)\right] dt\end{aligned} \tag{1.13}$$

where I^{sx} and φ^{sx} are the separatrix solution, and $[\cdot,\cdot]$ is the Poisson bracket. The whisker map for the forced pendulum (e.g., Zaslavsky and Filonenko, 1968; Chirikov, 1979; Lichtenberg and Lieberman, 1992) is

$$w_{i+1} = w_i + W \sin\varphi_i, \text{ and } \varphi_{i+1} = \varphi_i + \tfrac{\Omega}{\omega}\log(\tfrac{32}{|w_{i+1}|}) \bmod 2\pi; \tag{1.14}$$

where $w = E/\omega^2 - 1$. W is related to the magnitude of energy change ΔH_0. Both E and ω are the Hamiltonian energy and natural frequency of the unperturbed system, respectively. Ω is excitation frequency.

For the stochastic instability of trapped particles, Zaslavsky and Filonenko (1968) also linearized the whisker map to obtain a standard map in the neighborhood of an assumed resonance (e.g.,Chirikov,1979; Lichtenberg and Lieberman, 1992; Reichl,1992). Vecheslavov (1996) used a similar procedure to investigate chaotic motions in vicinity of the separatrix under high-frequency excitations. In addition to the above-mentioned three assumptions, two more assumptions are used in derivation of the standard map:

(iv) The whisker map is approximated at its period-1 fixed point based on a primary resonance.
(v) The primary resonant condition is assumed but not derived from Eq.(1.1) (i.e., equation (1.3) is not used).

From Assumptions (iv)-(v), the standard map (e.g., Chirikov, 1979; Lichtenberg and Lieberman, 1992) is

$$I_{i+1} = I_i + K\sin\varphi_i, \text{ and } \varphi_{i+1} \approx \varphi_i + I_{i+1}; \tag{1.15}$$

where $I = -(w - w_r)\Omega/\omega w_r$ and $K = -\Omega W/w_r\omega$. w_r is the resonant value based on the arbitrary resonant condition. Greene (1968, 1979) developed a method to numerically determine the strength of the stochasticity (i.e., $K^* \approx 0.9716\cdots$) when the transition to global stochasticity for Eq.(1.15) occurs. Other estimates for the strength of stochasticity parameter can be referred to the references (e.g., Chirikov, 1979 and Lichtenberg and Lieberman, 1992). Treschev (1998) presented a qualitative relationship between the width (w) of the stochastic layer and the width (d) of the lobe domain, such as $w/d \sim 1/\log\mu$ and μ is a multiplier at the corresponding hyperbolic point. From the energy analysis and numerical simulations, the resonance in regions separated by the separatrix in phase space are distinguishing themselves (e.g., Luo, 1995; Han and Luo, 1998; Luo and Han, 1999). Therefore, the Assumption (v) is inadequate for the derivation of the standard map. Further, the established standard map approach cannot provide a satisfactory prediction of the stochastic layer for the original system in Eq.(1.1). Although the approximate separatrix map (i.e., Eq.(1.14)) is not accurate to model Hamiltonian systems, it provides a better prediction of the stochastic layer than the standard map. Therefore, in recent years, the approximate whisker map was investigated extensively (e.g., Rom-Keder, 1990, 1994, 1995; Zaslavsky and Abdullaev, 1995; Abdullaev and Zaslavsky, 1995, 1996; Ahn et al, 1996; Iomin and Fishman, 1996). Rom-Kedar (1990) derived the approximate whisker map through the Melnikov function, and used such an approximate map to predict transport rates statistically and numerically. The escape rates in the vicinity of homoclinic tangles were investigated. However, the approximate whisker map in Rom-Kedar (1994) was further investigated. Rom-Kedar (1995) determined the secondary homoclinic bifurcation in the stochastic layer by setting the Hamiltonian energy at the second iteration of the approximate whisker map to zero. For an improved computation of energy change, Zaslavsky and Abdullaev (1995) introduced a shifted separatrix map. The procedure of the conventional derivation of whisker and standard maps was given in Luo and Han (1999) (also see, Zaslavsky and Filonenko, 1968; Luo, 2001). Even if so, the aforementioned approach cannot provide an appropriate prediction of the onset of resonance in the stochastic layer. Therefore, Luo (2001) developed an improved standard map approach based on the accurate whisker map for such a prediction.

