Chapter 6

Multiple Loop Feedback Amplifiers

6.1.0. Introduction

Chapters 4 and 5, and particularly Chapter 5, exploit signal flow analytical methods to circumvent the limitations inherent to the ideal feedback model in the course of studying the properties of feedback amplifiers. The idealized feedback model is useful only if the fundamental constituents of a feedback structure can be separated into the basic amplifier, $\mu(s)$, and the feedback network, $\beta(s)$. The procedure is difficult and sometimes virtually impossible, because the forward path may not be strictly unilateral, the feedback path is usually bilateral, and the input and output coupling networks are often complicated. Thus, the ideal feedback model may not be adequate to represent a practical amplifier. In this chapter, Bode's classic feedback theory is developed and assessed and in the process, a firm theoretical foundation is imparted to the disclosures in Chapter 5. Since Bode's technique is applicable to general feedback network configurations, it avoids the explicit necessity of identifying the forward and feedback transfer functions, $\mu(s)$ and $\beta(s)$, respectively.

Bode's feedback theory is based on the concept of return difference, which is defined in this chapter in terms of network determinants. The return difference, which is a generalization of the feedback factor concept implicit to the ideal feedback model, is a physical performance metric in that it can be measured directly. The null return difference and its physical significance follow straightforwardly from the return difference. Since Bode's feedback theory is formulated in terms of the first and second order cofactors of the elements of the indefinite admittance matrix of a feedback circuit, it is appropriate to review the mathematical concepts that underpin the indefinite admittance matrix for a linear network.

6.2.0. Indefinite Admittance Matrix

Figure 6.1 is an n-terminal network, N, composed of an arbitrary number of active and passive network elements connected in any electrically meaningful manner. Let $V_1, V_2, \ldots, V_n$ be the Laplace-transformed potentials measured between terminals $1, 2, \ldots, n$ and some arbitrary, but unspecified, reference point, and furthermore, let $I_1, I_2, \ldots, I_n$ be the Laplace-transformed currents entering the terminals $1, 2, \ldots, n$ from outside the network. Since network N and its load are linear, the terminal currents and voltages are related by the equation

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} + \begin{bmatrix} J_1 \\ J_2 \\ \vdots \\ J_n \end{bmatrix} \tag{6-1}$$

or more compactly,

$$\boldsymbol{I}(s) = \boldsymbol{Y}(s)\boldsymbol{V}(s) + \boldsymbol{J}(s) \tag{6-2}$$

where $\boldsymbol{Y}(s)$, is called the *indefinite admittance matrix* because the reference point for all network potentials is an arbitrary node extrinsic to the network, The current, J_k, ($k = 1, 2, \ldots, n$) denotes the current flowing into the kth terminal under the special condition of all terminals of N grounded to the reference point. This short circuit current can be construed as independent sources deriving from initial energy conditions established in the network interior. When no initial conditions prevail, $\boldsymbol{J}(s)$ is necessarily a null vector,

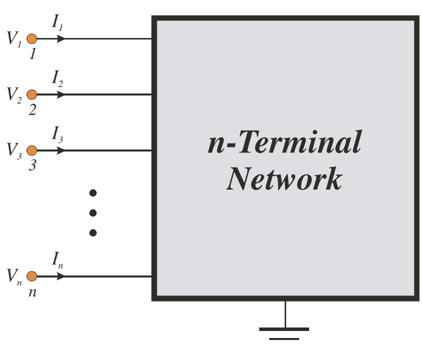

Figure 6.1. The general symbolic representation of an n-terminal network. All of the indicated terminal voltages are referenced to the system ground.

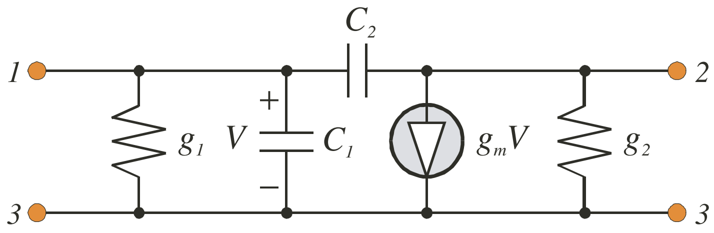

Figure 6.2. A small signal equivalent circuit of a transistor. The model is generally applicable to both bipolar and MOS technology devices.

whence

$$\boldsymbol{I}(s) = \boldsymbol{Y}(s)\boldsymbol{V}(s), \tag{6-3}$$

where the elements, y_{ij}, of $\boldsymbol{Y}(s)$ can be obtained as

$$y_{ij} = \left.\frac{I_i}{V_j}\right|_{V_x=0, x\neq j} \tag{6-4}$$

To illustrate, consider the small signal equivalent model of a transistor shown in Fig. 6.2. Its indefinite admittance matrix is

$$\boldsymbol{Y}(s) = \begin{bmatrix} g_1 + sC_1 + sC_2 & -sC_2 & -g_1 - sC_1 \\ g_m - sC_2 & g_2 + sC_2 & -g_2 - g_m \\ -g_1 - sC_1 - g_m & -g_2 & g_1 + g_2 + g_m + sC_1 \end{bmatrix} \tag{6-5}$$

Observe, as has been demonstrated in general in Chapter 2, that the sum of elements of each row and column is equal to zero.

If $\boldsymbol{Y}_{uv}$ denotes the submatrix obtained from an indefinite admittance matrix, $\boldsymbol{Y}(s)$, by deleting the uth row and vth column, the *first order cofactor,* denoted by the symbol Y_{uv}, of the element y_{uv} in $\boldsymbol{Y}(\boldsymbol{s})$, is

$$Y_{uv} = (-1)^{u+v} \det \boldsymbol{Y}_{uv}. \tag{6-6}$$

Because of the aforementioned zero row sum and zero column sum properties, all cofactors of the elements of the indefinite admittance matrix are equal. Such a matrix is referred to as an *equicofactor matrix*. It follows that if Y_{uv} and Y_{ij} are any two cofactors of an equicofactor matrix, $\boldsymbol{Y}(s)$,

$$Y_{uv} = Y_{ij} \tag{6-7}$$

for all u, v, i, and j. For the indefinite admittance matrix, $\boldsymbol{Y}(s)$, of Eq. (6-5), it is easily verified that all of its nine cofactors are equal to

$$Y_{uv} = s^2 C_1 C_2 + s\,(C_1 g_2 + C_2 g_1 + C_2 g_2 + g_m C_2) + g_1 g_2 \tag{6-8}$$

for u, $v = 1, 2, 3$.

Denote by $\boldsymbol{Y}_{rp,sq}$ the submatrix obtained from $\boldsymbol{Y}(s)$ by striking out rows r and s and columns p and q. Then the *second order cofactor*, denoted by the symbol $Y_{rp,sq}$ of the elements y_{rp} and y_{sq} of $\boldsymbol{Y}(s)$ is the scalar quantity defined by the expression,

$$Y_{rp,sq} = \operatorname{sgn}(r-s)\operatorname{sgn}(p-q)(-1)^{r+p+s+q} \det \boldsymbol{Y}_{rp,sq} \tag{6-9}$$

where $r \neq s$ and $p \neq q$, and

$$\left.\begin{aligned} \operatorname{sgn} u &= +1 \quad \text{if } u > 0 \\ \operatorname{sgn} u &= -1 \quad \text{if } u < 0 \\ \operatorname{sgn} u &\triangleq 0 \quad\;\; \text{if } u = 0 \end{aligned}\right\rangle. \tag{6-10}$$

In attempt to forestall confusion, the reader should understand that in Eq. (6-9) and related other equations, boldface type refers to a matrix, where conventional type denotes a scalar quantity.

As a further example, consider the hybrid-pi equivalent network of a bipolar junction transistor shown in Fig. 6.3. Assume that each node is an accessible terminal of a four-terminal network. Its indefinite admittance matrix is

$$\boldsymbol{Y}(s) = \begin{bmatrix} 0.02 & 0 & -0.02 & 0 \\ 0 & 5\times10^{-12}s & 0.2 - 5\times10^{-12}s & -0.2 \\ -0.02 & -5\times10^{-12}s & 0.024 + 105\times10^{-12}s & -0.004 - 10^{-10}s \\ 0 & 0 & -0.204 - 10^{-10}s & 0.204 + 10^{-10}s \end{bmatrix}. \tag{6-11}$$

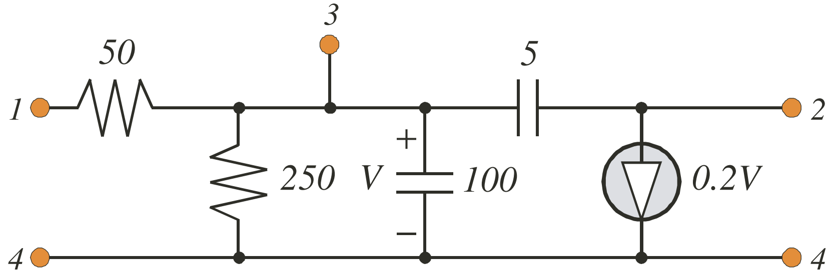

Figure 6.3. The hybrid-pi equivalent circuit of a bipolar junction transistor. All resistances in the model are in units of ohms, all capacitances are in picofarads, and the transconductance multiplier in the voltage controlled current source is in units of mhos, or siemens.

The second-order cofactor $Y_{31,42}$ and $Y_{11,34}$ of the elements of $\boldsymbol{Y}(s)$ of Eq. (6-11) are computed as follows:

$$Y_{31,42} = \operatorname{sgn}(3-4)\operatorname{sgn}(1-2)(-1)^{3+1+4+2} \times \det \begin{bmatrix} -0.02 & 0 \\ 0.2 - 5\times 10^{-12}s & -0.2 \end{bmatrix} = 0.004. \quad (6\text{-}12)$$

$$\begin{aligned} Y_{11,34} &= \operatorname{sgn}(1-3)\operatorname{sgn}(1-4)(-1)^{1+1+3+4} \\ &\quad \times \det \begin{bmatrix} 5\times 10^{-12}s & 0.2 - 5\times 10^{-12}s \\ 0 & -0.204 - 10^{-10}s \end{bmatrix} \\ &= 5\times 10^{-12}s\left(0.204 + 10^{-10}s\right). \end{aligned} \quad (6\text{-}13)$$

The engineering utility of the indefinite admittance matrix lies in the fact that it facilitates the computation of the driving point impedance presented by any pair of nodes or the transfer function from any nodal pair to any other pair. This contention is demonstrated by the material that follows.

Consider Fig. 6.4, which abstracts a linear network for which a current source is connected between any two nodes r and s so that a current I_{sr} is injected into the rth node and at the same time is extracted from the sth node. An ideal voltmeter is connected from node p to node q to monitor the potential rise from q to p. Then the *transfer impedance*, or *transimpedance*, $z_{rp,sq}$, between the node pairs rs and pq of the subject network is simply

$$z_{rp,sq} = \frac{V_{pq}}{I_{sr}}, \quad (6\text{-}14)$$

where it is understood that all initial conditions and associated independent energy sources inside N are set to zero; specifically, $\boldsymbol{J}(s) = 0$ in Eq. (6-2). The representation is, of course, quite general. When $r = p$ and $s = q$,

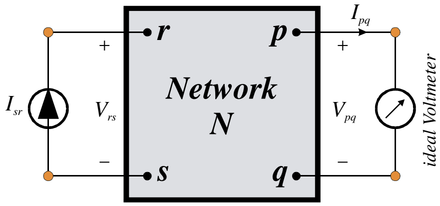

Figure 6.4. System abstraction for the measurement of a network transfer impedance, or transimpedance.

the transfer impedance $z_{rp,sq}$ becomes the *driving point impedance* $z_{rr,ss}$ between the terminal pair *rs*.

In Fig. 6.4, choose terminal q to be the zero voltage reference node for all other terminals. In terms of the equations of Eq. (6-1), this operation, coupled with the zero initial condition constraint, is tantamount to setting $\boldsymbol{J}(\boldsymbol{s}) = 0$, $V_q = 0$, $I_x = 0$ for $x \neq r, s$ and $I_r = -I_s = I_{sr}$. Since $\boldsymbol{Y}(s)$ is an equicofactor matrix, the equations of Eq. (6-1) are not linearly independent; that is, one of these equations is superfluous. To this end, suppress the sth equation in Eq. (6-1), which resultantly reduces to

$$\boldsymbol{I}_{-s} = \boldsymbol{Y}_{sq}\boldsymbol{V}_{-q}, \tag{6-15}$$

where $\boldsymbol{I}_{-s}$ and $\boldsymbol{V}_{-q}$ denote the subvectors obtained from $\boldsymbol{I}$ and $\boldsymbol{V}$ of Eq. (6-2) by deleting the sth and qth rows. Applying Cramer's rule, the voltage V_p, referenced to terminal q, is

$$V_p = \frac{\det \tilde{\mathbf{Y}}_{sq}}{\det \boldsymbol{Y}_{sq}}, \tag{6-16}$$

where $\tilde{\boldsymbol{Y}}_{sq}$ is the matrix derived from $\boldsymbol{Y}_{sq}$ by replacing the column corresponding to V_p by $\boldsymbol{I}_{-s}$. It should be noted that $\boldsymbol{I}_{-s}$ is in the pth column if $p < q$ but in the $(p-1)$th column if $p > q$. Furthermore, the row in which I_{sr} appears is the rth row if $r < s$ but in the $(r-1)$th row if $r > s$. Thus,

$$(-1)^{s+q} \det \tilde{\mathbf{Y}}_{sq} = I_{sr} Y_{rp,sq}. \tag{6-17}$$

In addition,

$$\det \boldsymbol{Y}_{sq} = (-1)^{s+q} Y_{sq}. \tag{6-18}$$

Substituting these relationships into Eq. (6-16),

$$z_{rp,sq} = \frac{Y_{rp,sq}}{Y_{uv}} \tag{6-19}$$

$$z_{rr,ss} = \frac{Y_{rr,ss}}{Y_{uv}} \tag{6-20}$$

where the fact that $Y_{sq} = Y_{uv}$ has been exploited.

The *voltage gain*, denoted by $g_{rp,sq}$, between node pairs *rs* and *pq* of the network of Fig. 6.4 is

$$g_{rp,sq} = \frac{V_{pq}}{V_{rs}}, \tag{6-21}$$

provided null initial conditions prevail in network N. It follows from Eqs. (6-19) and (6-20) that

$$g_{rp,sq} = \frac{z_{rp,sq}}{z_{rr,ss}} = \frac{Y_{rp,sq}}{Y_{rr,ss}}. \tag{6-22}$$

In an attempt to avoid confusion, it is worthwhile to review the mathematical logistics underpinning the subscripts invoked in Eqs. (6-19) through (6-22). In the numerators of these relationships, r is the current injecting node, p symbolizes the voltage measurement node, s denotes the current extracting node, and q represents the voltage reference node. Moreover, nodes r and p correspond to input and output transfer measurement.

To highlight the engineering utility of the foregoing theoretical disclosures, consider the hybrid-pi transistor equivalent network offered in Fig. 6.5. A 100-Ω load resistor is incident at nodes 2 and 4, and a voltage source, V_{14}, excites the amplifier input port. In the interest of analytical simplicity, let p denote the normalized frequency, $p = 10^{-9}s$. The indefinite admittance matrix of the amplifier is found to be

$$\boldsymbol{Y}(s) = \begin{bmatrix} 0.02 & 0 & -0.02 & 0 \\ 0 & 0.01 + 0.005p & 0.2 - 0.005p & -0.21 \\ -0.02 & -0.005p & 0.024 + 0.105p & -0.004 - 0.1p \\ 0 & -0.01 & -0.204 - 0.1p & 0.214 + 0.1p \end{bmatrix}. \tag{6-23}$$

To compute the voltage gain $g_{12,44}$, Eq. (6-22) can be exploited to obtain

$$g_{12,44} = \frac{V_{24}}{V_{14}} = \frac{Y_{12,44}}{Y_{11,44}} = \frac{p - 40}{5p^2 + 21.7p + 2.4}. \tag{6-24}$$

The input impedance facing the voltage source V_{14} is determined as

$$z_{11,44} = \frac{V_{14}}{I_{41}} = \frac{Y_{11,44}}{Y_{uv}} = \frac{Y_{11,44}}{Y_{44}} = \frac{50p^2 + 217p + 24}{p^2 + 4.14p + 0.08}. \tag{6-25}$$

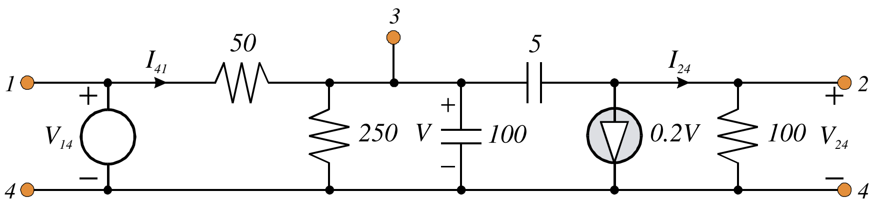

Figure 6.5. Model of a transistor amplifier used to illustrate the computation of the voltage gain, $g_{rp,sq}$. All resistances in the model are in units of ohms, all capacitances are in picofarads, and the transconductance multiplier in the voltage controlled current source is in units of siemens.

On the other hand, the current gain, defined as the ratio of current I_{24} in the 100-Ω resistor to the input current, I_{41}, Eq. (6-19) can be applied to arrive at

$$\frac{I_{24}}{I_{41}} = 0.01\frac{V_{24}}{I_{41}} = 0.01z_{12,44} = 0.01\frac{Y_{12,44}}{Y_{44}} = \frac{0.1p - 4}{p^2 + 4.14p + 0.08}. \qquad (6\text{-}26)$$

Finally, to compute the transfer admittance, defined as the ratio of load current I_{24} to input voltage V_{14}, we appeal to Eq. (6-22), which delivers

$$\frac{I_{24}}{V_{14}} = 0.01\frac{V_{24}}{V_{14}} = 0.01g_{12,44} = 0.01\frac{Y_{12,44}}{Y_{11,44}} = \frac{p - 40}{500p^2 + 2170p + 240}. \qquad (6\text{-}27)$$

It is notable that the delineation of the voltage gain, current gain, input impedance, and I/O transfer admittance metrics for a network entails a mere straightforward application of the indefinite admittance matrix of the network undergoing examination.

