

# Contents

<i>Preface</i>	v
<i>List of Symbols</i>	ix
1. REVIEW ON LINEAR ALGEBRAS	1
1.1 Linear Space and Basis Vector . . . . .	1
1.2 Linear Transformations and Linear Operators . . . . .	3
1.3 Similarity Transformation . . . . .	5
1.4 Eigenvectors and Diagonalization of a Matrix . . . . .	8
1.5 Inner Product of Vectors . . . . .	9
1.6 The Direct Product of Matrices . . . . .	12
1.7 Exercises . . . . .	13
2. GROUP AND ITS SUBSETS	17
2.1 Symmetry . . . . .	17
2.2 Group and its Multiplication Table . . . . .	19
2.3 Subsets in a Group . . . . .	27
2.3.1 Subgroup . . . . .	27
2.3.2 Cosets . . . . .	28
2.3.3 Conjugate Elements and the Class . . . . .	29
2.3.4 Invariant Subgroup . . . . .	31
2.4 Homomorphism of Two Groups . . . . .	33
2.5 Proper Symmetric Group of a Regular Polyhedron . . . . .	35
2.5.1 Tetrahedron, Octahedron, and Cube . . . . .	36
2.5.2 The Group Table of $\mathbf{T}$ . . . . .	38
2.5.3 The Group Table of $\mathbf{O}$ . . . . .	40

2.5.4	Regular Icosahedron and the Group Table of $\mathbf{I}$ . . .	40
2.6	Direct Product of Groups and Improper Point Groups . . .	44
2.6.1	The Direct Product of Two Groups . . . . .	44
2.6.2	Improper Point Groups . . . . .	44
2.7	Exercises . . . . .	46
3.	THEORY OF LINEAR REPRESENTATIONS OF GROUPS	49
3.1	Linear Representations of a Group . . . . .	49
3.1.1	Definition of a Linear Representation . . . . .	49
3.1.2	Group Algebra and the Regular Representation . . . . .	50
3.1.3	Class Operator and Class Space . . . . .	53
3.2	Transformation Operators for a Scalar Function . . . . .	54
3.3	Equivalent Representations . . . . .	57
3.4	Inequivalent and Irreducible Representations . . . . .	60
3.4.1	Irreducible Representations . . . . .	60
3.4.2	Schur Theorem . . . . .	61
3.4.3	Orthogonal Relation . . . . .	63
3.4.4	Completeness of Representations . . . . .	65
3.4.5	Character Tables of Finite Groups . . . . .	67
3.4.6	The Character Table of the Group $\mathbf{T}$ . . . . .	70
3.4.7	The Character Table of the Group $\mathbf{O}$ . . . . .	71
3.4.8	Self-conjugate Representation . . . . .	72
3.5	Subduced and Induced Representations . . . . .	73
3.6	Applications in Physics . . . . .	78
3.6.1	Classification of Static Wave Functions . . . . .	78
3.6.2	Clebsch–Gordan Series and Coefficients . . . . .	80
3.6.3	Wigner–Eckart Theorem . . . . .	81
3.6.4	Normal Degeneracy and Accidental Degeneracy . . . . .	83
3.6.5	An Example of Application . . . . .	85
3.7	Irreducible Bases in Group Algebra . . . . .	88
3.7.1	Ideal and Idempotent . . . . .	89
3.7.2	Primitive Idempotent . . . . .	90
3.7.3	Two-side Ideal . . . . .	92
3.7.4	Standard Irreducible Basis Vectors . . . . .	93
3.8	Exercises . . . . .	102
4.	THREE-DIMENSIONAL ROTATION GROUP	107
4.1	Three-dimensional Rotations . . . . .	107