To provide a more adequate prediction of the stochastic and resonance layers, Luo (1995) presented the incremental energy method. This method is based on the accurate whisker mapping rather than the standard mapping technique (also see, Han and Luo, 1998; Luo and Han, 2001). The criterion presented in Luo and Han (2001) can be expressed through the action variables and natural frequency of the unperturbed Hamiltonian systems, as in Arnold (1989). The onset of a specified resonance in the stochastic layer is predicted through the incremental energy approach. This approach is also applicable for strong excitations when the energy increments still maintain in good accuracy. For a better prediction of resonance interactions in the stochastic layer, a new computational method for the incremental energy should be developed. For the numerical prediction of stochastic layer, Luo et al (1999) developed an energy spectrum technique, and the resonant characteristics in stochastic layers were investigated through the energy spectra. This technique computes the maximum and minimum energies of the Poincaré mapping points. A comparison of analytical and numerical predictions was presented in Luo and Han (2001). From numerical results, the standard mapping approach developed in Luo and Han (2001) cannot provide a very good prediction for strong excitations.

1.2.3. Chirikov resonance overlap criterion

Chirikov (1979) presented a resonant overlap criterion, and pointed out that the transverse, homoclinic intersections, caused by the resonant overlap, transfer the energy from one resonant orbit to another one. For the transition of local stochasticity to chaos (or global stochasticity), Chirikov postulated that the last KAM surface between two lowest-order primary resonance is destroyed when the sum of the half-widths of the two island separatrices formed by the resonance is equal to the distance between the two resonance. Using this criterion, Reichl and Zheng (1984a,b) estimated the width of the stochastic layer (or excitation strength) for the undamped Duffing oscillator, and such estimates quantitatively agree with the predictions of excitation strength through the established standard map. The Chirikov overlap criterion usually gives the correct order of magnitude for excitation strength though the self-similarity of resonance is ignored. Consider a Hamiltonian pertaining to the standard map (e.g., Zaslavsky and Filonenko,1968; Zaslvasky, 1998), i.e.,

$$H = \tfrac{1}{2} y^2 + K\cos\theta + 2K\cos\theta\cos(\Omega t), \tag{1.16}$$

where $y = \dot{\theta}$. The third term of Eq.(1.16) is considered as a perturbation. The unperturbed Hamiltonian of Eq.(1.16) is

$$H_0 = \tfrac{1}{2} y^2 + K\cos\theta. \tag{1.17}$$

The half-width of the island separatrix at $\cos\theta = -1$ is

$$\Delta y_{\max} = 2\sqrt{K}. \tag{1.18}$$

Therefore, three overlap criteria are as follows:

(i) For the simple overlap, the critical condition is obtained by

$$2\Delta y_{\max} = 2\pi \;\Rightarrow\; K=\tfrac{\pi^2}{4} \approx 2.46740. \tag{1.19}$$

(ii) For the first and second-order resonance overlap, the critical value is

$$2\sqrt{K} + \tfrac{1}{2}K = \pi \;\Rightarrow\; K \approx 1.46. \tag{1.20}$$

(iii) For the improved overlap, the critical value is improved as

$$2\left[\sqrt{K} + \tfrac{(2\pi)^2}{K^2}\exp(-\tfrac{\pi^2}{\sqrt{K}})\right] + \tfrac{K}{2} = \pi \;\Rightarrow\; K \approx 1.2. \tag{1.21}$$