6.2.1. Return Difference

The study of feedback amplifier responses generally entails how a particular element or parameter of the subject amplifier affects that response. The selected element or parameter is crucial either in terms of its effect on the entire system or possibly, it is of engineering concern because of routinely encountered monolithic processing vagaries or manufacturing uncertainties. Ordinarily, the transfer function of an active device, the gain of an amplifier, or the immittance of a one-port network is of particular interest with respect to an assessment of the response sensitivity of a feedback network. For the present, assume that the selected crucial, or critical, parameter is x, the controlling parameter of a voltage controlled current source; that is,

$$I = xV. \qquad (6\text{-}28)$$

To focus attention on parameter x, Fig. 6.6 is the general configuration of a feedback amplifier in which the controlled source is highlighted as a two-port network connected to a general four-port network, along with the input source combination of I_s and admittance Y_1, and load admittance Y_2. The two-port representation of a controlled source in Eq. (6-28) is quite general. It includes the special situation where a one-port element is characterized

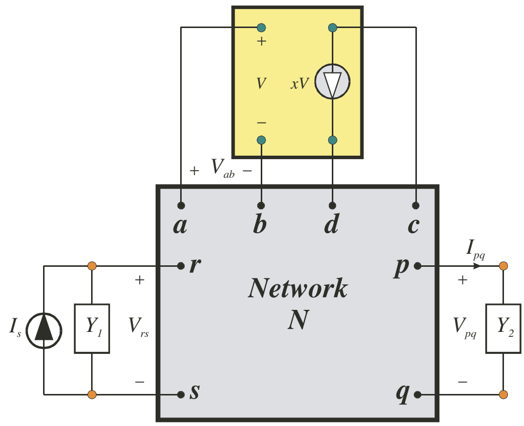

Figure 6.6. A generalized abstraction of a feedback circuit. The abstraction delineates local feedback imposed between two arbitrary internal ports of Network N.

by its immittance. In this case, the controlling voltage, V, is the terminal voltage of the controlled current source, I, whence x becomes the one-port admittance.

The *return difference,* $F(x)$, of a feedback amplifier with respect to a parameter x is defined as the ratio of the two functional values assumed by the first order cofactor of an element of its indefinite admittance matrix, under the condition that parameter x assumes its nominal value and the condition that parameter x assumes a null value. To emphasize the importance of the feedback critical parameter, x, express the indefinite admittance matrix $\boldsymbol{Y}$ of the amplifier as a function of x, even though it is also a function of the complex frequency variable s, and write $\boldsymbol{Y} = \boldsymbol{Y}(x)$. Then

$$F(x) \equiv \frac{Y_{uv}(x)}{Y_{uv}(0)}, \tag{6-29}$$

where

$$Y_{uv}(0) = Y_{uv}(x)|_{x=0}. \tag{6-30}$$

The physical significance of the return difference can now be addressed. In the network of Fig. 6.6, the input, the output, the controlling branch, and the controlled source are labeled as indicated. Then, parameter x enters the indefinite admittance matrix, $\boldsymbol{Y}(x)$, in a rectangular pattern as shown below:

$$\boldsymbol{Y}(x) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \!\!\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left[\begin{array}{cccc} & & & \\ & & & \\ x & -x & & \\ -x & x & & \end{array}\right] \end{array}. \tag{6-31}$$

If in Fig. 6.6 the controlled current source, xV, is supplanted by an independent current source of x amperes and the excitation current source, I_s, is set to zero, the indefinite admittance matrix of the resulting network is simply $\boldsymbol{Y}(0)$. By appealing to Eq. (6-24), the new voltage V'_{ab} appearing at terminals a and b of the controlling branch is found to be

$$V'_{ab} = x\frac{Y_{da,cb}(0)}{Y_{uv}(0)} = -x\frac{Y_{ca,db}(0)}{Y_{uv}(0)}. \tag{6-32}$$

Notice that the current injecting point is terminal d, not terminal c.

The foregoing operation is reflected electrically by the schematic diagram in Fig. 6.7. Observe that the controlling branch for the voltage controlled current source is severed and unity voltage is applied to the right of the break. This 1-volt sinusoidal voltage of a fixed angular frequency produces a current of x amperes at the controlled current source. The voltage established at the left of the circuit break by this 1-volt excitation is V'_{ab}, as

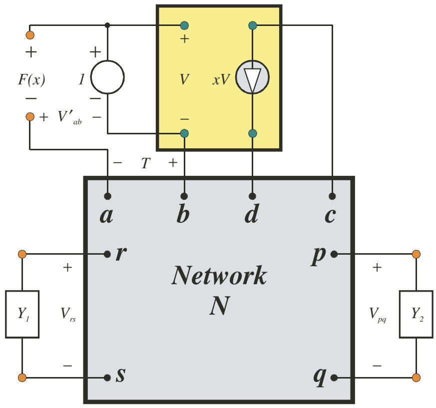

Figure 6.7. The physical interpretation of the return difference, $F(x)$, with respect to the controlling parameter, x, of a voltage controlled current source, xV.

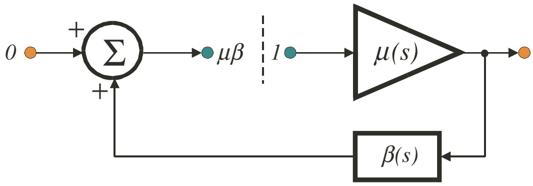

Figure 6.8. System level interpretation of the loop transmission metric.

indicated. This returned voltage, V'_{ab}, has the same physical significance as the loop transmission $\mu\beta$ defined for the ideal feedback model. A confirmation of this contention derives from setting the input excitation to the ideal feedback model to zero, breaking the forward path, and applying a unit input to the right of the break, as suggested by Fig. 6.8. The signal appearing at the left of the break is precisely the loop transmission.

The foregoing analytical disclosures all promote the introduction of the concept of the *return ratio, T*, which is defined as the negative of the voltage appearing at the controlling branch when the controlled current source is replaced by an independent current source of x amperes and the input excitation is set to zero. Thus, return ratio T is simply the negative of the returned voltage, V'_{ab} or $T = -V'_{ab}$. In view of this observation, the difference between the 1-volt excitation and the returned voltage, V'_{ab}, can be evaluated as

$$1 - V'_{ab} = 1 + x\frac{Y_{ca,db}}{Y_{uv}(0)} = \frac{Y_{uv}(0) + xY_{ca,db}}{Y_{uv}(0)} = \frac{Y_{db}(0) + xY_{ca,db}}{Y_{db}(0)}$$
$$= \frac{Y_{db}(x)}{Y_{db}(0)} = \frac{Y_{uv}(x)}{Y_{uv}(0)} = F(x), \tag{6-33}$$

in which the identities, $Y_{uv} = Y_{ij}$, and

$$Y_{db}(x) = Y_{db}(0) + xY_{ca,db} \tag{6-34}$$

have been invoked.

Of particular pertinence is the ability to write $Y_{ca,db}(x)$ as $Y_{ca,db}$ because $Y_{ca,db}(x)$ is independent of x. In other words, the return difference, $F(x)$, is simply the difference of the 1-volt excitation and the returned voltage, V'_{ab}, as illustrated in Fig. 6.7. Since

$$F(x) = 1 + T = 1 - \mu\beta, \tag{6-35}$$

the return difference can be seen to be identical to the feedback factor of the ideal feedback model. The significance of the above analytical disclosures is

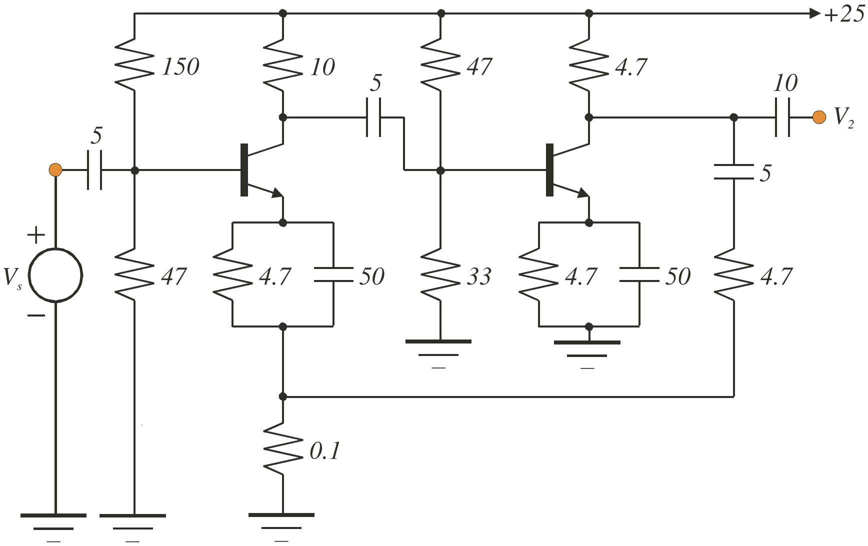

Figure 6.9. A series-shunt feedback amplifier shown with its requisite biasing and signal coupling circuitry. All resistances are in units of kilo-ohms, all capacitances are in units of microfarads, and the biasing supply is in units of volts.

the suggestion that the return ratio, T, or $-\mu\beta$, is deterministic via direct measurement. Once the return ratio is measured, other critical feedback network metrics, such as return difference and loop transmission, follow forthwith.

As an example of the engineering utility of the preceding arguments, consider the series-shunt feedback amplifier of Fig. 6.9. Assume that the two transistors are identical and are characterized by the following small signal hybrid h-parameters:

$$h_{ie} = 1.1\,\mathrm{K}\Omega, \quad h_{fe} = 50, \quad h_{re} = h_{oe} = 0. \tag{6-36}$$

After the biasing and coupling circuitry have been removed, the equivalent network is represented by the model in Fig. 6.10. The effective load of the first transistor is composed of the parallel combination of the 10-KΩ, 33-KΩ, 47-KΩ and 1.1-KΩ resistors. The effects of the 150-KΩ and 47-KΩ resistors can be ignored; they are included in the equivalent network only to show their insignificance in the computation.

To simplify notation, let

$$\tilde{\alpha}_k = \alpha_k \times 10^{-4} = \frac{h_{fe}}{h_{ie}} = 455 \times 10^{-4}, \quad k = 1, 2. \tag{6-37}$$

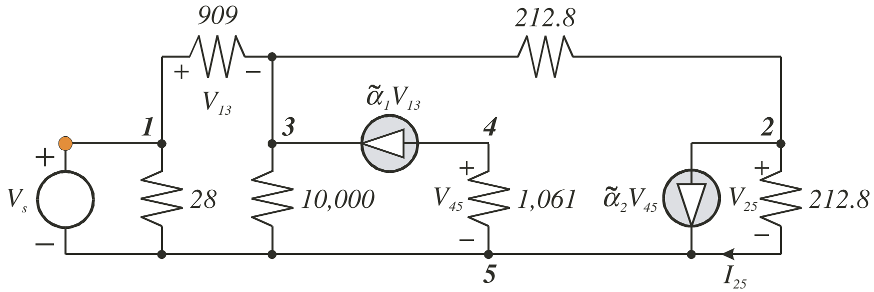

Figure 6.10. A low frequency equivalent circuit of the feedback amplifier in Figure 6.9. The resistive elements are represented as conductances with values in units of microsiemens.

The subscript k is used to distinguish the transconductances of the first and the second transistors. The indefinite admittance matrix of the feedback amplifier of Fig. 6.9 is found to be

$$\boldsymbol{Y} = 10^{-4}\begin{bmatrix} 9.37 & 0 & -9.09 & 0 & -0.28 \\ 0 & 4.256 & -2.128 & \alpha_2 & -2.128-\alpha_2 \\ -9.09-\alpha_1 & -2.128 & 111.218+\alpha_1 & 0 & -100 \\ \alpha_1 & 0 & -\alpha_1 & 10.61 & -10.61 \\ -0.28 & -2.128 & -100 & -10.61-\alpha_2 & 113.018+\alpha_2 \end{bmatrix}. \tag{6-38}$$

By applying Eq. (6-22), the voltage gain of the amplifier is found to be

$$g_{12,55} = \frac{V_{25}}{V_s} = \frac{Y_{12,55}}{Y_{11,55}} = \frac{211.54\times 10^{-7}}{4.66\times 10^{-7}} = 45.39. \tag{6-39}$$

To calculate the return differences with respect to the transconductances, $\tilde{\alpha}_k$, of the transistor, the input signal source, V_s, is short circuited. The resulting indefinite admittance matrix is obtained from Eq. (6-38) by adding the first row to the fifth row and the first column to the fifth column and then deleting the first row and column. Its first order cofactor is simply $Y_{11,55}$. Thus, the return differences with respect to $\tilde{\alpha}_k$ are

$$\left.\begin{aligned} F(\tilde{\alpha}_1) &= \frac{Y_{11,55}(\tilde{\alpha}_1)}{Y_{11,55}(0)} = \frac{466.1\times 10^{-9}}{4.97\times 10^{-9}} = 93.70 \\ F(\tilde{\alpha}_2) &= \frac{Y_{11,55}(\tilde{\alpha}_2)}{Y_{11,55}(0)} = \frac{466.1\times 10^{-9}}{25.52\times 10^{-9}} = 18.26 \end{aligned}\right\rangle. \tag{6-40}$$

6.2.2. Null Return Difference

The *null return difference,* $\hat{F}(x)$, of a feedback amplifier, with respect to a parameter x is an important feedback network metric from at least the perspective of network sensitivity with respect to a crucial parameter, x. It is defined to be the ratio of the two functional values assumed by the second order cofactor, $Y_{rp,sq}$, of the elements of the network indefinite admittance matrix, $\boldsymbol{Y}$, under the condition that element x assumes its nominal value and the condition that element x assumes a null value. It is to be understood that r and s are the input terminals, and p and q are the output terminals of the amplifier undergoing study. It follows that

$$\hat{F}(x) = \frac{Y_{rp,sq}(x)}{Y_{rp,sq}(0)}. \tag{6-41}$$

Similarly, the *null return ratio,* $\hat{T}$, with respect to a voltage controlled current source, $I = xV$, is the negative of the voltage appearing at the controlling branch when the controlled current source is replaced by an independent current source of x amperes and when the input excitation is adjusted so that the output of the amplifier is identically zero.

The null return difference is simply the return difference in the network under the constraint that the input excitation, I_s, has been adjusted so that the output is identically zero. In the network of Fig. 6.6, suppose that the controlled current source is supplanted by an independent current source of x amperes. Then by applying Eq. (6-19) and the superposition principle, the output current, I_{pq}, at the load is found to be

$$I_{pq} = Y_2\left[I_s\frac{Y_{rp,sq}(0)}{Y_{uv}(0)} + x\frac{Y_{dp,cq}(0)}{Y_{uv}(0)}\right]. \tag{6-42}$$

Setting $I_{pq} = 0$ or $V_{pq} = 0$ yields

$$I_s \equiv I_0 = -x\frac{Y_{dp,cq}(0)}{Y_{rp,sq}(0)}, \tag{6-43}$$

in which $Y_{dp,cq}$ is independent of x. This adjustment is possible only if there is a direct transmission from the input to the output when x is set to zero. Thus, in the network of Fig. 6.7, if an independent current source of strength I_0 is connected at the network input port, the voltage, V'_{ab}, is the negative of the null return ratio $\hat{T}$. Using Eq. (6-19),

$$\begin{aligned}\hat{T} &= -V'_{ab} = -x\frac{Y_{da,cb}(0)}{Y_{uv}(0)} - I_0\frac{Y_{ra,sb}(0)}{Y_{uv}(0)} \\ &= -\frac{x\left[Y_{da,cb}(0)Y_{rp,sq}(0) - Y_{ra,sb}(0)Y_{dp,cq}(0)\right]}{Y_{uv}(0)Y_{rp,sq}(0)} \\ &= \frac{x\dot{Y}_{rp,sq}}{Y_{rp,sq}(0)} = \frac{Y_{rp,sq}(x)}{Y_{rp,sq}(0)} - 1,\end{aligned} \tag{6-44}$$

where

$$\dot{Y}_{rp,sq} \equiv \frac{dY_{rp,sq}(x)}{dx}. \tag{6-45}$$

Consequently

$$\hat{F}(x) = 1 + \hat{T} = 1 - V'_{ab}, \tag{6-46}$$

which shows that the null return difference, $\hat{F}(x)$, is simply the difference of the 1-volt excitation applied to the right of the severed controlling branch of the controlled source and the returned voltage, V'_{ab}, appearing at the left of the break under the situation that the input signal, I_s, is adjusted to null the output port response to zero.