4.2	Fundamental Concept of a Lie Group . . . . .	111
4.2.1	The Composition Functions of a Lie Group . . . . .	111
4.2.2	The Local Property of a Lie Group . . . . .	112
4.2.3	Generators and Differential Operators . . . . .	113
4.2.4	The Adjoint Representation of a Lie Group . . . . .	114
4.2.5	The Global Property of a Lie Group . . . . .	115
4.3	The Covering Group of $SO(3)$ . . . . .	117
4.3.1	The Group $SU(2)$ . . . . .	117
4.3.2	Homomorphism of $SU(2)$ onto $SO(3)$ . . . . .	118
4.3.3	The Group Integral . . . . .	120
4.4	Irreducible Representations of $SU(2)$ . . . . .	124
4.4.1	Euler Angles . . . . .	124
4.4.2	Linear Representations of $SU(2)$ . . . . .	127
4.4.3	Spherical Harmonics Functions . . . . .	131
4.5	The Lie Theorems . . . . .	134
4.6	Clebsch–Gordan Coefficients of $SU(2)$ . . . . .	141
4.6.1	Direct Product of Representations . . . . .	141
4.6.2	Calculation of Clebsch–Gordan Coefficients . . . . .	144
4.6.3	Applications . . . . .	147
4.6.4	Sum of Three Angular Momentums . . . . .	149
4.7	Tensors and Spinors . . . . .	153
4.7.1	Vector Fields . . . . .	153
4.7.2	Tensor Fields . . . . .	155
4.7.3	Spinor Fields . . . . .	156
4.7.4	Total Angular Momentum Operator . . . . .	158
4.8	Irreducible Tensor Operators and Their Application . . . . .	160
4.8.1	Irreducible Tensor Operators . . . . .	160
4.8.2	Wigner–Eckart Theorem . . . . .	163
4.8.3	Selection Rule and Relative Intensity . . . . .	
	of Radiation . . . . .	164
4.8.4	Landé Factor and Zeeman Effects . . . . .	166
4.9	An Isolated Quantum $n$ -body System . . . . .	169
4.9.1	Separation of the Motion of Center-of-Mass . . . . .	169
4.9.2	Quantum Two-body System . . . . .	171
4.9.3	Quantum Three-body System . . . . .	172
4.9.4	Quantum $n$ -body System . . . . .	176
4.10	Exercises . . . . .	180

5.	SYMMETRY OF CRYSTALS	185
5.1	Symmetric Group of Crystals . . . . .	185
5.2	Crystallographic Point Groups . . . . .	187
5.2.1	Elements in a Crystallographic Point Group . . .	187
5.2.2	Proper Crystallographic Point Groups . . . . .	189
5.2.3	Improper Crystallographic Point Group . . . . .	193
5.3	Crystal Systems and Bravais Lattice . . . . .	195
5.3.1	Restrictions on Vectors of Crystal Lattice . . . . .	195
5.3.2	Triclinic Crystal System . . . . .	198
5.3.3	Monoclinic Crystal System . . . . .	198
5.3.4	Orthorhombic Crystal System . . . . .	199
5.3.5	Trigonal and Hexagonal Crystal System . . . . .	200
5.3.6	Tetragonal Crystal System . . . . .	204
5.3.7	Cubic Crystal System . . . . .	205
5.4	Space Group . . . . .	208
5.4.1	Symmetric Elements . . . . .	208
5.4.2	Symbols of a Space Group . . . . .	211
5.4.3	Method for Determining the Space Groups . . . . .	213
5.4.4	Example for the Space Groups in Type A . . . . .	215
5.4.5	Example for the Space Groups in Type B . . . . .	216
5.4.6	Analysis of the Symmetry of a Crystal . . . . .	218
5.5	Linear Representations of Space Groups . . . . .	220
5.5.1	Irreducible Representations of $\mathcal{T}$ . . . . .	220
5.5.2	Star of Wave Vectors and Group of Wave Vectors . . . . .	222
5.5.3	Representation Matrices of Elements in $\mathcal{S}$ . . . . .	224
5.5.4	Irreducible Representations of $\mathcal{S}(\mathbf{k}_1)$ . . . . .	225
5.5.5	The Bloch Theorem . . . . .	227
5.5.6	Energy Band in a Crystal . . . . .	228
5.6	Exercises . . . . .	229
6.	PERMUTATION GROUPS	231
6.1	Multiplication of Permutations . . . . .	231
6.1.1	Permutations . . . . .	231
6.1.2	Cycles . . . . .	233
6.1.3	Classes in a Permutation Group . . . . .	234
6.1.4	Alternating Subgroups . . . . .	236
6.1.5	Transposition of Two Neighbored Objects . . . . .	237