Consider a perturbed Hamiltonian system, the rescaled energy near the resonant orbit is:

$$H = \tfrac{1}{2}y^2 - M\cos\theta - P\cos\nu(\theta - \Omega t). \tag{1.22}$$

For $P = 0$, the half-width of the resonance at $y = 0$ is $\Delta y_{\max} = 2\sqrt{M}$. Similarly, when $M = 0$, the half-width of the resonance at $y = 1$ is $\Delta y_{\max} = 2\sqrt{P}$. The Chirikov overlap criterion in Reichl (1992) is

$$2\sqrt{M} + 2\sqrt{P} = 1. \tag{1.23}$$

Using the foregoing equation, Reichl and Zheng (1984a,b) estimated the widths of the stochastic layers (actually, excitation strengths) for the undamped Duffing oscillator (also see, Luo and Han, 1999, 2000). Luo and Han (1999) used the Chirikov overlap criterion to predict the onset of the resonant layer. Compared to the numerical prediction, this criterion gives a rough estimate of the resonant layer.

1.2.4. Renormalization group technique

Owing to the self-similar structure of resonance, the renormalization group theory (e.g., Escande and Doveil, 1981; Escande, 1985) was developed for prediction of the excitation strength when the last KAM torus surface between two lowest-order primary resonances is destroyed. Following such a theory in Escande (1985) and assuming $M > P$, the energy H in Eq. (1.22) is separated into two parts as

$$H_0 = \tfrac{1}{2}y^2 - M\cos\theta = E_0, H_1 = -P\cos\nu(\theta - \Omega t); \tag{1.24}$$

where E_0 is constant. The first one of the forgoing equation gives

$$\theta = 2\,\mathrm{am}(\tfrac{K(k)\varphi}{\pi}, k), y = \pm 2k\sqrt{M}\,\mathrm{cn}(\tfrac{2K(k)\varphi}{\pi}, k); \tag{1.25}$$

where am and cn are the elliptic amplitude and the Jacobi elliptic functions, and

$$k^2 = \frac{2M}{E_0 + M}, \; J = \frac{4\sqrt{M}}{\pi k} K(k). \tag{1.26}$$

Substitution of Eq. (1.25) into Eq. (1.22) yields

$$H = H_0(J) - P\sum_{-\infty}^{\infty} V_n(J)\cos[(\nu + n)\varphi - \nu\Omega t], \tag{1.27}$$

where $V_n(J)$ is a coefficient of the n^{th}-order term in the Fourier series. From Eq.(1.27), we have the n^{th}-order resonant condition

$$\omega = \frac{\pi\sqrt{M}}{kK(k)} = \frac{\nu\Omega}{\nu + n + \delta}, \tag{1.28}$$

where $\delta = \{0,1\}$. The energy renormalization in *Eq.*(1.27) for the n^{th}*-order* resonance yields,

$$\bar{H} = \frac{\bar{y}^2}{2} - \bar{M}\cos\bar{\theta} - \bar{P}\cos\bar{\nu}(\bar{\theta} - \bar{\Omega}t), \tag{1.29}$$

where

$$\left.\begin{aligned} \bar{M} &= \frac{PV_{n+\delta}\omega(\nu + n)^2(\nu + n + 1)^2}{\nu^2}, \\ \bar{P} &= \frac{PV_{n+1-\delta}\omega(\nu + n)^2(\nu + n + 1)^2}{\nu^2}, \\ \bar{\nu} &= \frac{(\nu + n + 1 - \delta)}{\nu + n + \delta}, \bar{\Omega} = \frac{(2\delta - 1)\nu\Omega}{\nu + n + 1 - \delta}, \\ \bar{\theta} &= (\nu + n + \delta)\varphi - \nu\Omega t. \end{aligned}\right\} \tag{1.30}$$

Solving Eq.(1.30) as the self-similar structures gives

$$2\sqrt{\bar{M}} + 2\sqrt{\bar{P}} \approx 0.7. \tag{1.31}$$

Note that Eqs.(1.23) and (1.31) have a similar form, but only two of the infinite primary resonances are modeled (e.g., Luo, 1995; Luo and Han, 1999; Luo et al, 1995). Han and Luo(1998) developed an improved standard map approach for predicting the onset and destruction of resonant separatrix layers. The Chirikov overlap criterion in Luo and Han (1999) and the renormalization group technique (e.g.,Reichl and Zheng, 1984; Lin and Reichl,1986; Luo et al, 1995) were modified for the prediction of the appearance of resonant layers. The renormalization group technique can give a good prediction of the resonance interaction when the excitation perturbation is very weak. For strong perturbations, the renormalization technique may not be adequate. Therefore, Luo (1995) presented an incremental energy method to give an approximate prediction (also see, Han and Luo, 1998), and the corresponding energy spectrum technique was developed in Luo (2002). Using the energy spectrum method, the appearance and disappearance of the resonant layer can be obtained.