Reconsider the series-shunt feedback amplifier of Fig. 6.9, for which the pertinent small signal model appears in Fig. 6.10. Using the indefinite admittance matrix of Eq. (6-38) in conjunction with Eq. (6-37), the null return differences with respect to $\tilde{\alpha}_k$ are

$$\left.\begin{aligned}\hat{F}(\tilde{\alpha}_1) &= \frac{Y_{12,55}(\tilde{\alpha}_1)}{Y_{12,55}(0)} = \frac{211.54 \times 10^{-7}}{205.24 \times 10^{-12}} = 103.07 \times 10^3 \\ \hat{F}(\tilde{\alpha}_2) &= \frac{Y_{12,55}(\tilde{\alpha}_2)}{Y_{12,55}(0)} = \frac{211.54 \times 10^{-7}}{104.79 \times 10^{-10}} = 2018.70\end{aligned}\right\rangle. \tag{6-47}$$

Alternatively, $\hat{F}(\tilde{\alpha}_1)$ can be computed directly through use of its physical interpretation. To this end, replace the controlled source, $\tilde{\alpha}_1 V_{13}$, in Fig. 6.10 by an independent current source of $\tilde{\alpha}_1$ amperes. Adjust the voltage source, V_s, so that the output current, I_{25}, is identically zero. Let I_0 be the input current resulting from this adjusted signal source. The corresponding network

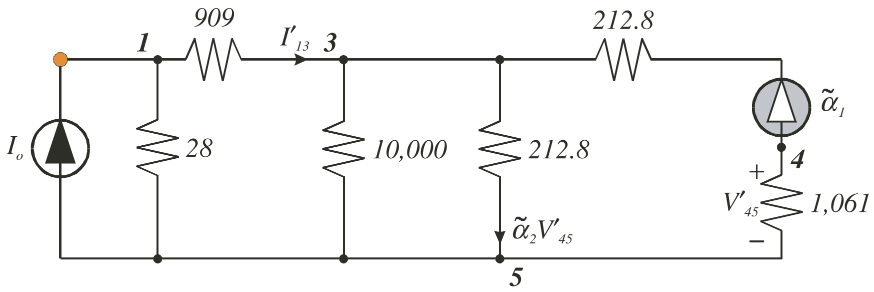

Figure 6.11. The network used to compute the null return difference with respect to parameter α_1. The resistive elements are represented as conductances with values in units of microsiemens.

is the structure in Fig. 6.11. From this network, an analysis reveals

$$\hat{F}(\tilde{\alpha}_1) = 1 + \hat{T} = 1 - V'_{13} = 1 - \frac{100V'_{35} + \alpha_2 V'_{45} - \alpha_1}{9.09} = 103.07 \times 10^3. \tag{6-48}$$

An analogous procedure computes the return difference, $\hat{F}(\tilde{\alpha}_2)$.

6.3.0. Network Functions and Feedback

Refer to the generalized feedback configuration of Fig. 6.6. Let w be a transfer function. As before, to emphasize the importance of the feedback element x, write $w = w(x)$. To be definitive, let $w(x)$ be the current gain, which is intrinsically dependent on critical parameter x, from the input port to the output port. Then from Eq. (6-19),

$$w(x) = \frac{I_{pq}}{I_s} = \frac{Y_2 V_{pq}}{I_s} = \frac{Y_{rp,sq}(x)}{Y_{uv}(x)} Y_2, \tag{6-49}$$

which produces

$$\frac{w(x)}{w(0)} = \frac{Y_{rp,sq}(x)}{Y_{uv}(x)} \frac{Y_{uv}(0)}{Y_{rp,sq}(0)} = \frac{\hat{F}(x)}{F(x)}, \tag{6-50}$$

provided that $w(0) \neq 0$. It follows that the current gain of interest is expressible as the simple relationship,

$$w(x) = w(0)\frac{\hat{F}(x)}{F(x)}. \tag{6-51}$$

Equation (6-51) remains valid if $w(x)$ denoted the transfer impedance $z_{rp,sq} = V_{pq}/I_s$, as opposed to the current gain.

6.3.1. Blackman's Formula

When $r = p$ and $s = q$, $w(x)$ represents the driving point impedance $z_{rr,ss}(x)$ established at the r-s terminal pair. In this case, $F(x)$ is the return difference with respect to parameter x under the condition, $I_s = 0$. Thus, $F(x)$ is the return difference for the case when the port at which the input impedance is defined is left open circuited; that is, the port is left without current source excitation. For this case, clarity compels writing $F(x) = F$(input open circuited). Likewise, from Fig. 6.6, $\hat{F}(x)$ is the return difference with respect to x for the input excitation I_s and output response V_{rs} under the condition that I_s is adjusted to clamp voltage response V_{rs} to zero. In other words, $\hat{F}(x)$ is the return difference for the situation when the port at which the input impedance is defined is short circuited. Accordingly, write $\hat{F}(x) = F$ (input short circuited). Resultantly, the input impedance, $Z(x)$, looking into a terminal pair can be conveniently expressed as

$$Z(x) = Z(0)\frac{F(\text{input short circuited})}{F(\text{input open circuited})}. \qquad \text{(6-52)}$$

Equation (6-52) is the well-known *Blackman's formula* for computing an impedance presented at any terminal pair of any linear, active or passive, network. The formula is extremely useful because its right hand side can usually be determined easily. If x represents the controlling parameter of a controlled source in a single loop feedback amplifier, setting $x = 0$ opens the feedback loop and $Z(0)$ is simply the zero feedback value of the impedance of interest. The return difference for x when the input port is short circuited or open circuited is relatively simple to compute because shorting or opening a terminal pair frequently breaks the feedback loop. In addition, Blackman's formula can be used to determine the return difference by measurements. Because it involves two return differences, only one of them can be identified and the other must be known in advance. In the case of a single loop feedback amplifier, it is usually possible to choose a terminal pair so that either the numerator or the denominator on the right hand side of Eq. (6-52) is unity. If F(input short circuited) $= 1$, F(input open circuited) becomes the return difference under normal operating conditions, whence

$$F(x) = \frac{Z(0)}{Z(x)}. \qquad \text{(6-53)}$$

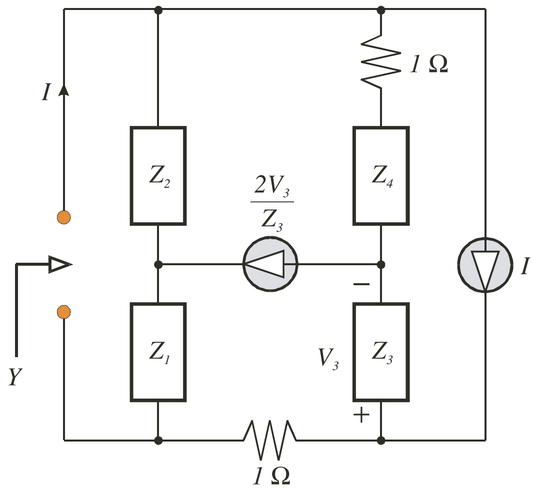

Figure 6.12. A general active RC one port realization of a rational function.

On the other hand, if F(input open circuited) $= 1$, F(input short circuited) becomes the return difference under normal operating conditions and thus,

$$F(x) = \frac{Z(x)}{Z(0)}. \tag{6-54}$$

Example 6.1. The network of Fig. 6.12 is a normalized active RC one port realization of a rational impedance. Use Blackman's formula to verify that the indicated input admittance of the subject network is given by

$$Y = 1 + \frac{Z_3 - Z_4}{Z_1 - Z_2}.$$

Solution 6.1.
Appealing to Eq. (6-52), the input admittance, $Y = Y(x)$, can be written as

$$Y(x) = Y(0)\frac{F(\text{input open circuited})}{F(\text{input short circuited})}, \tag{E6-1}$$

where $x = 2/Z_3$. By setting x to zero, the network used to compute $Y(0)$ becomes the topology offered in Fig. 6.13. Its input admittance is found to be

$$Y(0) = \frac{Z_1 + Z_2 + Z_3 + Z_4 + 2}{Z_1 + Z_2}. \tag{E6-2}$$

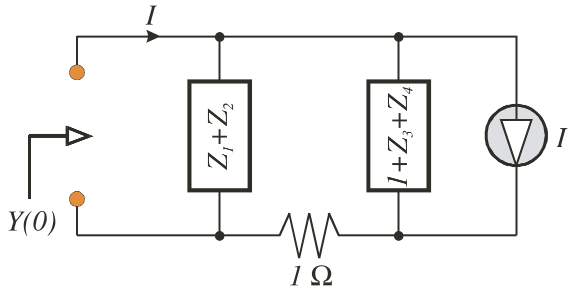

Figure 6.13. The network used to compute the admittance, $Y(0)$.

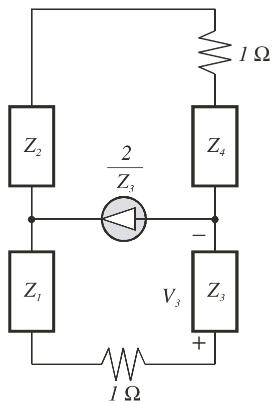

Figure 6.14. The network used to compute the function F(*input open circuited*).

When the input port is open circuited, the network of Fig. 6.1 degenerates to that of Fig. 6.14. The return difference with respect to x is found to be

$$F(\text{input open circuited}) = 1 - V_3' = \frac{Z_1 + Z_3 - Z_2 - Z_4}{2 + Z_1 + Z_2 + Z_3 + Z_4}, \tag{E6-3}$$

where the returned voltage, V_3', at the controlling branch is

$$V_3' = \frac{2\,(1 + Z_2 + Z_4)}{2 + Z_1 + Z_2 + Z_3 + Z_4}. \tag{E6-4}$$

To compute the return difference when the input port is short circuited, use the network in Fig. 6.15, to deduce

$$F(\text{input short circuited}) = 1 - V_3'' = \frac{Z_1 - Z_2}{Z_1 + Z_2}, \tag{E6-5}$$

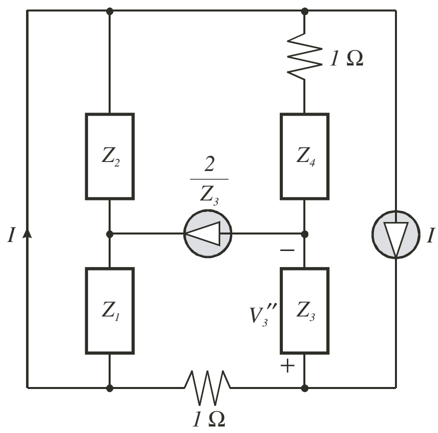

Figure 6.15. The network used to compute the function F(*input short circuited*).

where the returned voltage, V_3'', at the controlling branch is

$$V_3'' = \frac{2Z_2}{Z_1 + Z_2}. \tag{E6-6}$$

Substituting Eqs. (E6-2), (E6-3) and (E6-5) into Eq. (E6-1) yields the desired result,

$$Y = 1 + \frac{Z_3 - Z_4}{Z_1 - Z_2}. \tag{E6-7}$$

To determine the effect of feedback on the input and output impedances, consider the series-shunt feedback configuration given in Fig. 6.16. Since

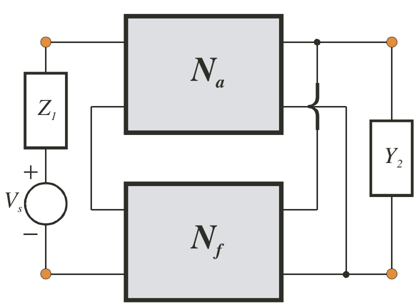

Figure 6.16. Generalized series-shunt feedback configuration.

shorting the terminals of Y_2 interrupts the feedback loop, Eq. (6-53) applies and the output impedance across load admittance Y_2 becomes

$$Z_{\text{out}}(x) = \frac{Z_{\text{out}}(0)}{F(x)}, \tag{6-55}$$

which verifies that the impedance measured across the path of the feedback is reduced by the factor that is the normal value of the return difference with respect to the element x. For the input impedance of the amplifier seen by the voltage source, V_s, in Fig. 6.16, by open-circuiting or removing the voltage source V_s, the feedback loop is broken. Thus, Eq. (6-6) applies, and the input impedance becomes

$$Z_{\text{in}}(x) = F(x)Z_{\text{in}}(0), \tag{6-56}$$

which suggests that the impedance measured in series lines is increased by the return difference, $F(x)$. Similar conclusions can be reached for other types of configurations.

Refer once again to the general feedback configuration of Fig. 6.6. If $w(x)$ represents either the voltage gain, V_{pq}/V_{rs}, or the transfer admittance, I_{pq}/V_{rs}, Eq. (6-22) offers

$$\frac{w(x)}{w(0)} = \frac{Y_{rp,sq}(x)}{Y_{rp,sq}(0)}\frac{Y_{rr,ss}(0)}{Y_{rr,ss}(x)}. \tag{6-57}$$

The first term in the product on the right hand side of this expression is the null return difference, $\hat{F}(x)$, with respect to x for the input terminals r and s and output terminals p and q. The second term is the reciprocal of the null return difference with respect to x for the same input and output port at terminals r and s. This reciprocal can then be interpreted as the return difference with respect to x when the input port of the amplifier is short circuited. Thus, the voltage gain or the transfer admittance can be expressed as

$$w(x) = w(0)\frac{\hat{F}(x)}{F(\text{input short circuited})}. \tag{6-58}$$

If $w(x)$ denotes the short circuit current gain I_{pq}/I_s as Y_2 approaches infinity,

$$\frac{w(x)}{w(0)} = \frac{Y_{rp,sq}(x)}{Y_{rp,sq}(0)}\frac{Y_{pp,qq}(0)}{Y_{pp,qq}(x)}. \tag{6-59}$$

The second term in the product on the right hand side in Eq. (6-59) is the reciprocal of the return difference with respect to x when the output port of the amplifier is short circuited, thereby enabling an expression for the short circuit current gain in the form of

$$w(x) = w(0)\frac{\hat{F}(x)}{F(\text{output short circuited})}. \tag{6-60}$$

Consider once again the series-shunt feedback amplifier of Fig. 6.9, the equivalent network of which is depicted in Fig. 6.10. The return differences, $F(\tilde{\alpha}_k)$, the null return differences, $\hat{F}(\tilde{\alpha}_k)$, and the voltage gain, w, have been computed as Eqs. (6-40), (6-47) and (6-39), respectively, and are repeated herewith for reader convenience; namely,

$$\left.\begin{aligned} &F(\tilde{\alpha}_1) = 93.70, \quad F(\tilde{\alpha}_2) = 18.26 \\ &\hat{F}(\tilde{\alpha}_1) = 103.07 \times 10^3, \quad \hat{F}(\tilde{\alpha}_2) = 2018.70 \\ &w = \frac{V_{25}}{V_S} = w(\tilde{\alpha}_1) = w(\tilde{\alpha}_2) = 45.39 \end{aligned}\right\rangle. \tag{6-61}$$

Applying Eq. (6-21), the voltage gain, w, evolves as

$$\begin{aligned} w(\tilde{\alpha}_1) &= w(0)\frac{\hat{F}(\tilde{\alpha}_1)}{F(\text{input short circuited})} \\ &= 0.04126\frac{103.07 \times 10^3}{93.699} = 45.39, \end{aligned} \tag{6-62}$$

where

$$\left.\begin{aligned} &w(0) = \left.\frac{Y_{12,55}(\tilde{\alpha}_1)}{Y_{11,55}(\tilde{\alpha}_1)}\right|_{\tilde{\alpha}_1=0} = \frac{205.24 \times 10^{-12}}{497.41 \times 10^{-11}} = 0.04126 \\ &F(\text{input short circuited}) = \frac{Y_{11,55}(\tilde{\alpha}_1)}{Y_{11,55}(0)} = \frac{466.07 \times 10^{-9}}{4.9741 \times 10^{-9}} = 93.699 \end{aligned}\right\rangle, \tag{6-63}$$

and

$$w(\tilde{\alpha}_2) = w(0)\frac{\hat{F}(\tilde{\alpha}_2)}{F(\text{input short circuited})} = 0.41058\frac{2018.70}{18.26} = 45.39, \tag{6-64}$$

with the understanding that

$$\left.\begin{aligned} w(0) &= \left.\frac{Y_{12,55}(\tilde{\alpha}_2)}{Y_{11,55}(\tilde{\alpha}_2)}\right|_{\tilde{\alpha}_2=0} = \frac{104.79 \times 10^{-10}}{255.22 \times 10^{-10}} = 0.41058 \\ F(\text{input short circuited}) &= \frac{Y_{11,55}(\tilde{\alpha}_2)}{Y_{11,55}(0)} = \frac{466.07 \times 10^{-9}}{25.52 \times 10^{-9}} = 18.26 \end{aligned}\right\rangle. \tag{6-65}$$

6.3.2. Sensitivity Function

One of the most important effects of negative feedback is its ability to make an amplifier less sensitive to parametric variations caused by the ravages of aging, temperature variations, and other environmental changes. A useful quantitative measure for the degree of dependence of an amplifier on a particular parameter is known as the sensitivity. The *sensitivity function,* written as $\mathcal{S}(x)$, for a given transfer function with respect to a parameter, x, is defined as the ratio of the fractional change in a transfer function to the fractional change in x for the circumstance in which all of interest are differentially small. Thus, if $w(x)$ is the transfer function, the sensitivity of this function with respect to small perturbations in parameter x is

$$\mathsf{S}(x) = \lim_{\Delta x \to 0} \frac{\Delta w / w}{\Delta x / x} = \frac{x}{w}\frac{\partial w}{\partial x} = x\frac{\partial \ln w}{\partial x} \tag{6-66}$$

Refer to the general feedback configuration of Fig. 6.6, and let $w(x)$ represent either the current gain, I_{pq}/I_s, or the transfer impedance, V_{pq}/I_s. Then from Eq. (6-19),

$$w(x) = Y_2 \frac{Y_{rp,sq}(x)}{Y_{uv}(x)}. \tag{6-67}$$

Letting

$$\left.\begin{aligned} \dot{Y}_{uv}(x) &\triangleq \frac{\partial Y_{uv}(x)}{\partial x} \\ \dot{Y}_{rp,sq}(x) &\triangleq \frac{\partial Y_{rp,sq}(x)}{\partial x} \end{aligned}\right\rangle, \tag{6-68}$$

which precipitates

$$\left.\begin{aligned} Y_{uv}(x) &= Y_{uv}(0) + x\dot{Y}_{uv}(x) \\ Y_{rp,sq}(x) &= Y_{rp,sq}(0) + x\dot{Y}_{rp,sq}(x) \end{aligned}\right\rangle. \tag{6-69}$$

Substituting Eq. (6-22) in Eq. (6-21) in conjunction with Eq. (6-24) yields

$$\mathsf{S}(x) = x\frac{\dot{Y}_{rp,sq}(x)}{Y_{rp,sq}(x)} - x\frac{\dot{Y}_{uv}(x)}{Y_{uv}(x)} = \frac{Y_{rp,sq}(x) - Y_{rp,sq}(0)}{Y_{rp,sq}(x)} - \frac{Y_{uv}(x) - Y_{uv}(0)}{Y_{uv}(x)}$$

$$= \frac{Y_{uv}(0)}{Y_{uv}(x)} - \frac{Y_{rp,sq}(0)}{Y_{rp,sq}(x)} = \frac{1}{F(x)} - \frac{1}{\hat{F}(x)}. \tag{6-70}$$

Recalling Eq. (6-3),

$$\mathsf{S}(x) = \frac{1}{F(x)}\left[1 - \frac{w(0)}{w(x)}\right]. \tag{6-71}$$

Note that for $w(0)= 0$, Eq. (6-71) becomes

$$\mathsf{S}(x) = \frac{1}{F(x)}, \tag{6-72}$$

which implies that the aforementioned sensitivity metric with respect to a parameter x is equal to the reciprocal of the return difference, also evaluated with respect to x. For the ideal feedback model, the feedback path is unilateral. Hence, $w(0) = 0$ and

$$\mathsf{S} = \frac{1}{F} = \frac{1}{1+T} = \frac{1}{1-\mu\beta}. \tag{6-73}$$

In a practical amplifier, $w(0)$ is usually very much smaller than $w(x)$ in the passband, and $F \approx 1/\mathsf{S}$ may be used as a reasonable estimate of the reciprocal of the sensitivity in the same frequency band. A single loop feedback amplifier comprised of a cascade of common emitter stages with a passive network providing the desired feedback fulfills this requirement. If in such a structure any one of the transistors fails, the forward transmission is nearly zero and $w(0)$ is practically zero. Accordingly, if the failure of any element interrupts signal transmission through the amplifier to nearly zero, the sensitivity is approximately equal to the reciprocal of the return difference with respect to that element. In the case of driving point impedance, $w(0)$ is generally not much smaller than $w(x)$, which renders the reciprocity approximation invalid.