6.2	Young Patterns, Young Tableaux, and Young Operators . . . . .	237
6.2.1	Young Patterns . . . . .	237
6.2.2	Young Tableaux . . . . .	238
6.2.3	Young Operators . . . . .	240
6.2.4	Fundamental Property of Young Operators . . . . .	242
6.2.5	Products of Young Operators . . . . .	244
6.3	Irreducible Representations of $S_n$ . . . . .	246
6.3.1	Primitive Idempotents in the Group Algebra . . . . . of $S_n$ . . . . .	246
6.3.2	Orthogonal Primitive Idempotents of $S_n$ . . . . .	249
6.3.3	Calculation of Representation Matrices for $S_n$ . . . . .	253
6.3.4	Calculation of Characters by Graphic Method . . . . .	257
6.3.5	The Permutation Group $S_3$ . . . . .	259
6.3.6	Inner Product of Irreducible Representations . . . . . of $S_n$ . . . . .	261
6.4	Real Orthogonal Representation of $S_n$ . . . . .	262
6.5	Outer Product of Irreducible Representations of $S_n$ . . . . .	268
6.5.1	Representations of $S_{n+m}$ and Its Subgroup . . . . . $S_n \otimes S_m$ . . . . .	268
6.5.2	Littlewood–Richardson Rule . . . . .	271
6.6	Exercises . . . . .	273
7.	LIE GROUPS AND LIE ALGEBRAS . . . . .	277
7.1	Lie Algebras and its Structure Constants . . . . .	277
7.1.1	The Global Property of a Lie Group . . . . .	277
7.1.2	The Local Property of a Lie Group . . . . .	278
7.1.3	The Lie Algebra . . . . .	281
7.1.4	The Killing Form and the Cartan Criteria . . . . .	283
7.2	The Regular Form of a Semisimple Lie Algebra . . . . .	285
7.2.1	The Inner Product in a Semisimple Lie Algebra . . . . .	285
7.2.2	The Cartan Subalgebra . . . . .	286
7.2.3	Regular Commutative Relations of Generators . . . . .	287
7.2.4	The Inner Product of Roots . . . . .	289
7.2.5	Positive Roots and Simple Roots . . . . .	292
7.3	Classification of Simple Lie Algebras . . . . .	295
7.3.1	Angle between Two Simple Roots . . . . .	295
7.3.2	Dynkin Diagrams . . . . .	296
7.3.3	The Cartan Matrix . . . . .	302
7.4	Classical Simple Lie Algebras . . . . .	302

7.4.1	The $SU(N)$ Group and its Lie Algebra . . . . .	302
7.4.2	The $SO(N)$ Group and its Lie Algebra . . . . .	307
7.4.3	The $USp(2\ell)$ Group and its Lie Algebra . . . . .	309
7.5	Representations of a Simple Lie Algebra . . . . .	313
7.5.1	Representations and Weights . . . . .	313
7.5.2	Weight Chain and Weyl Reflections . . . . .	316
7.5.3	Mathematical Property of Representations . . . . .	319
7.5.4	Fundamental Dominant Weights . . . . .	320
7.5.5	The Casimir Operator of Order 2 . . . . .	321
7.6	Main Data of Simple Lie Algebras . . . . .	322
7.6.1	Lie Algebra $A_\ell$ and Lie Group $SU(\ell + 1)$ . . . . .	323
7.6.2	Lie Algebra $B_\ell$ and Lie Group $SO(2\ell + 1)$ . . . . .	324
7.6.3	Lie Algebra $C_\ell$ and Lie Group $USp(2\ell)$ . . . . .	325
7.6.4	Lie Algebra $D_\ell$ and Lie Group $SO(2\ell)$ . . . . .	325
7.6.5	Lie Algebra $G_2$ . . . . .	327
7.6.6	Lie Algebra $F_4$ . . . . .	327
7.6.7	Lie Algebra $E_6$ . . . . .	328
7.6.8	Lie Algebra $E_7$ . . . . .	329
7.6.9	Lie Algebra $E_8$ . . . . .	330
7.7	Block Weight Diagrams . . . . .	331
7.7.1	Chevalley Bases . . . . .	331
7.7.2	Orthonormal Basis States . . . . .	333
7.7.3	Method of Block Weight Diagram . . . . .	335
7.7.4	Some Representations of $A_2$ . . . . .	337
7.7.5	Some Representations of $C_3$ . . . . .	338
7.7.6	Planar Weight Diagrams . . . . .	342
7.8	Clebsch–Gordan Coefficients . . . . .	343
7.8.1	Representations in the CG Series . . . . .	344
7.8.2	Method of Dominant Weight Diagram . . . . .	345
7.8.3	Reductions of Direct Product Representations in $A_2$ . . . . .	347
7.9	Exercises . . . . .	350
8.	UNITARY GROUPS . . . . .	353
8.1	Irreducible Representations of $SU(N)$ . . . . .	353
8.1.1	Reduction of a Tensor Space . . . . .	354
8.1.2	Basis Tensors in the Tensor Subspace . . . . .	356
8.1.3	Chevalley Bases of Generators in $SU(N)$ . . . . .	362