1.3. Dissipative nonlinear systems

For dissipative nonlinear dynamical systems, the perturbation analysis is an important tool to determine the periodic motion behaviors. The earliest approximation method is the method of averaging, and the idea of averaging originates from Lagrange (1788) to investigate the three-body problem. In the end of the 19^{th} century, Poincaré (1892) used the similar ideas to develop the perturbation theory in 2-dimensional dynamic systems, one has applied the Poincaré perturbation methods for periodic motions in nonlinear dynamical systems. The classic perturbation methods for nonlinear oscillators were presented (e.g. Stoker 1950, Minorsky, 1962; Hayashi, 1964). Because the time for periodic motions in nonlinear oscillation is finite, the perturbation analysis can provide an approximate estimate of periodic motions in nonlinear oscillators. However, the time for chaotic motions in nonlinear dynamical systems is infinite, so the perturbation method may not provide an adequate analysis of chaotic motions. In recent decades, chaos in dissipative nonlinear dynamical system was extensively investigated. For instance, Ueda (1980) gave the numerical simulation of regular and chaotic motions in the damped Duffing oscillator. So far, chaotic motions in dissipated nonlinear systems with one degree of freedom are analytically predicted by the Melnikov method. The detailed discussion of the Melnikov method can be referred to Guckenheimer and Holmes (1983). In fact, the Melnikov function (1963) can only give an acceptable, approximate estimate of the width for the separatrix splitting for a weak perturbation. Such a function may not be adequate for the global transversality of flows to the homoclinic or heteroclinic orbit in nonlinear dynamical systems. In this book, the author would like to discuss this question.

Based on the work of Melnikov (1963), Greenspan (1981) extended the Melnikov's ideas to the dissipative dynamical systems (also see, Greenspan and Holmes, 1982; Guckenheimer and Holmes, 1983). Further, the Melnikov method was developed for the global transversality in dissipative nonlinear systems. Once the global transversality to the separatrix exists, one thought that the Smale horseshoe presented in Smale (1967) may exist, and further chaos in such a nonlinear dynamical system may occur. In recent years, many researches were carried out from such a thought. However, from such a prediction based on the