Assume now that $w(x)$ represents the voltage gain. Substituting Eq. (6-22) in Eq. (6-21) results in

$$\mathsf{S}(x) = x\frac{\dot{Y}_{rp,sq}(x)}{Y_{rp,sq}(x)} - x\frac{\dot{Y}_{rr,ss}(x)}{Y_{rr,ss}(x)} = \frac{Y_{rp,sq}(x) - Y_{rp,sq}(0)}{Y_{rp,sq}(x)} - \frac{Y_{rr,ss}(x) - Y_{rr,ss}(0)}{Y_{rr,ss}(x)}$$

$$= \frac{Y_{rr,ss}(0)}{Y_{rr,ss}(x)} - \frac{Y_{rp,sq}(0)}{Y_{rp,sq}(x)} = \frac{1}{F(\text{input short circuited})} - \frac{1}{\hat{F}(x)}. \tag{6-74}$$

Combining this result with Eq. (6-71) gives

$$\mathsf{S}(x) = \frac{1}{F(\text{input short circuited})}\left[1 - \frac{w(0)}{w(x)}\right]. \tag{6-75}$$

Finally, if $w(x)$ denotes the short circuit current gain, I_{pq}/I_s, as Y_2 approaches infinity, the sensitivity function can be written as

$$\mathsf{S}(x) = \frac{Y_{pp,qq}(0)}{Y_{pp,qq}(x)} - \frac{Y_{rp,sq}(0)}{Y_{rp,sq}(x)} = \frac{1}{F(\text{output short circuited})} - \frac{1}{\hat{F}(x)}, \tag{6-76}$$

which when combined with Eq. (6-71) yields

$$\mathsf{S}(x) = \frac{1}{F(\text{output short circuited})}\left[1 - \frac{w(0)}{w(x)}\right]. \tag{6-77}$$

As an example, consider the network of Fig. 6.17, which schematically portrays a common emitter transistor amplifier. After removing the biasing circuit and using the common emitter hybrid model for the transistor at low frequencies, an appropriate equivalent circuit of the amplifier is the structure presented in Fig. 6.18 with

$$\left.\begin{aligned} I'_s &= \frac{V_s}{R_1 + r_x} \\ G'_1 &= \frac{1}{R'_1} = \frac{1}{R_1 + r_x} + \frac{1}{r_\pi} \\ G'_2 &= \frac{1}{R'_2} = \frac{1}{R_2} + \frac{1}{R_c} \end{aligned}\right\rangle. \tag{6-78}$$

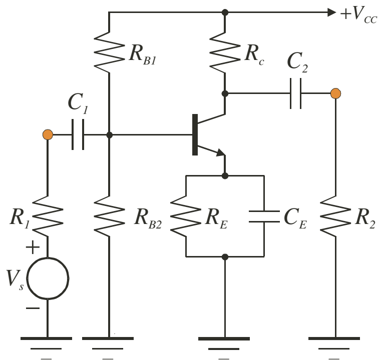

Figure 6.17. A common emitter transistor feedback amplifier.

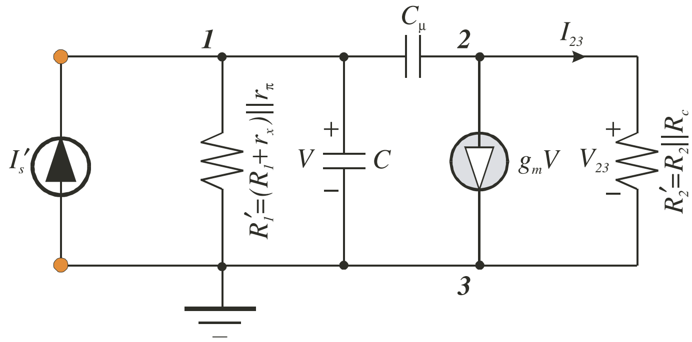

Figure 6.18. Equivalent circuit of the feedback amplifier in Figure 6.17.

The indefinite admittance matrix of the amplifier is seen to be

$$\boldsymbol{Y} = \begin{bmatrix} G_1' + sC_\pi + sC_\mu & -sC_\mu & -G_1' - sC_\pi \\ g_m - sC_\mu & G_2' + sC_\mu & -G_2' - g_m \\ -G_1' - sC_\pi - g_m & -G_2' & G_1' + G_2' + sC_\pi + g_m \end{bmatrix} \tag{6-79}$$

Assume that the controlling transconductance, g_m, is the critical element of immediate interest. The return difference and the null return difference with respect to g_m in Fig. 6.18, with I_s' applied to the input port and R_2' terminating the output port, are found to be

$$F(g_m) = \frac{Y_{33}(g_m)}{Y_{33}(0)} = \frac{(G_1' + sC_\pi)(G_2' + sC_\mu) + sC_\mu(G_2' + g_m)}{(G_1' + sC_\pi)(G_2' + sC_\mu) + sC_\mu G_2'} \tag{6-80}$$

and

$$\hat{F}(g_m) = \frac{Y_{12,33}(g_m)}{Y_{12,33}(0)} = \frac{sC_\mu - g_m}{sC_\mu} = 1 - \frac{g_m}{sC_\mu}. \tag{6-81}$$

The current gain, I_{23}/I_s', as defined in Fig. 6.18, is computed as

$$w(g_m) = \frac{Y_{12,33}(g_m)}{R_2' Y_{33}(g_m)} = \frac{sC_\mu - g_m}{R_2'\left[\left(G_1' + sC_\pi\right)\left(G_2' + sC_\mu\right) + sC_\mu\left(G_2' + g_m\right)\right]}. \tag{6-82}$$

Substituting these in Eq. (6-74) or Eq. (6-75) gives

$$\mathsf{S}(g_m) = -\frac{g_m\left(G_1' + sC_\pi + sC_\mu\right)\left(G_2' + sC_\mu\right)}{\left(sC_\mu - g_m\right)\left[\left(G_1' + sC_\pi\right)\left(G_2' + sC_\mu\right) + sC_\mu\left(G_2' + g_m\right)\right]}. \tag{6-83}$$

Finally, compute the sensitivity for the driving point impedance facing the current source, I_s'. From Eq. (6-74),

$$\begin{aligned}\mathsf{S}(g_m) &= \frac{1}{F(g_m)}\left[1 - \frac{Z(0)}{Z(g_m)}\right] \\ &= -\frac{sC_\mu g_m}{\left(G_1' + sC_\pi\right)\left(G_2' + sC_\mu\right) + sC_\mu\left(G_2' + g_m\right)},\end{aligned} \tag{6-84}$$

where

$$Z(g_m) = \frac{Y_{11,33}(g_m)}{Y_{33}(g_m)} = -\frac{G_2' + sC_\mu}{\left(G_1' + sC_\pi\right)\left(G_2' + sC_\mu\right) + sC_\mu\left(G_2' + g_m\right)}. \tag{6-85}$$

6.4.0. Measurement of Return Difference

The zeros of a network determinant are called the *natural frequencies* of the network. Their locations in the complex frequency plane are extremely important in that they determine the stability and both the frequency domain and time domain responses of the network. A network is said to be *stable* if all of its natural frequencies are restricted to the open left-half of the complex frequency plane (LHS). If a network determinant is known, its roots can readily be computed explicitly with the aid of a computer if necessary, and the stability problem can then be settled directly. However, for a physical network there remains the difficulty of getting an accurate formulation of the network determinant itself, because every equivalent network is, to a greater or lesser extent, an idealization of physical reality. As frequency is increased, parasitic effects of the physical elements must be taken into account. What is really needed is some kind of experimental verification that the network is stable and remains stable under prescribed operational and environmental conditions. The measurement of the return difference provides an elegant solution to this problem.

The return difference with respect to an element x in a feedback amplifier is given as

$$F(x) = \frac{Y_{uv}(x)}{Y_{uv}(0)}. \tag{6-86}$$

Since $Y_{uv}(x)$ denotes the nodal determinant and $Y_{uv}(0)$ is said determinant evaluated for $x = 0$, the zeros of the return difference are exactly the same as the zeros of the nodal determinant, provided that there is no cancellation of common factors between $Y_{uv}(x)$ and $Y_{uv}(0)$. Therefore, if $Y_{uv}(0)$ is known to have no zeros in the closed right-half of the complex frequency plane (RHS), which is usually the case in a single loop feedback amplifier, $F(x)$ gives precisely the same information about the stability of a feedback amplifier, as does the nodal determinant itself. The difficulty inherent in the

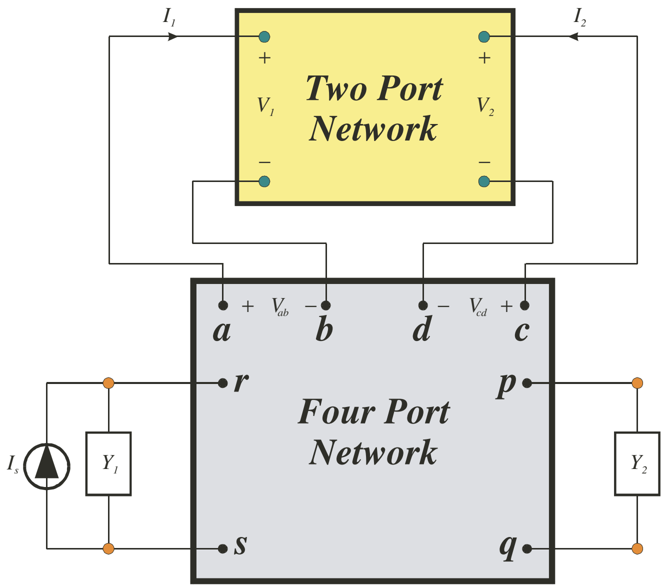

Figure 6.19. The general configuration of a feedback amplifier with a two port device.

measurement of the return difference with respect to the controlling parameter of a controlled source is that, in a physical system, the controlling branch and the controlled source invariably form part of a single device, such as a transistor, and cannot be physically separated. The measurement schema presented herewith does not require the physical decomposition of a device.

Let a device of interest be brought out as a two-port network connected to a general four-port network as shown in Fig. 6.19. Assume that this device can be characterized by its y-parameters, thereby enabling the electrical representation in Fig. 6.20, where parameter y_{21} controls signal transmission in the forward direction through the device while y_{12} gives the reverse transmission deriving from intrinsic device feedback. The fundamental objective here is the measurement of the return difference with respect to the forward short circuit transfer admittance y_{21}.

6.4.1. Blecher's Procedure

Let the two-port device in question be a transistor operated in the common emitter configuration with terminals a, b, c, and d respectively representing, the base, emitter, collector, and ground terminals. To simplify notation, let $a = 1$, $b = d = 3$ and $c = 2$, as exhibited in Fig. 6.21.

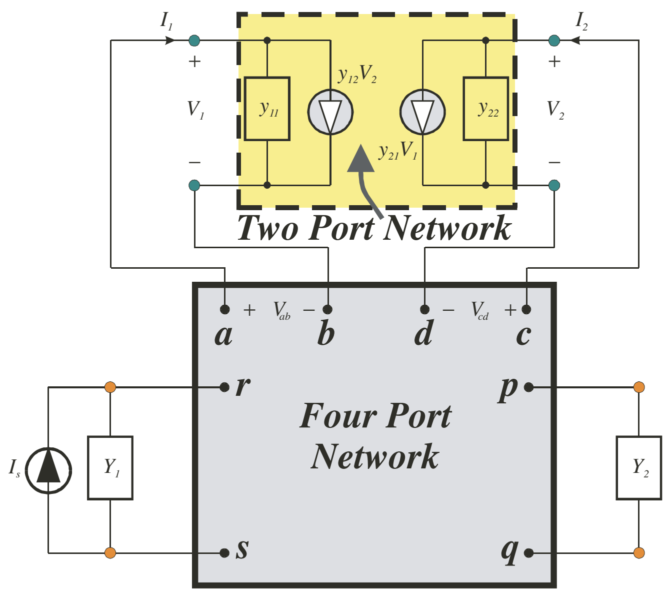

Figure 6.20. The system of Figure 6.19 with the two port network represented by its short circuit admittance (y-) parameters.

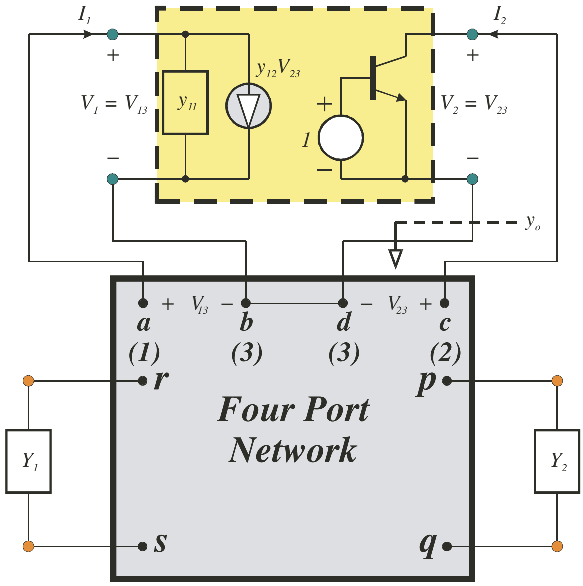

Figure 6.21. Physical interpretation of the return difference, $F(y_{11})$, for a transistor operated in common emitter mode and modeled by its short circuit admittance parameters.

To measure $F(y_{21})$, break the base terminal of the transistor and apply a 1-V excitation at its input as diagrammed in Fig. 6.21. To ensure that the controlled current source, $y_{21}V_{13}$, drives a replica of the load it sees during normal operation, connect an active one-port network composed of a parallel combination of the admittance, y_{11}, and a controlled current source, $y_{12}V_{23}$, at terminals 1 and 3. The returned voltage, V_{13}, is precisely the negative of the return ratio with respect to the element, y_{21}. If the externally applied feedback is large compared with the internal feedback of the transistor over the frequency passband of interest, the controlled source, $y_{12}V_{23}$, can be ignored. On the other hand, if internal device feedback cannot be ignored, its effects can be simulated by using an additional transistor, connected as shown in Fig. 6.22. This additional transistor must be matched as closely as possible to the one in question. The one-port admittance y_o denotes the admittance presented to the output port of the transistor under consideration as indicated in Figs. 6.21 and 6.22. For a common emitter state, it is reasonable to assume that $|y_o| \gg |y_{12}|$ and $|y_{11}| \gg |y_{12}|$. In view of these presumptions, it is straightforward to show that the Norton equivalent network looking into the two-port network at terminals 1 and

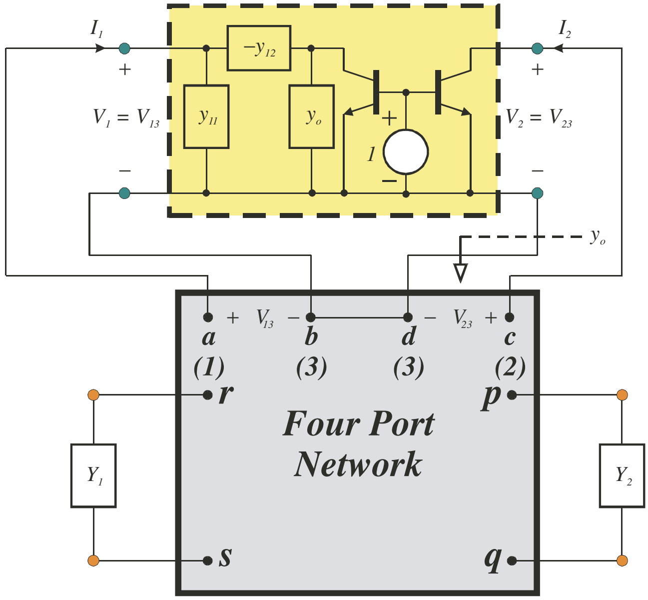

Figure 6.22. The measurement of the return difference, $F(y_{21})$, for a transistor operated as a common emitter amplifier and represented electrically by its short circuit admittance parameters.