8.1.4	Inequivalent and Irreducible Representations . . . . .	363
8.1.5	Dimensions of Representations of $SU(N)$ . . . . .	364
8.1.6	Subduced Representations with Respect . . . . . to Subgroups . . . . .	366
8.2	Orthonormal Irreducible Basis Tensors . . . . .	367
8.2.1	Orthonormal Basis Tensors in $\mathcal{T}_\mu^{[\lambda]}$ . . . . .	368
8.2.2	Orthonormal Basis Tensors in $S_n$ . . . . .	373
8.3	Direct Product of Tensor Representations . . . . .	373
8.3.1	Outer Product of Tensors . . . . .	373
8.3.2	Covariant and Contravariant Tensors . . . . .	377
8.3.3	Traceless Mixed Tensors . . . . .	379
8.3.4	Adjoint Representation of $SU(N)$ . . . . .	382
8.4	$SU(3)$ Symmetry and Wave Functions of Hadrons . . . . .	383
8.4.1	Quantum Numbers of Quarks . . . . .	384
8.4.2	Planar Weight Diagrams . . . . .	386
8.4.3	Mass Formulas . . . . .	391
8.4.4	Wave Functions of Mesons . . . . .	393
8.4.5	Wave Functions of Baryons . . . . .	395
8.5	Exercises . . . . .	397
9.	REAL ORTHOGONAL GROUPS . . . . .	399
9.1	Tensor Representations of $SO(N)$ . . . . .	399
9.1.1	Tensors of $SO(N)$ . . . . .	399
9.1.2	Irreducible Basis Tensors of $SO(2\ell + 1)$ . . . . .	403
9.1.3	Irreducible Basis Tensors of $SO(2\ell)$ . . . . .	408
9.1.4	Dimensions of Irreducible Tensor . . . . . Representations . . . . .	411
9.1.5	Adjoint Representation of $SO(N)$ . . . . .	414
9.1.6	Tensor Representations of $O(N)$ . . . . .	415
9.2	$\Gamma$ Matrix Groups . . . . .	416
9.2.1	Property of $\Gamma$ Matrix Groups . . . . .	417
9.2.2	The Case $N = 2\ell$ . . . . .	418
9.2.3	The Case $N = 2\ell + 1$ . . . . .	421
9.3	Spinor Representations of $SO(N)$ . . . . .	422
9.3.1	Covering Groups of $SO(N)$ . . . . .	422
9.3.2	Fundamental Spinors of $SO(N)$ . . . . .	425
9.3.3	Direct Products of Spinor Representations . . . . .	426
9.3.4	Spinor Representations of Higher Ranks . . . . .	427
9.3.5	Dimensions of the Spinor Representations . . . . .	430

9.4	Rotational Symmetry in $N$ -Dimensional Space . . . . .	432
9.4.1	Orbital Angular Momentum Operators . . . . .	432
9.4.2	Spherical Harmonic Functions . . . . .	432
9.4.3	Schrödinger Equation for a Two-body System . . . . .	434
9.4.4	Schrödinger Equation for a Three-body System . . . . .	435
9.4.5	Dirac Equation in $(N + 1)$ -dimensional . . . . .	
	Space–time . . . . .	438
9.5	The $SO(4)$ Group and the Lorentz Group . . . . .	444
9.5.1	Irreducible Representations of $SO(4)$ . . . . .	445
9.5.2	Single-valued Representations of $O(4)$ . . . . .	448
9.5.3	The Lorentz Group . . . . .	450
9.5.4	Irreducible Representations of $L_p$ . . . . .	451
9.5.5	The Covering Group of $L_p$ . . . . .	454
9.5.6	Classes of $L_p$ . . . . .	456
9.5.7	Irreducible Representations of $L_h$ . . . . .	457
9.6	Exercises . . . . .	459
10.	THE SYMPLECTIC GROUPS . . . . .	461
10.1	Irreducible Representations of $USp(2\ell)$ . . . . .	461
10.1.1	Decomposition of the Tensor Space of $USp(2\ell)$ . . . . .	461
10.1.2	Orthonormal Irreducible Basis Tensors . . . . .	463
10.1.3	Dimensions of Irreducible Representations . . . . .	468
10.2	Physical Application . . . . .	471
10.3	Exercises . . . . .	472
Appendix A	Identities on Combinatorics . . . . .	473
Appendix B	Covariant and Contravariant Tensors . . . . .	475
Appendix C	The Space Groups . . . . .	477
	<i>Bibliography</i> . . . . .	481
	<i>Index</i> . . . . .	487