Melnikov method, one cannot observe the global transversality in nonlinear dynamical systems. The Smale horseshoe theory may not be adequate for nonlinear dynamical systems rather than the topological structure, the author does not proceed to discuss whether the Smale horseshoe theory is right or wrong in nonlinear dynamical systems. From the perturbation analysis, the Melnikov function was obtained for Hamiltonian systems with a small perturbation. One used such a function to analytically predict global behaviors (e.g., chaos) in Hamiltonian systems with a small perturbation. Due to the perturbation analysis, the Melnikov method can give a reasonable analysis of the global behavior only when the perturbation is very small and close to zero. However, the perturbation is very small to zero, chaos in the nonlinear dynamical systems may not occur. So the Melnikov method may not help us understand the global behaviors of nonlinear dynamical systems. Luo (1995) used the Chirikov criterion to determine Hamiltonian chaos and applied the Melnikov function to investigate the global transversality (also see, Luo and Han, 1999). It was found that the Melnikov method cannot provide an adequate prediction of chaotic motions in the dissipative system. For a better understanding of the Melnikov method, the paper of Melnikov (1963) should be revisited. In 1963, Melnikov presented a perturbation analysis to estimate the width of the separatrix splitting. Indeed, the width of the separatrix can be approximately estimated, but it cannot be used for prediction of the existence of chaos. In fact, the Melnikov function is an approximate energy increment during a certain time period, which can be found in references (e.g., Arnold, 1964; Chirikov,1979; Luo and Han, 2001). If the Melnikov function is zero, from physical point of view, it implies that the system energy is conserved during a certain time period. Such a zero value of the Melnikov function does not imply that the flow has any global transversality to the separatrix. To resolve this puzzle, Luo (2007a) used the concepts in discontinuous dynamical systems in Luo (2005) (also see, Luo, 2006a) to investigate the global transversality and tangency to the separatrix in nonlinear dynamical systems with a continuous vector field. The concepts of global tangency and transversality to the separatrix are introduced. It is very easy for us to observe that complex behaviors in nonlinear systems are involved with the separatrix. This is because the singularity originates from separatrix with zero frequency. The phase space of a dynamical system is divided by the separatrix into many sub-domains. In each individual sub-domain, the dynamical system has the similar dynamic behaviors which, however, are different in the other sub-domains. The differences of dynamic behaviors in different sub-domains cause the complexity of motion near the separatrix in nonlinear dynamical systems. Luo (2007a) developed a general theory for the global tangency and transversality of flows in n-dimensional nonlinear dynamical systems. Such a general theory is applied to a dissipative dynamical system (e.g., a periodically forced, damped Duffing oscillator) in Luo (2007b), and the global tangency and transversality of the periodic and chaotic motions to the separatrix were discussed.

The detailed discussion was given in Luo (2007c). It was observed that the transversality and tangency to the separatrix in nonlinear dynamical systems are independent of the Melnikov function (or the energy increments). Furthermore, the conditions for the global transversality and tangency of periodic and chaotic motions were given, which may help us understand the complexity of motions in nonlinear dynamic systems with separatrix.

In dynamic systems, the stability of equilibrium and periodic solutions is a very important issue. This is another key to understand the complexity of flows in nonlinear dynamical systems correctly. The stability by linearization is extensively used, which is completed through the eigenvalue analysis of the linearized system of equilibriums or periodic orbits in the corresponding neighborhoods. Such a stability analysis is local. To determine the non-local stability of nonlinear dynamical systems, the Lyapunov method should be employed. Lyapunov (1907) generalized the work of the Torricelli, Huygens and Lagrange about the ideas of the stability in mechanics, and developed the Lyapunov direct method based on differential equations rather than the potential energy or an energy-alike quantity in general. From the stability definition, the stability conditions in the Lyapunov direct method are too strong. So far, except for numerical simulations of dissipative systems, almost no analytical methods can be used to predict chaotic motions. The most popular method is the method of Lyapunov exponents, which is based on the theory of linear differential equations. To use the linearization of instantaneous point of a given trajectory of the nonlinear dynamical system, the characteristic exponents are computed. One thought that such a trajectory is chaotic if the maximum exponent is greater than zero. In other words, to use the ideas of the local linearization, the Lyapunov exponents of a given trajectory characterize the mean exponential rate of divergence of the trajectories in the vicinity of such a given trajectory. Honen and Heiles (1964) introduced the divergence of near trajectories to investigate the stochasticity of a trajectory in phase space. Zaslvasky and Chirikov (1972) gave the further studies of such stochasticity of a given trajectory in nonlinear oscillation systems. The connection between Lyapunov exponents and exponential divergence was presented by Benettin et al (1976), and the comprehensive presentation about the Lyaponuv characteristic exponents can be found in Benettin et al (1980a,b). Wolf et al (1985) presented a simplest computational scheme to determine the Lyapunov exponents from a time series. The continuous orthogalization methods were properly implemented for computing the Lyapunov exponents of continuous dynamical systems (e.g., Goldhirsch et al, 1987; Dieci et al, 1997), which is much better than the method of Wolf et al (1985). Based on the QR method, the further improvement of methods for computing Lyapunov exponents can be found in references (e.g., Dieci et al, 1994, 1995; Udwadia and von Bremen, 2001). Bartler (1999) used the idea of tracking Lyapunov vectors to determine Lyapunov exponents. From Bartler's inspiration, Yang et al (2005) modified the Lyapunov vector method. Lu et al (2005) gave the further mathematic development of the Lyapunov vector method, and a numerical integration scheme was derived that can automatically preserve the orthogonality between any two consecutive vectors. In such a method, the vectors are orthogonal but not necessary orthonormal. Except for the numerical method based on the linearized equation of the dynamical system, Luo (2007a) pointed out the G-function to the first integral manifold can be used to determine stability of the system. This stability conditions similar to the Lyapunov stability is very strong. Therefore, it was suggested that the first integral quantity increment is used to determine the stability of periodic motions in nonlinear dynamical systems in Luo(2007a). The chaotic motions in nonlinear dynamical systems can be investigated through the corresponding iterations given by the first integral quantity increment. For this method, the key is to determine the first integral manifold, which can be done from Lie group analysis. Sometimes, it is very difficult to obtain the first integral surface of a specific, nonlinear dynamical system. In recent years, Ao (2004) constructed the potential function for stochastic differential equations (see, Ao, 2005, Yin and Ao, 2006). Once the potential function can be found, the first integral quantity for stochastic dynamical systems can be developed. Such an issue needs further discussion.