3 of Fig. 6.22 can be approximated by the parallel combination of y_{11} and $y_{12}V_{23}$ as indicated in Fig. 6.21. In Fig. 6.22, voltage sources having very low internal impedances can be joined together at the two transistor base terminals, which can then be driven by a single voltage source of low internal impedance. But for the foregoing measurement procedure to be feasible, it must be demonstrated that the admittances, y_{11} and $-y_{12}$, can be realized as positive real admittances presented by suitable one-port passive networks.

Consider the small signal hybrid-pi model of a common emitter transistor shown in Fig. 6.23. The short circuit admittance matrix of the this circuit model is

$$\boldsymbol{Y}_{sc} = \frac{1}{g_x + g_\pi + sC_\pi + sC_\mu} \times \begin{bmatrix} g_x\left(g_\pi + sC_\pi + sC_\mu\right) & -g_x sC_\mu \\ g_x\left(g_n - sC_\mu\right) & sC_\mu\left(g_x + g_\pi + sC_\pi + g_m\right) \end{bmatrix}. \quad (6\text{-}87)$$

It is easy to confirm that the admittances, y_{11} and $-y_{12}$, can be realized by the one-port networks of Fig. 6.24.

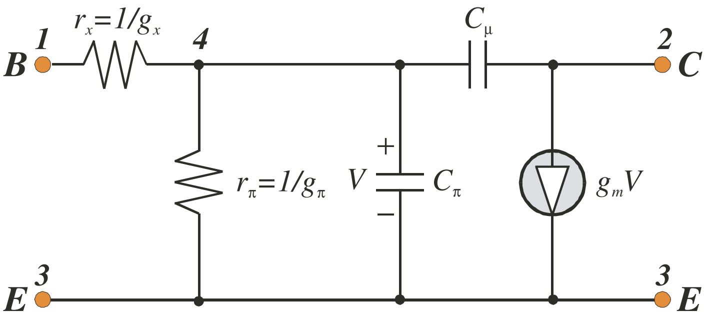

Figure 6.23. The approximate hybrid-pi equivalent circuit of a transistor operated as a common emitter amplifier.

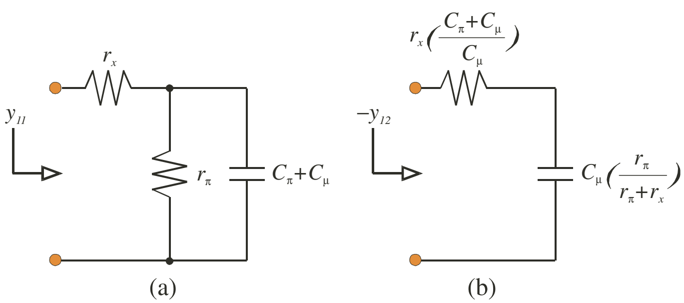

Figure 6.24. (a) The realization of the common emitter short circuit admittance parameter, y_{11}. (b) The realization of the short circuit admittance function, $-y_{12}$.

6.4.2. Impedance Measurements

Refer again to the general feedback configuration of Fig. 6.6. Suppose that the return difference with respect to the forward short circuit transfer admittance y_{21}, is to be evaluated. The controlling parameters, y_{12} and y_{21}, enter the indefinite admittance matrix, $\boldsymbol{Y}$, in the rectangular patterns abstracted below:

$$\boldsymbol{Y}(x) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} & & y_{12} & -y_{12} \\ & & -y_{12} & y_{12} \\ y_{21} & -y_{21} & & \\ -y_{21} & y_{21} & & \end{bmatrix} \end{array}. \tag{6-88}$$

To emphasize the importance of y_{12} and y_{21}, write $Y_{uv}(x)$ as $Y_{uv}(y_{12}, y_{21})$ and $z_{aa,bb}(x)$ as $z_{aa,bb}(y_{12}, y_{21})$. By appealing to Eq. (6-20), the impedance looking into terminals a and b of Fig. 6.6 is

$$z_{aa,bb}(y_{12}, y_{21}) = \frac{Y_{aa,bb}(y_{12}, y_{21})}{Y_{dd}(y_{12}, y_{21})}. \tag{6-89}$$

The return difference with respect to parameter y_{21} follows as

$$F(y_{21}) = \frac{Y_{dd}(y_{12}, y_{21})}{Y_{dd}(y_{12}, 0)}. \tag{6-90}$$

Combining Eqs. (6-89) and (6-90),

$$\begin{aligned} F(y_{21})z_{aa,bb}(y_{12}, y_{21}) &= \frac{Y_{aa,bb}(y_{12}, y_{21})}{Y_{dd}(y_{12}, 0)} = \frac{Y_{aa,bb}(0, 0)}{Y_{dd}(y_{12}, 0)} \\ &= \frac{Y_{aa,bb}(0, 0)}{Y_{dd}(0, 0)} \frac{Y_{dd}(0, 0)}{Y_{aa,bb}(y_{12}, 0)} = \frac{z_{aa,bb}(0, 0)}{F(y_{12})|_{y_{21}=0}}, \end{aligned} \tag{6-91}$$

which engenders

$$F(y_{12})|_{y_{21}=0} F(y_{21}) = \frac{z_{aa,bb}(0, 0)}{z_{aa,bb}(y_{12}, y_{21})} \tag{6-92}$$

as an interrelationship among the return differences and the driving point impedances. $F(y_{12})|_{y_{21}=0}$ is the return difference with respect to y_{12} when y_{21} is set to zero. This quantity can be measured by the arrangement of Fig. 6.25. In this diagram, $z_{aa,bb}(y_{12}, y_{21})$ is the driving point impedance looking into terminals a and b of the network of Fig. 6.6. Finally, $z_{aa,bb}(0, 0)$ is the impedance to which $z_{aa,bb}(y_{12}, y_{21})$ reduces when the controlling parameters, y_{12} and y_{21}, are both set to zero. This impedance can be measured by the configuration of Fig. 6.26. Note that in all three measurements, the independent current source, I_s, is removed.

Suppose now that the return difference, $F(y_{21})$, with respect to the forward transfer admittance, y_{21}, of the common emitter transistor shown in Fig. 6.20 is to be measured. For all practical purposes, the return difference, $F(y_{12})$, when y_{21} is set to zero, is indistinguishable from unity. Therefore, Eq. (6-92) reduces to the simpler form,

$$F(y_{21}) \approx \frac{z_{11,33}(0, 0)}{z_{11,33}(y_{12}, y_{21})}, \tag{6-93}$$

which shows that the return difference, $F(y_{21})$, is effectively the ratio of the two functional values assumed by the driving point impedance looking into terminals 1 and 3 of Fig. 6.6 under the conditions that the controlling parameters, y_{12} and y_{21}, are both set to zero and that these parameters assume their nominal values. These two impedances can be measured by the network arrangements of Figs. 6.27 and 6.28.

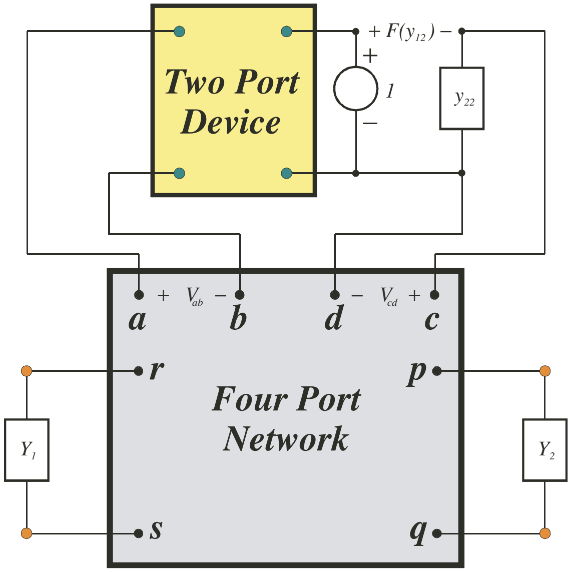

Figure 6.25. The measurement of the return difference, $F(y_{12})$, with short circuit admittance parameter y_{21} set to zero.

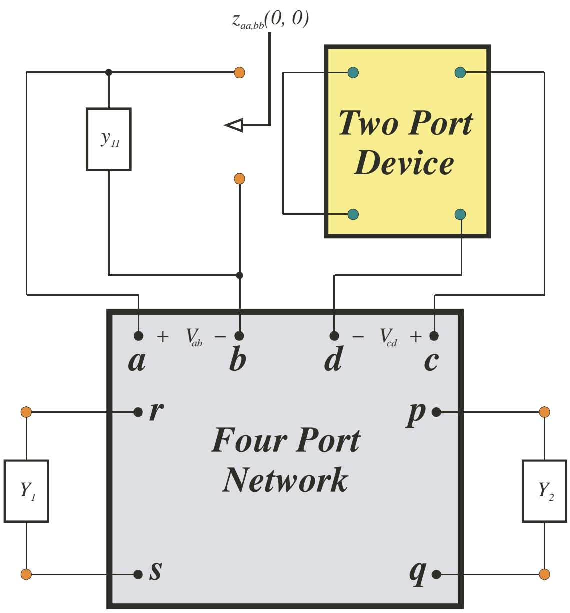

Figure 6.26. The measurement of the driving point impedance, $z_{aa,bb}(0, 0)$.

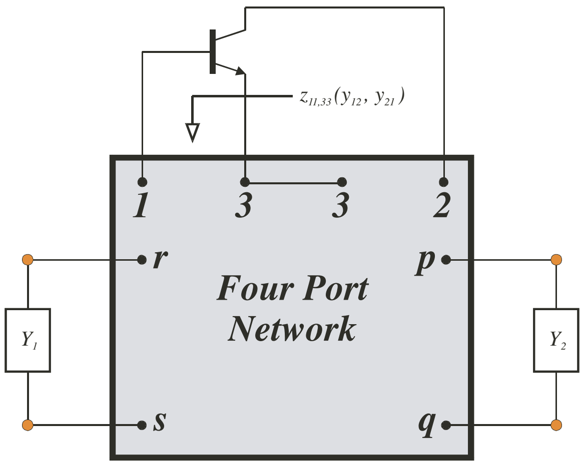

Figure 6.27. The measurement of the driving point impedance, $z_{11,33}(y_{12}, y_{21})$.

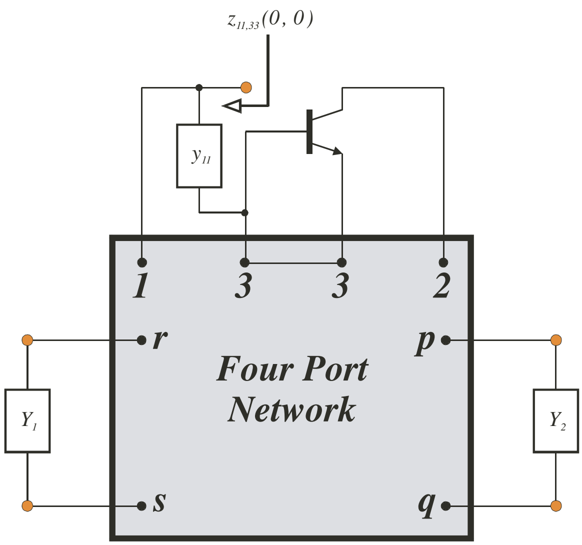

Figure 6.28. The measurement of the driving point impedance, $z_{11,33}(0, 0)$.

6.5.0. Multiloop Feedback

At this juncture, single loop feedback amplifiers have been studied in depth, wherein the return difference has been postured as a pivotally important characteristic. The return difference is the difference between a unit applied signal and the resultantly returned signal. The returned signal has the same physical meaning as the loop transmission in the ideal feedback model. It plays an important role in the study of amplifier stability, the sensitivity of amplifier responses to variations of parameters, and the determination of feedback amplifier transfer and driving point impedances. The fact that the return difference can be measured experimentally for most practical amplifiers suggests that relevant parasitic effects in assessing relative network stability and other amplifier performance measures.

In this section, amplifiers containing a multiplicity of inputs, outputs, and feedback loops are studied. These networks are referred to as the *multiple loop feedback amplifiers*. As might be expected, the traditional concept of a return difference with respect to an element is no longer applicable because of the presence of multiple loops and numerous potentially critical parameters. Accordingly, the return difference concept for a controlled source must be expanded to the notion of a return difference matrix for a multiplicity of controlled sources. For measurement situations, the null return difference matrix is introduced, and its engineering significance is discussed. It is shown herewith that the determinant of the overall transfer function matrix of a multiple loop feedback network can be expressed explicitly in terms of the determinants of the return difference and the null return difference matrices, thereby allowing a generalization of Blackman's impedance formula.

6.5.1. Multiloop Feedback Theory

The general configuration of a multiple input, multiple output, multiple loop feedback amplifier is abstracted in Fig. 6.29, in which the input, output, and feedback variables may be either currents or voltages. For the specific arrangement of Fig. 6.29, the input and output variables are represented by an n-dimensional vector $\boldsymbol{u}$ and an m-dimensional vector $\boldsymbol{y}$ as

$$\boldsymbol{u}(s) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \\ u_{k+1} \\ u_{k+2} \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} I_{s1} \\ I_{s2} \\ \vdots \\ I_{sk} \\ V_{s1} \\ V_{s2} \\ \vdots \\ V_{s(n-k)} \end{bmatrix}, \quad \boldsymbol{y}(s) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \\ y_{r+1} \\ y_{r+2} \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_r \\ V_{r+1} \\ V_{r+2} \\ \vdots \\ V_m \end{bmatrix}, \qquad (6\text{-}94)$$

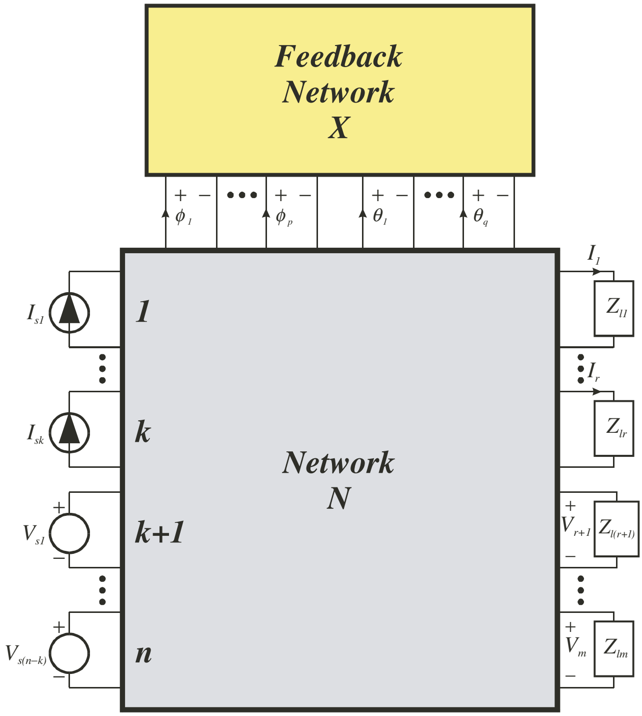

Figure 6.29. General configuration of a multiple input, multiple output, multiple loop feedback amplifier.

respectively. The elements of interest can be represented by a rectangular matrix, $\boldsymbol{X}$, of order $q \times p$ relating the controlled and controlling variables by the matrix equation

$$\boldsymbol{\Theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_q \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ x_{q1} & x_{q2} & \cdots & x_{qp} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{bmatrix} = \boldsymbol{X\Phi}, \tag{6-95}$$

where the p-dimensional vector, $\boldsymbol{\Phi}$, is called the *controlling vector*, and the q-dimensional vector, $\boldsymbol{\Theta}$, the *controlled vector*. The controlled variables, θ_k, and the controlling variables, ϕ_k, can either be currents or voltages. The matrix $\boldsymbol{X}$ can represent either a transfer function matrix or a driving point function matrix. If $\boldsymbol{X}$ represents a driving point function matrix, the vectors, $\boldsymbol{\Theta}$ and $\boldsymbol{\Phi}$, are of the same dimension ($q = p$), and their components are the currents and voltages of a p-port network.

The general configuration of Fig. 6.29 can be represented equivalently by the block digraph of Fig. 6.30, in which N is a $(p+q+m+n)$-port network and the elements of interest are exhibited by block $\boldsymbol{X}$. For the $(p+q+m+n)$-port network N, the vectors, $\boldsymbol{u}$ and $\boldsymbol{\Theta}$, are its inputs, and the vectors, $\boldsymbol{\Phi}$ and $\boldsymbol{y}$, its outputs. Since N is linear, the input and output vectors are related by the matrix equation

$$\left.\begin{aligned} \boldsymbol{\Phi} &= \boldsymbol{A\Theta} + \boldsymbol{Bu} \\ \boldsymbol{y} &= \boldsymbol{C\Theta} + \boldsymbol{Du} \end{aligned}\right\rangle, \tag{6-96}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are transfer function matrices of orders $p \times q$, $p \times n$, $m \times q$ and $m \times n$, respectively. The vectors, $\boldsymbol{\Theta}$ and $\boldsymbol{\Phi}$, are not independent and are related by

$$\boldsymbol{\Theta} = \boldsymbol{X\Phi}. \tag{6-97}$$

The relationships among the above three linear matrix equations can also be represented by the matrix signal flow graph depicted in Fig. 6.31, which is known as the *fundamental matrix feedback flow graph*. The overall

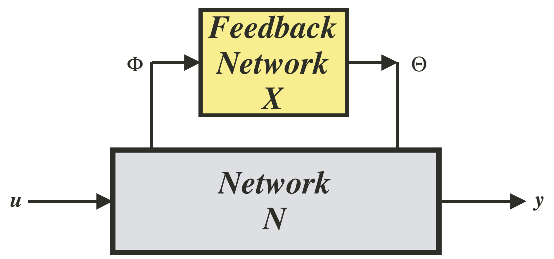

Figure 6.30. The block diagram representation of the general feedback configuration of Figure 6.29.

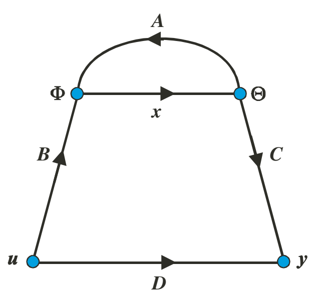

Figure 6.31. The fundamental matrix feedback signal flow graph.

closed loop *transfer function matrix*, $\boldsymbol{W}(\boldsymbol{X})$, of the multiple loop feedback amplifier derives from the equation

$$\boldsymbol{y} = \boldsymbol{W}(\boldsymbol{X})\boldsymbol{u}, \tag{6-98}$$

where $\boldsymbol{W}(\boldsymbol{X})$ is of order $m \times n$. As before, to emphasize the importance of $\boldsymbol{X}$, the matrix $\boldsymbol{W}$ is written as $\boldsymbol{W}(\boldsymbol{X})$ for the present discussion, even though it is also a function of the complex frequency variable, s. Combining the above matrix equations, the transfer function matrix is found to be

$$\left.\begin{aligned} \boldsymbol{W}(\boldsymbol{X}) &= \boldsymbol{D} + \boldsymbol{C}\boldsymbol{X}\left(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X}\right)^{-1}\boldsymbol{B} \\ \boldsymbol{W}(\boldsymbol{X}) &= \boldsymbol{D} + \boldsymbol{C}\left(\mathbf{1}_q - \boldsymbol{X}\boldsymbol{A}\right)^{-1}\boldsymbol{X}\boldsymbol{B} \end{aligned}\right\rangle, \tag{6-99}$$

where 1_p denotes the identity matrix of order p. Clearly,

$$\boldsymbol{W}(\mathbf{0}) = \boldsymbol{D}. \tag{6-100}$$

In particular, when $\boldsymbol{X}$ is square and nonsingular, Eq. (6-99) can be written as

$$\boldsymbol{W}(\boldsymbol{X}) = \boldsymbol{D} + \boldsymbol{C}\left(\boldsymbol{X}^{-1} - \boldsymbol{A}\right)^{-1}\boldsymbol{B}. \tag{6-101}$$

As an illustration, consider the feedback amplifier of Fig. 6.9. An equivalent network is shown in Fig. 6.32, in which the two transistors are presumed identical with $h_{ie} = 1.1\,\mathrm{k}\Omega$, $h_{fe} = 50$, and $h_{re} = h_{oe} = 0$. Let the controlling parameters of the two controlled sources be the elements of

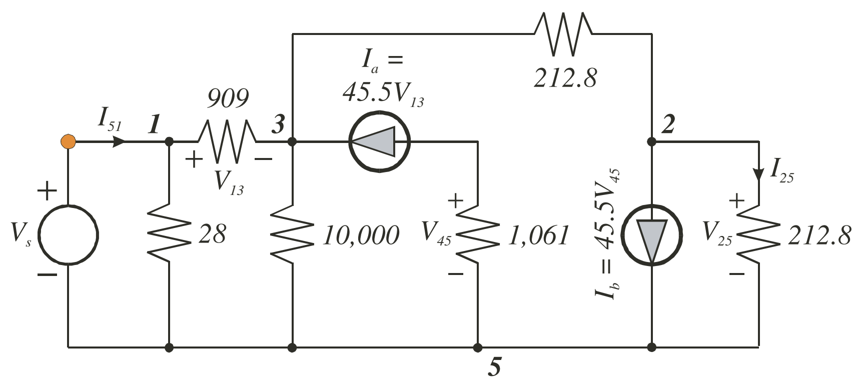

Figure 6.32. A low frequency equivalent circuit of the feedback amplifier offered in Figure 6.9. The resistive branch elements are represented as conductances with values in units of microsiemens, while the transconductance parameters associated with the voltage controlled current sources are in units of millisiemens.

interest. Then,

$$\boldsymbol{\Theta} = \begin{bmatrix} I_a \\ I_b \end{bmatrix} = 10^{-4} \begin{bmatrix} 455 & 0 \\ 0 & 455 \end{bmatrix} \begin{bmatrix} V_{13} \\ V_{45} \end{bmatrix} = \boldsymbol{X}\boldsymbol{\Phi}. \tag{6-102}$$

Assume that the output voltage, V_{25}, and the input current, I_{51}, are the output variables. Then the seven-port network, N, defined by the variables V_{13}, V_{45}, V_{25}, I_{51}, I_a, I_b and V_s can be characterized by the matrics,

$$\left.\begin{aligned} \boldsymbol{\Phi} &= \begin{bmatrix} V_{13} \\ V_{45} \end{bmatrix} = \begin{bmatrix} -90.782 & 45.391 \\ -942.507 & 0 \end{bmatrix} \begin{bmatrix} I_a \\ I_b \end{bmatrix} \\ &\quad + \begin{bmatrix} 0.91748 \\ 0 \end{bmatrix} [V_s] = \boldsymbol{A}\Theta + \boldsymbol{B}u \\ \boldsymbol{y} &= \begin{bmatrix} V_{25} \\ I_{51} \end{bmatrix} = \begin{bmatrix} 45.391 & -2372.32 \\ -0.08252 & 0.04126 \end{bmatrix} \begin{bmatrix} I_a \\ I_b \end{bmatrix} \\ &\quad + \begin{bmatrix} 0.041260 \\ 0.000862 \end{bmatrix} [V_s] = \boldsymbol{C}\boldsymbol{\Theta} + \boldsymbol{D}\boldsymbol{u} \end{aligned}\right\rangle. \tag{6-103}$$

According to Eq. (6-98), the transfer function matrix of the amplifier is defined by the matrix equation

$$\boldsymbol{y} = \begin{bmatrix} V_{25} \\ I_{51} \end{bmatrix} = \begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix} [V_s] = \boldsymbol{W}(\boldsymbol{X})\boldsymbol{u}. \tag{6-104}$$

Since $\boldsymbol{X}$ is square and nonsingular, Eq. (6-101) can be exploited to calculate $\boldsymbol{W}(\boldsymbol{X})$:

$$\boldsymbol{W}(\boldsymbol{X}) = \boldsymbol{D} + \boldsymbol{C}\left(\boldsymbol{X}^{-1} - \boldsymbol{A}\right)^{-1}\boldsymbol{B} = \begin{bmatrix} 45.387 \\ 0.369 \times 10^{-4} \end{bmatrix} = \begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix}, \tag{6-105}$$

where

$$\left(\boldsymbol{X}^{-1} - \boldsymbol{A}\right)^{-1} = 10^{-4} \begin{bmatrix} 4.856 & 10.029 \\ -208.245 & 24.914 \end{bmatrix}. \tag{6-106}$$

These relationships can be used to obtain the closed loop voltage gain, w_{11}, and the input impedance, Z_{in}, facing the voltage source, V_s. In particular,

$$w_{11} = \frac{V_{25}}{V_s} = 45.387, \quad Z_{\text{in}} = \frac{V_s}{I_{51}} = \frac{1}{w_{21}} = 27.1\,\text{K}\Omega. \tag{6-107}$$

6.5.2. Return Difference Matrix

In the fundamental feedback flow graph of Fig. 6.31, break the input of the branch with transmittance $\boldsymbol{X}$, set the input excitation vector, $\boldsymbol{u}$, to zero, and apply a signal p-vector, $\boldsymbol{g}$, to the right of the break, as depicted in Fig. 6.33.

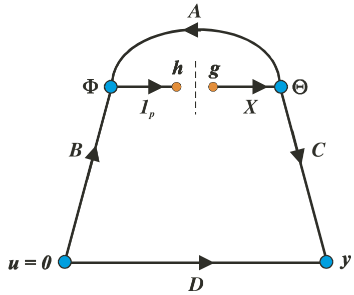

Figure 6.33. The engineering interpretation of the loop transmission matrix.

The resultant signal p-vector, $\boldsymbol{h}$, returned to the left of the break is

$$\boldsymbol{h} = \boldsymbol{A}\boldsymbol{X}\boldsymbol{g}. \tag{6-108}$$

The square matrix, $\boldsymbol{AX}$, is called the *loop transmission matrix*, and its negative, referred to as the *return ratio matrix*, is denoted by

$$\boldsymbol{T}(\boldsymbol{X}) = -\boldsymbol{A}\boldsymbol{X}. \tag{6-109}$$

The difference between the applied signal vector, $\boldsymbol{g}$, and the returned signal vector, $\boldsymbol{h}$, is

$$\boldsymbol{g} - \boldsymbol{h} = \left(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X}\right)\boldsymbol{g}. \tag{6-110}$$

The square matrix, $[\mathbf{1}_p - \boldsymbol{AX}]$, relating the applied signal vector, $\boldsymbol{g}$, to the difference of applied signal vector $\boldsymbol{g}$ and the returned signal vector, $\boldsymbol{h}$, is called the *return difference matrix* with respect to $\boldsymbol{X}$ and is symbolized by

$$\boldsymbol{F}(\boldsymbol{X}) = 1_p - \boldsymbol{A}\boldsymbol{X}. \tag{6-111}$$

Combining this expression with Eq. (6-109) gives

$$\boldsymbol{F}(\boldsymbol{X}) = \mathbf{1}_p + \boldsymbol{T}(\boldsymbol{X}). \tag{6-112}$$

For the series-shunt feedback amplifier of Fig. 6.32, let the controlling parameters of the two controlled current sources be the elements of interest. The corresponding return ratio matrix is found from Eqs. (6-102) and (6-103) to be

$$\boldsymbol{T}(\boldsymbol{X}) = -\boldsymbol{A}\boldsymbol{X} = -\begin{bmatrix} -90.782 & 45.391 \\ -942.507 & 0 \end{bmatrix}\begin{bmatrix} 455 \times 10^{-4} & 0 \\ 0 & 455 \times 10^{-4} \end{bmatrix}$$
$$= \begin{bmatrix} 4.131 & -2.065 \\ 42.884 & 0 \end{bmatrix}, \tag{6-113}$$

which yields

$$\boldsymbol{F}(\boldsymbol{X}) = \mathbf{1}_2 + \boldsymbol{T}(\boldsymbol{X}) = \begin{bmatrix} 5.131 & -2.065 \\ 42.884 & 1 \end{bmatrix}. \tag{6-114}$$

6.5.3. Null Return Difference Matrix

A direct extension of the null return difference for the single loop feedback amplifier is the null return difference matrix for the multiple loop feedback networks. Refer again to the fundamental matrix feedback flow graph of Fig. 6.31. Once again, break the branch with transmittance $\boldsymbol{X}$ and apply a signal p-vector, $\boldsymbol{g}$, to the right of the break, as illustrated in Fig. 6.34. Next, adjust the input excitation n-vector, $\boldsymbol{u}$, so that the total output m-vector, $\boldsymbol{y}$, resulting from the inputs $\boldsymbol{g}$ and $\boldsymbol{u}$ is null. From Fig. 6.34, the required input excitation, $\boldsymbol{u}$, is

$$\boldsymbol{D}\boldsymbol{u} + \boldsymbol{C}\boldsymbol{X}\boldsymbol{g} = \mathbf{0}, \tag{6-115}$$

whence

$$\boldsymbol{u} = -\boldsymbol{D}^{-1}\boldsymbol{C}\boldsymbol{X}\boldsymbol{g}, \tag{6-116}$$

provided that matrix $\boldsymbol{D}$ is square and nonsingular. An underlying prerequisite to this constraint is that output $\boldsymbol{y}$ be of the same dimension as input $\boldsymbol{u}$; namely, $m = n$. Physically, this requirement is reasonable because the effects at the output caused by $\boldsymbol{g}$ can be neutralized by a unique input excitation $\boldsymbol{u}$ only when $\boldsymbol{u}$ and $\boldsymbol{y}$ are of the same dimension. Given these inputs, $\boldsymbol{u}$ and $\boldsymbol{g}$, the returned signal, $\boldsymbol{h}$, to the left of the break in Fig. 6.34 computes as

$$\boldsymbol{h} = \boldsymbol{B}\boldsymbol{u} + \mathbf{A}\boldsymbol{X}\boldsymbol{g} = \left(-\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}\boldsymbol{X} + \boldsymbol{A}\boldsymbol{X}\right)\boldsymbol{g}, \tag{6-117}$$

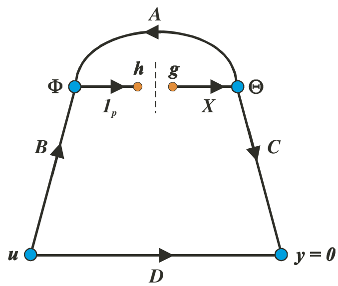

Figure 6.34. The engineering interpretation of the null return difference matrix.

whereby

$$\boldsymbol{g} - \boldsymbol{h} = \left(\mathbf{1}_p - \boldsymbol{AX} + \boldsymbol{BD}^{-1}\boldsymbol{CX}\right)\boldsymbol{g}. \tag{6-118}$$

The square matrix,

$$\hat{\boldsymbol{F}}(\boldsymbol{X}) = \mathbf{1}_p + \hat{\boldsymbol{T}}(\boldsymbol{X}) = \mathbf{1}_p - \boldsymbol{AX} + \boldsymbol{BD}^{-1}\boldsymbol{CX} = \mathbf{1}_p - \hat{\boldsymbol{A}}\boldsymbol{X}, \tag{6-119}$$

relating input signal vector $\boldsymbol{g}$ to the difference of input signal vector $\boldsymbol{g}$ and returned signal vector $\boldsymbol{h}$ is commonly referred to as the *null return difference matrix* with respect to $\boldsymbol{X}$, where

$$\left.\begin{aligned}&\hat{\boldsymbol{T}}(\boldsymbol{X}) = -\boldsymbol{AX} + \boldsymbol{BD}^{-1}\boldsymbol{CX} = -\hat{\boldsymbol{A}}\boldsymbol{X}\\&\hat{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{BD}^{-1}\boldsymbol{C}\end{aligned}\right\rangle. \tag{6-120}$$

The square matrix, $\hat{\boldsymbol{T}}(\boldsymbol{X})$, is the *null return ratio matrix*.

As yet another example, consider the voltage-series feedback amplifier of Fig. 6.9, for which the equivalent network is the structure in Fig. 6.32. Assume that the voltage, V_{25}, is the output variable. From Eq. (6-103),

$$\left.\begin{aligned}\boldsymbol{\Phi} &= \begin{bmatrix} V_{13} \\ V_{45} \end{bmatrix}\\ &= \begin{bmatrix} -90.782 & 45.391 \\ -942.507 & 0 \end{bmatrix}\begin{bmatrix} I_a \\ I_b \end{bmatrix} + \begin{bmatrix} 0.91748 \\ 0 \end{bmatrix}[V_s] = \boldsymbol{A\Theta} + \boldsymbol{Bu}\\ \boldsymbol{y} &= [V_{25}]\\ &= \begin{bmatrix} 45.391 & -2372.32 \end{bmatrix}\begin{bmatrix} I_a \\ I_b \end{bmatrix} + [0.04126][V_s] = \boldsymbol{C\Theta} + \boldsymbol{Du}\end{aligned}\right\rangle. \tag{6-121}$$

Using the coefficient matrices in Eq. (6-120),

$$\hat{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{BD}^{-1}\boldsymbol{C} = \begin{bmatrix} -1100.12 & 52{,}797.6 \\ -942.507 & 0 \end{bmatrix}, \tag{6-122}$$

which gives the null return difference matrix with respect to $\boldsymbol{X}$ as

$$\hat{\boldsymbol{F}}(\boldsymbol{X}) = \mathbf{1}_2 - \hat{\boldsymbol{A}}\boldsymbol{X} = \begin{bmatrix} 51.055 & -2402.29 \\ 42.884 & 1 \end{bmatrix}. \tag{6-123}$$

Suppose that the input current, I_{51}, is chosen as the output variable. Then from Eq. (6-103),

$$\boldsymbol{y} = [I_{51}] = \begin{bmatrix} -0.08252 & 0.04126 \end{bmatrix}\begin{bmatrix} I_a \\ I_b \end{bmatrix} + [0.000862][V_s] = \boldsymbol{C\Theta} + \boldsymbol{Du}. \tag{6-124}$$

The corresponding null return difference matrix is

$$\hat{\boldsymbol{F}}(\boldsymbol{X}) = \mathbf{1}_2 - \hat{\boldsymbol{A}}\boldsymbol{X} = \begin{bmatrix} 1.13426 & -0.06713 \\ 42.8841 & 1 \end{bmatrix}, \tag{6-125}$$

where

$$\hat{\boldsymbol{A}} = \begin{bmatrix} -2.95085 & 1.47543 \\ -942.507 & 0 \end{bmatrix}. \tag{6-126}$$

6.5.4. Transfer Function Matrix

This section demonstrates the effect of feedback on the transfer function matrix, $\boldsymbol{W}(\boldsymbol{X})$. Specifically, express the determinant, $\det[\boldsymbol{W}(\boldsymbol{X})]$, in terms of the determinant, $\det[\boldsymbol{X}(0)]$, and the determinants of the return difference and null return difference matrices, thereby expanding Blackman's impedance formula for a single input to that of a formulation applicable to a multiplicity of inputs.

Before proceeding, it is necessary to state the following determinant identity for two arbitrary matrices, $\boldsymbol{M}$ and $\boldsymbol{N}$, of orders $m \times n$ and $n \times m$, respectively:

$$\det(\mathbf{1}_m + \boldsymbol{MN}) = \det(\mathbf{1}_n + \boldsymbol{NM})\,. \tag{6-127}$$

A proof of this identity relationship can be found in Chen [1991]. Armed with this stipulation, the following generalization of Blackman's formula for input impedance can be proffered.