1.4. Book layout

This book will systematically present a theory for n-dimensional nonlinear dynamics from a different point of view. The history and recent development of dynamics will be presented first. The differential geometrical relations between two flows in two different dynamical systems will be presented through G-functions. Based on the G-functions, the global transversality in nonlinear dynamical systems will be investigated. A theory for chaotic layer dynamics for nonlinear Hamiltonian systems will be discussed, which includes the resonant theory of stochastic layers and the stochasticity of resonant layers. The nonlinear dynamics on the $(2n\text{-}1)$-dimensional equi-energy surface will be discussed. For dissipative nonlinear dynamical systems, the stability and grazing bifurcations will be discussed in general. Finally, the global dynamics of 2-dimensional dynamical systems will be systematically discussed as an example. Switchability of a flow from a domain to an adjacent domain in discontinuous dynamical systems will be also discussed through the G-functions, and the first incremental increment for discontinuous dynamical system will be given. All the materials are scattered in ten chapters, as summarized as follows.

In Chapter 1, the brief development history of the modern theory of dynamics will be presented. The recent development of chaotic dynamics in nonlinear Hamiltonian systems was discussed first and the corresponding existing methodologies for approximate predictions of the onset, growth and destruction of chaotic layers were reviewed. The current researches on dissipative, nonlinear dynamical systems were addressed and the crucial, unsolved problems existing in nonlinear dynamical systems were briefly discussed. Finally, the summary and layout of contents in this book are given.

In Chapter 2, in two dynamical systems, the concepts for both *compared* and *reference* flows will be introduced to determine flow complexity. It is assumed that the flows of the reference dynamic system always exist on the certain reference surfaces. The time-change rate of the normal distance between two flows in the normal direction of the reference surface will be measured by a new function (i.e., G-function). Based on the reference flow, the k^{th}-order G-function for the non-contact and l^{th}-order contact flows in two different dynamical systems will be introduced in the normal direction of the reference surface. Through the new functions, the geometric relations between two flows in two dynamical systems will be presented for two flows with and without contact. Finally, the brief discussion of applications will be given. Such G-function quantity will be extensively used to measure the complexity properties of a flow in nonlinear dynamical systems.

In Chapter 3, the first integral quantity for nonlinear dynamical systems will be introduced for the global transversality of a flow to the separatrix. From the first integral quantity, the separatrix surface and initial sets of flows will be introduced, and the local and global flows to the separatrix surface will be discussed. The global transversality of flows to the separatrix surface in nonlinear dynamical systems will be discussed, and the corresponding necessary and sufficient conditions will be presented. The global transversality of a flow to the $2n$-dimensional Hamiltonian systems will be discussed comprehensively. The global transversality of a flow to the generalized separatrix will be also discussed. The global transversality of a flow in nonlinear dynamical system is a basic element to understand the trajectory complexity in phase space, which can very clearly explain the complexity of gradient dynamical systems.