Theorem 6.1. In a multiple loop feedback amplifier, if $\boldsymbol{W(0)} = \boldsymbol{D}$ is nonsingular, the determinant of the transfer function matrix, $\boldsymbol{W}(\boldsymbol{X})$, is related to the determinants of the return difference matrix, $\boldsymbol{F}(\boldsymbol{X})$, and the null return difference matrix, $\hat{\boldsymbol{F}}(\boldsymbol{X})$, by

$$\det \boldsymbol{W}(\boldsymbol{X}) = \det \boldsymbol{W}(\mathbf{0}) \frac{\det \hat{\boldsymbol{F}}(\boldsymbol{X})}{\det \boldsymbol{F}(\boldsymbol{X})}. \tag{6-128}$$

Proof. From Eq. (6-99),

$$\boldsymbol{W}(\boldsymbol{X}) = \boldsymbol{D}\left\lfloor \mathbf{1}_n + \boldsymbol{D}^{-1}\boldsymbol{C}\boldsymbol{X}\left(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X}\right)^{-1}\boldsymbol{B}\right\rfloor, \tag{6-129}$$

which yields

$$\begin{aligned}\det \boldsymbol{W}(\boldsymbol{X}) &= [\det \boldsymbol{W}(\boldsymbol{0})] \det \left\lfloor \mathbf{1}_n + \boldsymbol{D}^{-1}\boldsymbol{C}\boldsymbol{X}\left(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X}\right)^{-1}\boldsymbol{B} \right\rfloor \\ &= [\det \boldsymbol{W}(\boldsymbol{0})] \det \left\lfloor \mathbf{1}_p + \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}\boldsymbol{X}\left(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X}\right)^{-1} \right\rfloor \\ &= \frac{\det \boldsymbol{W}(\boldsymbol{0}) \det \hat{\boldsymbol{F}}(\boldsymbol{X})}{\det \boldsymbol{F}(\boldsymbol{X})}. \end{aligned} \tag{6-130}$$

The second line follows directly from Eq. (6-127).

As indicated in Eq. (6-52), the input impedance, $Z(x)$, looking into a terminal pair can be conveniently expressed as

$$Z(x) = Z(0)\frac{F(\text{input short circuited})}{F(\text{input open circuited})} \tag{6-131}$$

A similar expression can be derived from Eq. (6-128) if $\boldsymbol{W}(\boldsymbol{X})$ denotes the impedance matrix of the n-port network in Fig. 6.29. In this case, $\boldsymbol{F}(\boldsymbol{X})$ is the return difference matrix with respect to $\boldsymbol{X}$ for the situation when the n ports at which the impedance matrix is defined are left open circuited without any applied sources. Hence, $\boldsymbol{F}(\boldsymbol{X}) = \boldsymbol{F}$(input open circuited). Likewise, $\hat{\boldsymbol{F}}(\boldsymbol{X})$ is the return difference matrix with respect to $\boldsymbol{X}$ for the input port current vector $\boldsymbol{I}_s$ and the output port voltage vector $\boldsymbol{V}$ under the condition that $\boldsymbol{I}_s$ is adjusted so that the port voltage vector $\boldsymbol{V}$ is identically zero. In other words, $\hat{\boldsymbol{F}}(\boldsymbol{X})$ is the return difference matrix for the circumstance in which the n ports where the impedance matrix is defined are short circuited. Consequently, $\hat{\boldsymbol{F}}(\boldsymbol{X}) = \boldsymbol{F}$(input short circuited). Resultantly, the determinant of the impedance matrix, $\boldsymbol{Z}(\boldsymbol{X})$, of an n-port network can be expressed as, recalling Eq. (6-128),

$$\det[\boldsymbol{Z}(\boldsymbol{X})] = \det[\boldsymbol{Z}(\boldsymbol{0})]\frac{\det\left[\boldsymbol{F}(\text{input short circuited})\right]}{\det\left[\boldsymbol{F}(\text{input open circuited})\right]} \tag{6-132}$$

As an illustration, refer again to the feedback amplifier of Fig. 6.9 and its linearized model of Fig. 6.32. As computed in Eq. (6-114), the return difference matrix with respect to the two controlling parameters is given by

$$\boldsymbol{F}(\boldsymbol{X}) = \mathbf{1}_2 + \boldsymbol{T}(\boldsymbol{X}) = \begin{bmatrix} 5.131 & -2.065 \\ 42.884 & 1 \end{bmatrix}, \tag{6-133}$$

the determinant of which is found to be

$$\det[\boldsymbol{F}(\boldsymbol{X})] = 93.68646. \tag{6-134}$$

If V_{25} of Fig. 6.32 is chosen as the output and V_s as the input, the null return difference matrix is, from Eq. (6-123),

$$\hat{\boldsymbol{F}}(\boldsymbol{X}) = \mathbf{1}_2 - \hat{\boldsymbol{A}}\boldsymbol{X} = \begin{bmatrix} 51.055 & -2402.29 \\ 42.884 & 1 \end{bmatrix} \tag{6-135}$$

the determinant of which is found to be

$$\det\left[\hat{\boldsymbol{F}}(\boldsymbol{X})\right] = 103{,}071. \tag{6-136}$$

From Eq. (6-128), the feedback amplifier voltage gain, V_{25}/V_s, is seen to be

$$w(\boldsymbol{X}) = \frac{V_{25}}{V_s} = w(\mathbf{0})\frac{\det\left[\hat{\boldsymbol{F}}(\boldsymbol{X})\right]}{\det\left[\boldsymbol{F}(\boldsymbol{X})\right]} = 0.04126\,\frac{103{,}071}{93.68646} = 45.39, \tag{6-137}$$

where $w(\mathbf{0}) = 0.04126$, as given in Eq. (6-121).

Suppose, instead, that the input current, I_{51}, is chosen as the output and V_s as the input. Using Eq. (6-125), the null return difference matrix becomes

$$\hat{\boldsymbol{F}}(\boldsymbol{X}) = \mathbf{1}_2 - \hat{\boldsymbol{A}}\boldsymbol{X} = \begin{bmatrix} 1.13426 & -0.06713 \\ 42.8841 & 1 \end{bmatrix}, \tag{6-138}$$

for which the determinant is

$$\det\left[\hat{\boldsymbol{F}}(\boldsymbol{X})\right] = 4.01307. \tag{6-139}$$

By applying Eq. (6-128), the amplifier input admittance is

$$w(\boldsymbol{X}) = \frac{I_{51}}{V_s} = w(\mathbf{0})\frac{\det\left[\hat{\boldsymbol{F}}(\boldsymbol{X})\right]}{\det\left[\boldsymbol{F}(\boldsymbol{X})\right]} = 8.62 \times 10^{-4}\,\frac{4.01307}{93.68646} = 36.92\,\mu\text{mho} \tag{6-140}$$

or 27.2 KΩ, confirming Eq. (6-107), where $w(\mathbf{0}) = 862\,\mu$mho is found from Eq. (6-124).

Another application of the generalized Blackman's formula is its use as a basis of a procedure for the indirect measurement of return difference. Refer to the general feedback network of Fig. 6.20. Suppose the return difference, $F(y_{21})$, with respect to the forward short circuit transfer admittance, y_{21}, of a two-port device is to be measured. Choose the two controlling parameters, y_{21} and y_{12}, to be the elements of interest. From Fig. 6.21,

$$\boldsymbol{\Theta} = \begin{bmatrix} I_a \\ I_b \end{bmatrix} = \begin{bmatrix} y_{21} & 0 \\ 0 & y_{12} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \boldsymbol{X\Phi}, \tag{6-141}$$

where I_a and I_b are the currents of the voltage controlled current sources. By appealing to Eq. (6-132), the impedance looking into terminals a and b of Fig. 6.20 can be written as

$$z_{aa,bb}(y_{12}, y_{21}) = z_{aa,bb}(0,0)\,\frac{\det\left[\boldsymbol{F}(\text{input short circuited})\right]}{\det\left[\boldsymbol{F}(\text{input open circuited})\right]}. \tag{6-142}$$

When input terminals a and b are open circuited, the resulting return difference matrix is exactly the same as that found under normal operating conditions, and accordingly,

$$\boldsymbol{F}(\text{input open circuited}) = \boldsymbol{F}(\boldsymbol{X}) = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}. \tag{6-143}$$

Since

$$\boldsymbol{F}(\boldsymbol{X}) = \mathbf{1}_2 - \boldsymbol{AX}, \tag{6-144}$$

the elements, F_{11} and F_{21}, are calculated with $y_{12} = 0$, whereas F_{12} and F_{22} are evaluated with $y_{21} = 0$. When input terminals a and b are short circuited, the feedback loop is interrupted and only the second row and first column element of matrix $\boldsymbol{A}$ is nonzero, thereby resulting in

$$\det\left[\boldsymbol{F}(\text{input short circuited})\right] = 1. \tag{6-145}$$

Since $\boldsymbol{X}$ is diagonal, the return difference function, $F(y_{21})$, can be expressed in terms of $\det[\boldsymbol{F}(\boldsymbol{X})]$ and the cofactor of the first row and first column element of $\boldsymbol{F}(\boldsymbol{X})$ as

$$F(y_{21}) = \frac{\det\left[\boldsymbol{F}(\boldsymbol{X})\right]}{F_{22}}. \tag{6-146}$$

Substituting these results into Eq. (6-142) generates the result,

$$F(y_{12})|_{y_{21}=0}\,F(y_{21}) = \frac{z_{aa,bb}(0,0)}{z_{aa,bb}(y_{12}, y_{21})}, \tag{6-147}$$

where

$$F_{22} = 1 - a_{22}y_{12}|_{y_{21}=0} = F(y_{12})|_{y_{21}=0}, \tag{6-148}$$

and a_{22} is the second row and second column element of $\boldsymbol{A}$.

6.5.5. Sensitivity Matrix

The sensitivity with respect to a perturbation in a particular element of a transfer function has been studied for a single loop feedback network. In multiple loop feedback networks, interest generally focuses on the sensitivity of a transfer function with respect to the variation of a set of network elements. This set may include either elements that are inherently sensitive to variation or elements whose effect on the overall amplifier performance is of paramount concern to circuit designers. To this end, a sensitivity matrix is introduced, and formulas for computing multiparameter sensitivity functions for a multiple loop feedback amplifier are developed.

Figure 6.35 is the block diagram of a multivariable open loop control network having n inputs and m outputs, whereas Fig. 6.36 shows the corresponding general feedback structure. If all feedback signals derive from the output variable and if the controllers are linear, there is no loss of generality by assuming the controller to be of the form shown in Fig. 6.37.

Denote the set of Laplace transformed input signals by the n-vector $\boldsymbol{u}$, the set of inputs to the network $\boldsymbol{X}$ in the open loop configuration of Fig. 6.35 by the p-vector $\boldsymbol{\Phi}_o$, and the set of outputs of network $\boldsymbol{X}$ in Fig. 6.35 by the m-vector $\boldsymbol{y}_o$. Let the corresponding signals for the closed loop configuration of Fig. 6.37 be denoted by the n-vector, $\boldsymbol{u}$, the p-vector, $\boldsymbol{\Phi}_c$, and the

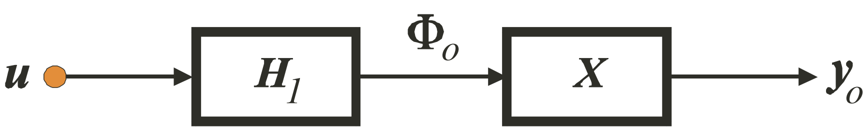

Figure 6.35. The block diagram of a multivariable open loop system.

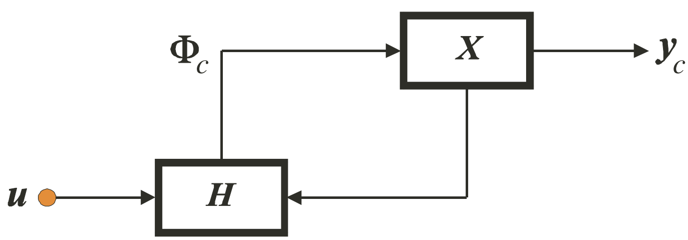

Figure 6.36. General feedback structure.

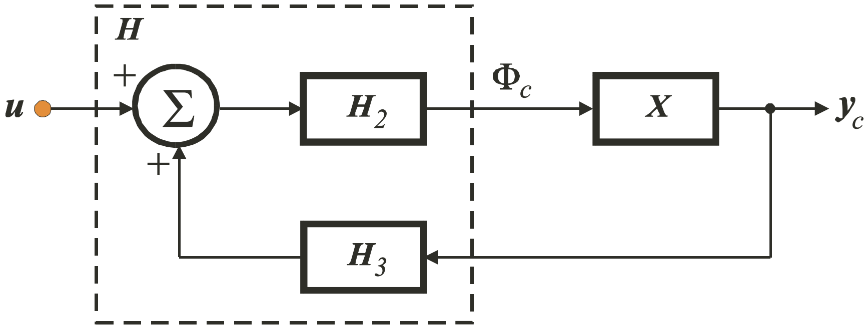

Figure 6.37. General feedback configuration.

m-vector, $\boldsymbol{y}_c$. Figures 6.35 and 6.37 produce the linear relationships,

$$\left.\begin{array}{l} \boldsymbol{y}_o = \boldsymbol{X}\boldsymbol{\Phi}_o \\ \boldsymbol{\Phi}_o = \boldsymbol{H}_1\boldsymbol{u} \\ \boldsymbol{y}_c = \boldsymbol{X}\boldsymbol{\Phi}_c \\ \boldsymbol{\Phi}_c = \boldsymbol{H}_2\left(\boldsymbol{u} + \boldsymbol{H}_3\boldsymbol{y}_c\right) \end{array}\right\rangle, \tag{6-149}$$

where the transfer function matrices, $\boldsymbol{X}, \boldsymbol{H}_1, \boldsymbol{H}_2$, and $\boldsymbol{H}_3$ are of orders $m \times p$, $p \times n$, $p \times n$ and $n \times m$, respectively. Combining Eqs. (6-149) and (6-110) results in

$$(\mathbf{1}_m - \boldsymbol{X}\boldsymbol{H}_2\boldsymbol{H}_3)\,\boldsymbol{y}_c = \boldsymbol{X}\boldsymbol{H}_2\boldsymbol{u} \tag{6-150}$$

from which

$$\boldsymbol{y}_c = (\mathbf{1}_m - \boldsymbol{X}\boldsymbol{H}_2\boldsymbol{H}_3)^{-1}\,\boldsymbol{X}\boldsymbol{H}_2\boldsymbol{u}. \tag{6-151}$$

Since the closed loop transfer function matrix, $\boldsymbol{W}(\boldsymbol{X})$, relating input vector $\boldsymbol{u}$ to output vector $\boldsymbol{y}_c$ is implicitly defined by

$$\boldsymbol{y}_c = \boldsymbol{W}(\boldsymbol{X})\boldsymbol{u}, \tag{6-152}$$

$$\boldsymbol{W}(\boldsymbol{X}) = (\mathbf{1}_m - \boldsymbol{X}\boldsymbol{H}_2\boldsymbol{H}_3)^{-1}\,\boldsymbol{X}\boldsymbol{H}_2. \tag{6-153}$$

If parameter matrix $\boldsymbol{X}$ is perturbed from $\boldsymbol{X}$ to $\boldsymbol{X} + \delta\boldsymbol{X}$, the outputs of the open loop and closed loop systems of Figs. 6.35 and 6.37 no longer assume their original nominal values. Distinguishing the new from the old variables by the superscript, $+$,

$$\left.\begin{array}{l} \boldsymbol{y}_o^+ = \boldsymbol{X}^+\boldsymbol{\Phi}_o \\ \boldsymbol{y}_c^+ = \boldsymbol{X}^+\boldsymbol{\Phi}_c^+ \\ \boldsymbol{\Phi}_c^+ = \boldsymbol{H}_2\left(\mathbf{u} + \boldsymbol{H}_3\boldsymbol{y}_c^+\right) \end{array}\right\rangle, \tag{6-154}$$

where $\boldsymbol{\Phi}_o$ retains its original value.

It is now necessary to compare the relative effects of the variations of $\boldsymbol{X}$ on the performance of the open loop and the closed loop systems. For a meaningful comparison, assume that $\boldsymbol{H}_1$, $\boldsymbol{H}_2$, and $\boldsymbol{H}_3$ are such that when there is no variation of $\boldsymbol{X}$, $\boldsymbol{y}_o = \boldsymbol{y}_c$. Define the error vectors resulting from the perturbation in $\boldsymbol{X}$ as

$$\left.\begin{aligned} \boldsymbol{E}_o &= \boldsymbol{y}_o - \boldsymbol{y}_o^+ \\ \boldsymbol{E}_c &= \boldsymbol{y}_c - \boldsymbol{y}_c^+ \end{aligned}\right\rangle. \tag{6-155}$$

The square matrix relating vector $\boldsymbol{E}_o$ to vector $\boldsymbol{E}_c$ is called the *sensitivity matrix*, $\mathcal{S}(\boldsymbol{X})$, for the transfer function matrix, $\boldsymbol{W}(\boldsymbol{X})$, with respect to the variations of $\boldsymbol{X}$; that is,

$$\boldsymbol{E}_c = \mathsf{S}(\boldsymbol{X})\boldsymbol{E}_o. \tag{6-156}$$

The subject sensitivity matrix, $\mathsf{S}(\boldsymbol{X})$, is expressible in terms of the system matrices, $\boldsymbol{X}$, $\boldsymbol{H}_2$, and $\boldsymbol{H}_3$.