In Chapter 4, a theory for chaotic layer dynamics will be presented. The small domain of flows to a specific, first integral manifold will be introduced in phase space. The first integral quantity increment will be introduced for periodic and chaotic flows in nonlinear dynamic systems. Based on different reference surfaces, all possible expressions for the first integral quantity increment will be given for different applications. The relations of the periodic and chaotic flows with the first integral quantity increments will be presented. The criteria for resonances in the stochastic and resonant layers will be presented through the first integral quantity increment. The first integral quantity increment for periodic flows in nonlinear dynamical systems is zero.

In Chapter 5, a resonant mechanism for stochastic layers in nonlinear Hamiltonian systems will be presented. The mathematical description of the stochastic layers will be given to help one understand the complexity of trajectory in the stochastic layer, and the theory and methodology for approximate predictions of the onset and vanishing of the resonances in stochastic layers will be discussed in detail. Several examples will be presented for demonstrating how to approximately predict the onset, growth and destruction of resonance in the stochastic layers. The maximum and minimum energy spectrum techniques will be presented, and the width computation of the stochastic layers will be estimated. From such a presentation, it will be observed that chaos in the stochastic layer of nonlinear Hamiltonian systems is caused by the resonant separatrix structure instead of the Smale's horseshoe structure.

In Chapter 6, the stochasticity of the resonant separatrix layers (or resonant layers) in nonlinear Hamiltonian systems will be discussed. The renormalization procedure in the vicinity of the resonant separatrix will be presented and the approximate conditions for the onset, growth and disappearance of the resonant layers will be given as well. The maximum and minimum energy increment spectrum method will be presented for numerically detecting resonant layers. Chaos in the resonant layer of nonlinear Hamiltonian systems is caused by the sub-resonant separatrix structure and resonance overlap.

In Chapter 7, the nonlinear dynamics on the $(2n-1)$-dimensional equi-energy space will be presented for $2n$-dimensional nonlinear Hamiltonian systems. A general methodology for quasi-periodic motions and chaos in such a Hamiltonian system will be presented. Such an idea can be extended to the dynamics on $(n-1)$-dimensional first integral invariant surface for n-dimensional dynamical systems. A nonlinear Hamiltonian system with two degrees of freedom will be investigated as a sampled problem.

In Chapter 8, the stability of equilibriums and periodic flows in dissipative dynamical systems will be discussed through the first integral quantity increment. Compared to the Lyapunov stability conditions, the weak stability conditions for equilibriums and periodic flows will be developed. Using the first integral quantity increment, the limit cycle in 2-dimensional nonlinear systems will be briefly discussed. Grazing bifurcation and the mapping structures based on the first integral manifold surface will be discussed for periodic and chaotic flows. The invariant set fragmentation of chaotic motions, caused by the grazing bifurcation, will be presented.

In Chapter 9, the analytical conditions for the global transversality of 2-dimensional, time-dependent, nonlinear dynamical systems will be presented. The first integral quantity increment (i.e., the energy increment) for a certain time interval will be achieved for periodic flows and chaos in the 2-dimensional nonlinear dynamical systems. Under perturbation assumptions and convergent conditions, the Melnikov function can be recovered from the first integral quantity increment. An example will be presented to show how to apply the newly developed theory.

In Chapter 10, the switchability of a flow from one domain to its adjacent domain in discontinuous dynamical systems will be presented. The G-function for discontinuous dynamical systems will be introduced. The imaginary flow in discontinuous dynamical systems will be employed. Based on such a G-function, the passability conditions for a flow from a domain to the adjacent domains will be investigated. Because of the discontinuity, the nonpassable flow to the separation boundary will be investigated. The tangential flow to the separation boundary will be addressed. Further, the switching bifurcation between the passable and non-passable flows to the separation boundary will be discussed. Finally, the first integral quantity increment for discontinuous dynamical systems will be presented instead of the approximate function (e.g., the Melnikov function) to develop the iterative mapping relations.