The input and output relationship for the perturbed system can be written as

$$\boldsymbol{y}_c^+ = \left(\mathbf{1}_m - \boldsymbol{X}^+\boldsymbol{H}_2\boldsymbol{H}_3\right)^{-1}\boldsymbol{X}^+\boldsymbol{H}_2\boldsymbol{u}. \tag{6-157}$$

Substituting Eq. (6-151) into Eq. (6-155) gives

$$\begin{aligned} \boldsymbol{E}_c = \boldsymbol{y}_c - \boldsymbol{y}_c^+ &= \big[(\mathbf{1}_m - \boldsymbol{X}\boldsymbol{H}_2\boldsymbol{H}_3)^{-1}\boldsymbol{X}\boldsymbol{H}_2 \\ &\quad - \left(\mathbf{1}_m - \boldsymbol{X}^+\boldsymbol{H}_2\boldsymbol{H}_3\right)^{-1}\boldsymbol{X}^+\boldsymbol{H}_2\big]\boldsymbol{u} \\ &= \left(\mathbf{1}_m - \boldsymbol{X}^+\boldsymbol{H}_2\boldsymbol{H}_3\right)^{-1} \\ &\quad \times \big\{[\mathbf{1}_m - (\boldsymbol{X} + \delta\boldsymbol{X})\,\boldsymbol{H}_2\boldsymbol{H}_3]\,(\mathbf{1}_m - \boldsymbol{X}\boldsymbol{H}_2\boldsymbol{H}_3)^{-1}\boldsymbol{X}\boldsymbol{H}_2 \\ &\quad - (\boldsymbol{X} + \delta\boldsymbol{X})\,\boldsymbol{H}_2\big\}\,\boldsymbol{u} \\ &= \left(\mathbf{1}_m - \boldsymbol{X}^+\boldsymbol{H}_2\boldsymbol{H}_3\right)^{-1} \\ &\quad \times \big[\boldsymbol{X}\boldsymbol{H}_2 - \delta\boldsymbol{X}\boldsymbol{H}_2\boldsymbol{H}_3\,(\mathbf{1}_m - \boldsymbol{X}\boldsymbol{H}_2\boldsymbol{H}_3)^{-1}\boldsymbol{X}\boldsymbol{H}_2 - \boldsymbol{X}\boldsymbol{H}_2 - \delta\boldsymbol{X}\boldsymbol{H}_2\big]\boldsymbol{u} \\ &= -\left(\mathbf{1}_m - \boldsymbol{X}^+\boldsymbol{H}_2\boldsymbol{H}_3\right)^{-1}\delta\boldsymbol{X}\boldsymbol{H}_2[\mathbf{1}_n + \boldsymbol{H}_3\boldsymbol{W}(\boldsymbol{X})]\boldsymbol{u}. \end{aligned} \tag{6-158}$$

From Eqs. (6-149) and (6-152),

$$\boldsymbol{\Phi}_c = \boldsymbol{H}_2[\mathbf{1}_n + \boldsymbol{H}_3\boldsymbol{W}(\boldsymbol{X})]\boldsymbol{u}. \tag{6-159}$$

Since $\boldsymbol{y}_o = \boldsymbol{y}_c$,

$$\boldsymbol{\Phi}_o = \boldsymbol{\Phi}_c = \boldsymbol{H}_2[\mathbf{1}_n + \boldsymbol{H}_3\boldsymbol{W}(\boldsymbol{X})]\boldsymbol{u}, \tag{6-160}$$

which yields

$$\begin{aligned} \boldsymbol{E}_o = \boldsymbol{y}_o - \boldsymbol{y}_o^+ &= \left(\boldsymbol{X} - \boldsymbol{X}^+\right)\boldsymbol{\Phi}_o = -\delta\boldsymbol{X}\boldsymbol{\Phi}_o \\ &= -\delta\boldsymbol{X}\boldsymbol{H}_2[\boldsymbol{1}_n + \boldsymbol{H}_3\boldsymbol{W}(\boldsymbol{X})]\boldsymbol{u}. \end{aligned} \tag{6-161}$$

Combining Eqs. (6-158) and (6-161) leads to an expression that interrelates the error vectors, $\boldsymbol{E}_c$ and $\boldsymbol{E}_o$, of the closed loop and open loop systems in accordance with

$$\boldsymbol{E}_c = \left(\boldsymbol{1}_m - \boldsymbol{X}^+\boldsymbol{H}_2\boldsymbol{H}_3\right)^{-1}\boldsymbol{E}_o, \tag{6-162}$$

thereby giving rise to the sensitivity matrix,

$$\mathsf{S}(\boldsymbol{X}) = \left(\boldsymbol{1}_m - \boldsymbol{X}^+\boldsymbol{H}_2\boldsymbol{H}_3\right)^{-1}. \tag{6-163}$$

For a small variation in $\boldsymbol{X}$, $\boldsymbol{X}^+$ is approximately equal to $\boldsymbol{X}$. It follows in Fig. 6.37 that if the matrix triple product, $\boldsymbol{X}\boldsymbol{H}_2\boldsymbol{H}_3$, is regarded as the *loop transmission matrix* and $-\boldsymbol{X}\boldsymbol{H}_2\boldsymbol{H}_3$ is interpreted as the *return ratio matrix*, the difference between the unit matrix and the loop transmission matrix,

$$\boldsymbol{1}_m - \boldsymbol{X}^+\boldsymbol{H}_2\boldsymbol{H}_3, \tag{6-164}$$

can be defined as the *return difference matrix*. Therefore, Eq. (6-163) is a direct extension of the sensitivity function defined for a single input, single output system to which focus is given to only a single parameter. This result is synergistic with Eq. (6-28), where it was demonstrated that, using the ideal feedback model, the sensitivity function of the closed loop transfer function with respect to the forward amplifier gain is equal to the reciprocal of its return difference with respect to the same parameter.

In particular, when $\boldsymbol{W}(\boldsymbol{X})$, $\delta\boldsymbol{X}$, and $\boldsymbol{X}$ are square and nonsingular, from Eqs. (6-149), (6-152), (6-155) combine to deliver

$$\left.\begin{aligned} \boldsymbol{E}_c = \boldsymbol{y}_c - \boldsymbol{y}_c^+ &= \left[\boldsymbol{W}(\boldsymbol{X}) - \boldsymbol{W}^+(\boldsymbol{X})\right]\boldsymbol{u} = -\delta\boldsymbol{W}(\boldsymbol{X})\boldsymbol{u} \\ \boldsymbol{E}_o = \boldsymbol{y}_o - \boldsymbol{y}_o^+ &= \left(\boldsymbol{X}H_1 - \boldsymbol{X}^+\boldsymbol{H}_1\right)\boldsymbol{u} = -\delta\boldsymbol{X}\boldsymbol{H}_1\boldsymbol{u} \end{aligned}\right\rangle. \tag{6-165}$$

If $\boldsymbol{H}_1$ is nonsingular, $\boldsymbol{u}$ in the expressions in Eq. (6-165) combine to give

$$\boldsymbol{E}_c = \delta\boldsymbol{W}(\boldsymbol{X})\boldsymbol{H}_1^{-1}(\delta\boldsymbol{X})^{-1}\boldsymbol{E}_o. \tag{6-166}$$

Since $\boldsymbol{y}_o = \boldsymbol{y}_c$,

$$\boldsymbol{X}\boldsymbol{H}_1 = \boldsymbol{W}(\boldsymbol{X}). \tag{6-167}$$

From Eq. (6-166),

$$\boldsymbol{E}_c = \delta\boldsymbol{W}(\boldsymbol{X})\boldsymbol{W}^{-1}(\boldsymbol{X})\boldsymbol{X}(\delta\boldsymbol{X})^{-1}\boldsymbol{E}_o, \tag{6-168}$$

whence

$$\mathsf{S}(\boldsymbol{X}) = \delta\boldsymbol{W}(\boldsymbol{X})\boldsymbol{W}^{-1}(\boldsymbol{X})\boldsymbol{X}(\delta\boldsymbol{X})^{-1}. \tag{6-169}$$

This result compares favorably with the scalar sensitivity function in Eq. (6-21), which can be put in the form

$$\mathsf{S}(x) = (\delta w)w^{-1}x(\delta x)^{-1}. \tag{6-170}$$

6.5.6. Multi-Parameter Sensitivity

In this section, formulas are derived to relate the effect of a change in $\boldsymbol{X}$ on a scalar transfer function $w(\boldsymbol{X})$. Let x_k, $k = 1, 2, \ldots, pq$, be the elements of $\boldsymbol{X}$. The multivariable Taylor series expansion of $w(\boldsymbol{X})$ with respect to x_k is

$$\delta w = \sum_{k=1}^{pq} \frac{\partial w}{\partial x_k}\delta x_k + \sum_{j=1}^{pq}\sum_{k=1}^{pq} \frac{\partial^2 w}{\partial x_j \partial x_k}\frac{\delta x_j \delta x_k}{2!} + \cdots. \tag{6-171}$$

The first order perturbation can then be written as

$$\delta w \approx \sum_{k=1}^{pq} \frac{\partial w}{\partial x_k}\delta x_k. \tag{6-172}$$

From Eq. (6-21),

$$\frac{\delta w}{w} \approx \sum_{k=1}^{pq} \mathsf{S}(x_k)\frac{\delta x_k}{x_k}. \tag{6-173}$$

This expression gives the fractional change of the transfer function, w, in terms of the scalar sensitivity functions $\mathsf{S}(x_k)$.

Refer once again to the fundamental matrix feedback flow graph of Fig. 6.31. If the amplifier has a single input and a single output, Eq. (6-129) gives the overall transfer function, $w(\boldsymbol{X})$, of the multiple loop feedback as

$$w(\boldsymbol{X}) = \boldsymbol{D} + \boldsymbol{C}\boldsymbol{X}\left(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X}\right)^{-1}\boldsymbol{B}. \tag{6-174}$$

When $\boldsymbol{X}$ is perturbed to $\boldsymbol{X}^+ = \boldsymbol{X} + \delta\boldsymbol{X}$, the corresponding expression to Eq. (6-174) is

$$w(\boldsymbol{X}) + \delta w(\boldsymbol{X}) = D + \boldsymbol{C}(\boldsymbol{X} + \delta\boldsymbol{X})\left(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X} - \boldsymbol{A}\delta\boldsymbol{X}\right)^{-1}\boldsymbol{B}, \tag{6-175}$$

or

$$\delta w(\boldsymbol{X}) = \boldsymbol{C}\lfloor(\boldsymbol{X} + \delta\boldsymbol{X})(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X} - \boldsymbol{A}\delta\boldsymbol{X})^{-1} - \boldsymbol{X}(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X})^{-1}\rfloor\boldsymbol{B}. \tag{6-176}$$

As $\delta \boldsymbol{X}$ approaches zero,

$$\begin{aligned}
\delta w(\boldsymbol{X}) &= \boldsymbol{C}\lfloor(\boldsymbol{X}+\delta\boldsymbol{X}) - \boldsymbol{X}\left(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X}\right)^{-1}\left(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X} - \boldsymbol{A}\delta\boldsymbol{X}\right)\rfloor \\
&\qquad \times \left(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X} - \boldsymbol{A}\delta\boldsymbol{X}\right)^{-1}\boldsymbol{B} \\
&= \boldsymbol{C}\lfloor \delta\boldsymbol{X} + \boldsymbol{X}\left(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X}\right)^{-1}\boldsymbol{A}\delta\boldsymbol{X}\rfloor \left(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X} - \boldsymbol{A}\delta\boldsymbol{X}\right)^{-1}\boldsymbol{B} \\
&= \boldsymbol{C}\left(\mathbf{1}_q - \boldsymbol{X}\boldsymbol{A}\right)^{-1}(\delta\boldsymbol{X})\left(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X} - \boldsymbol{A}\delta\boldsymbol{X}\right)^{-1}\boldsymbol{B} \\
&\approx \boldsymbol{C}\left(\mathbf{1}_q - \boldsymbol{X}\boldsymbol{A}\right)^{-1}(\delta\boldsymbol{X})\left(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X}\right)^{-1}\boldsymbol{B},
\end{aligned} \tag{6-177}$$

where $\boldsymbol{C}$ is a row q-vector and $\boldsymbol{B}$ is a column p-vector. Write

$$\left.\begin{aligned}
\boldsymbol{C} &= \left[\begin{matrix} c_1 & c_2 & \cdots & c_q \end{matrix}\right] \\
\boldsymbol{B}' &= \left[\begin{matrix} b_1 & b_2 & \cdots & b_p \end{matrix}\right] \\
\tilde{\boldsymbol{W}} &= \boldsymbol{X}\left(\mathbf{1}_p - \boldsymbol{A}\boldsymbol{X}\right)^{-1} = \left(\mathbf{1}_q - \boldsymbol{X}\boldsymbol{A}\right)^{-1}\boldsymbol{X} = \left[\tilde{w}_{ij}\right]
\end{aligned}\right\rangle. \tag{6-178}$$

The increment, $\delta w(\boldsymbol{X})$, can be expressed in terms of the elements of Eq. (6-178) and those of $\boldsymbol{X}$. In the case where $\boldsymbol{X}$ is diagonal with

$$\boldsymbol{X} = \operatorname{diag}\lfloor \begin{matrix} x_1 & x_2 & \cdots & x_p \end{matrix} \rfloor, \tag{6-179}$$

where $p = q$, the expression for $\delta w(\boldsymbol{X})$ can be compactly written as

$$\begin{aligned}
\delta w(\mathbf{X}) &= \sum_{i=1}^{p}\sum_{k=1}^{p}\sum_{j=1}^{p} c_i\left(\frac{w_{ik}}{x_k}\right)(\delta x_k)\left(\frac{w_{kj}}{x_k}\right) b_j \\
&= \sum_{i=1}^{p}\sum_{k=1}^{p}\sum_{j=1}^{p} \frac{c_i\tilde{w}_{ik}\tilde{w}_{kj}b_j}{x_k}\frac{\delta x_k}{x_k}
\end{aligned} \tag{6-180}$$

Comparing this equation with Eq. (6-173), obtain an explicit form for the single parameter sensitivity function becomes

$$\mathsf{S}(x_k) = \sum_{i=1}^{p}\sum_{j=1}^{p}\frac{c_i\tilde{w}_{ik}\tilde{w}_{kj}b_j}{x_k w(\boldsymbol{X})}. \tag{6-181}$$

Thus, knowing Eqs. (6-178) and (6-179), the multiparameter sensitivity function for the scalar transfer function, $\boldsymbol{w}(\boldsymbol{X})$, follows immediately.

As an example, consider again the voltage-series feedback amplifier of Fig. 6.9 and its small signal model in Fig. 6.32. Assume that V_s is the input and V_{25} is the output response. The transfer function of interest is the amplifier voltage gain V_{25}/V_s. The elements of concern are the two controlling parameters of the controlled sources. Thus, let

$$\boldsymbol{X} = \begin{bmatrix} \tilde{\alpha}_1 & 0 \\ 0 & \tilde{\alpha}_2 \end{bmatrix} = \begin{bmatrix} 0.0455 & 0 \\ 0 & 0.0455 \end{bmatrix}. \tag{6-182}$$

From Eq. (6-121),

$$\left.\begin{aligned} \boldsymbol{A} &= \begin{bmatrix} -90.782 & 45.391 \\ -942.507 & 0 \end{bmatrix} \\ \boldsymbol{B}' &= \begin{bmatrix} 0.91748 & 0 \end{bmatrix} \\ \boldsymbol{C} &= \begin{bmatrix} 45.391 & -2372.32 \end{bmatrix} \end{aligned}\right\rangle, \tag{6-183}$$

which produces

$$\tilde{\boldsymbol{W}} = \boldsymbol{X}(\mathbf{1}_2 - \boldsymbol{A}\boldsymbol{X})^{-1} = 10^{-4}\begin{bmatrix} 4.85600 & 10.02904 \\ -208.245 & 24.91407 \end{bmatrix}. \tag{6-184}$$

Also, from Eq. (6-107),

$$w(\boldsymbol{X}) = \frac{V_{25}}{V_s} = 45.387. \tag{6-185}$$

To compute the sensitivity functions with respect to $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, apply Eq. (6-181) to obtain

$$\left.\begin{aligned} \mathsf{S}(\tilde{\alpha}_1) &= \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{c_i \tilde{w}_{i1}\tilde{w}_{1j} b_j}{\alpha_1 w(\boldsymbol{X})} \\ &= \frac{c_1\tilde{w}_{11}\tilde{w}_{11}b_1 + c_1\tilde{w}_{11}\tilde{w}_{12}b_2 + c_2\tilde{w}_{21}\tilde{w}_{11}b_1 + c_2\tilde{w}_{21}\tilde{w}_{12}b_2}{\tilde{\alpha}_1 w} \\ &= 0.01066 \\ \mathsf{S}(\tilde{\alpha}_2) &= \frac{c_1\tilde{w}_{12}\tilde{w}_{21}b_1 + c_1\tilde{w}_{12}\tilde{w}_{22}b_2 + c_2\tilde{w}_{22}\tilde{w}_{21}b_1 + c_2\tilde{w}_{22}\tilde{w}_{22}b_2}{\tilde{\alpha}_2 w} \\ &= 0.05426 \end{aligned}\right\rangle. \tag{6-186}$$

As a check, Eq. (6-25) can be exploited to compute the foregoing sensitivities. From Eqs. (6-40) and (6-47),

$$\left.\begin{aligned} F(\tilde{\alpha}_1) &= 93.70 \\ F(\tilde{\alpha}_2) &= 18.26 \\ \hat{F}(\tilde{\alpha}_1) &= 103.07 \times 10^3 \\ \hat{F}(\tilde{\alpha}_2) &= 2018.70 \end{aligned}\right\rangle. \tag{6-187}$$

Substituting these results into Eq. (6-25), the pertinent sensitivities are

$$\left.\begin{aligned} \mathsf{S}(\tilde{\alpha}_1) &= \frac{1}{F(\tilde{\alpha}_1)} - \frac{1}{\hat{F}(\tilde{\alpha}_1)} = 0.01066 \\ \mathsf{S}(\tilde{\alpha}_2) &= \frac{1}{F(\tilde{\alpha}_2)} - \frac{1}{\hat{F}(\tilde{\alpha}_2)} = 0.05427 \end{aligned}\right\rangle, \tag{6-188}$$

which agree with Eq. (6-186).

Suppose that $\tilde{\alpha}_1$ is changed by 4% and $\tilde{\alpha}_2$ by 6%. The fractional change of the voltage gain, $w(\boldsymbol{X})$, is found from Eq. (6-173) to be

$$\frac{\delta w}{w} \approx \mathsf{S}(\tilde{\alpha}_1)\frac{\delta\tilde{\alpha}_1}{\tilde{\alpha}_1} + \mathsf{S}(\tilde{\alpha}_2)\frac{\delta\tilde{\alpha}_2}{\tilde{\alpha}_2} = 0.003683 \tag{6-189}$$

or 0.37% .

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