I Absolute invariants for open manifolds and bundles

For closed manifolds, there is a highly elaborated theory of number valued invariants. Examples are the characteristic numbers like Stiefel–Whitney, Chern and Pontrjagin numbers, the Euler number, the dimension of rational (co–)homology and homotopy groups, the signature. Moreover, we have invariants coming from surgery theory etc. Taking into account a Riemannian metric, we obtain global spectral invariants like analytic torsion, the eta invariant. On general open manifolds, more or less all of this fails. Characteristic numbers are not defined, (co–)homology groups can have infinite rank etc.

We have the following simple

Proposition 0.1 *Let $\mathfrak{M}^n$ be the set of all smooth oriented manifolds and V a vector space or an abelian group. There does not exist a nontrivial map $c : \mathfrak{M} \longrightarrow V$ such that*

1) *$M^n \cong M'^n$ by an orientation preserving diffeomorphism implies $c(M) = c(M')$ and*

2) *$c(M \# M') = c(M) + c(M')$.*

Proof. Assume at first $M^n \ncong \Sigma^n$, fix two points at M^n, then $M_\infty = M_1 \# M_2 \# \ldots$, $M_i = (M, i) \cong M$ has a well defined meaning. We can write $M_\infty = M_1 \# M_{\infty,2}$ with $M_{\infty,2} = M_2 \# M_3 \# \ldots$ and get $c(M_\infty) = c(M) + c(M_{\infty,2}) = c(M) + c(M_\infty)$ hence $c(M) = 0$.

Assume $M^n = \Sigma^n$, and ord $\Sigma^n = k > 1$ which yields

$$c(\Sigma^n \# \ldots \# \Sigma^n) = k \cdot c(\Sigma^n) = c(S^n), \quad c(\Sigma^n) = \frac{1}{k} c(S^n),$$

$$c(\Sigma^n) = c(\Sigma^n \# S^n) = \left(1 + \frac{1}{k}\right) c(S^n),$$

$$c(S^n) = 2c(S^n), \quad c(S^n) = 0, \quad (\Sigma^n) = 0.$$

□

The only real invariant, defined for all connected manifolds M^n known to the author is the dimension n. If one characterizes orientability / nonorientability by ± 1, then there are two such invariants. That is all. Looking at the classification theory, we see a deep distinction between the case of closed or open manifolds, respectively.

Denote by $\mathfrak{M}^n([cl])$ the set of all diffeomorphism classes of closed n–manifolds. Then we have

Proposition 0.2 $\#\mathfrak{M}^n([cl]) = \aleph_0$.

Proof. According to Cheeger, there are only finitely many diffeomorphism types for (M^n, g) with diam $(M^n, g) \leq D$, $r_{\text{inj}}(M^n, g) \leq i$, and bounded sectional curvature with bound K. Setting $D_\nu = K_\nu = i_\nu = \nu$ and considering $\nu \to \infty$, we count all diffeomorphism types of closed Riemannian n–manifolds, in particular all diffeomorphism types of closed manifolds. □

On the other hand, for open manifolds there holds

Proposition 0.3 *The cardinality of* $\mathfrak{M}([open])$ *is at least that of the continuum for* $n \geq 2$.

Proof. Assume $n \geq 3$, n odd, let $2 = p_1 < p_2 < \dots$ be the increasing sequence of prime numbers and let $L^n(p_\nu) = S^n/\mathbf{Z}_{p_\nu}$ be the corresponding lens space. Consider $M^n := d_1 \cdot L(p_1) \# d_2 \cdot L(p_2) \# \dots,\ d_\nu = 0, 1$. Then any $0,1$–sequence $(d_1, d_2, \dots)$ defines a manifold and different sequences define non diffeomorphic manifolds. If $n \geq 4$ is even multiply by S^1. For $n = 2$ the assertion follows from the classification theorem in [59]. □

There are simple methods to construct from only one closed manifold $M^n \neq \Sigma^n$ infinitely many nondiffeomorphic manifolds. This, proposition 0.3 and other considerations support the naive imagination, that "measure of $\mathfrak{M}([open])$: measure $\mathfrak{M}([cl]) =$

$\infty : 0$". We understand this as an additional hint how difficult any classification of open manifolds would be.

The deep distinction between the propositions 0.2 and 0.3 indicates that the chance to classify open manifolds (at least partially) by means of number valued invariants is very small. This is additionally supported by proposition 0.1.

Concerning number valued invariants, there are two ways out from this situation,

1. to consider only those Riemannian manifolds for which absolute characteristic numbers and other invariants are defined,

2. to give up the concept of absolute characteristic numbers and invariants and to establish a theory of relative invariants for pairs of manifolds and bundles.

In this chapter, we give an outline of absolute number valued invariants. In section 1 we introduce and discuss absolute characteristic numbers for open manifolds associated to a Riemmanian metric. These numbers are invariants of the component of the Sobolev manifold of metric connections. In the compact case, there is only one such a component and one gets back the well known independence of the metric. To define the Sobolev component we use the language of uniform structures. A comprehensive treatise of Sobolev uniform structures will be given in chapter II. We conclude section 1 with a short discussion of the Novikov conjecture for open manifolds.

Many characteristic numbers appear as the topological index of certain differential operators. An outline of index theorems for open manifolds will be the content of section 2. To define relative number valued invariants for pairs of manifolds and bundles will be the topic of the chapters IV, V and VI.

1 Absolute characteristic numbers for open manifolds

Let (M^{4k}, g) be closed, oriented, g an arbitrary Riemannian metric, $p_i(M, g)$ the associated by Chern–Weil construction Pontrjagin classes, $e(M, g)$ the Euler class, L_k the Hirzebruch polynomial. Then there are the well known equations

$$\sigma(M^{4k}) = \int L_k(M, g) = \int L_k(p_1(M, g), \ldots, p_k(M, g)) = \sigma(M, g) \tag{1.1}$$

and for (M^n, g) oriented

$$\chi(M^n) = \int e(M, g) = \chi(M, g). \tag{1.2}$$

These equations express in particular that the r.h.s. are in fact independent of g and are homotopy invariants. We proved that the space of Riemannian metrics on a manifold splits w.r.t. a canonical uniform structure into "many" components and that on a compact manifold there is only one component (cf. e.g. [32]). The independence of g can be reformulated as the r.h.s. depend only on comp(g), since the space of Riemannian metrics on closed manifolds consists only of one component. We will extend the definitions of the l.h.s. and the r.h.s. to certain classes of open manifolds. In some cases there even holds equality. The main questions connected with such an extension are

1) the invariance properties,

2) b applications, the geometrical meaning.

It is clear that the definition of characteristic numbers via Chern–Weil construction can be extended to an open manifold if the Chern–Weil integrand is $\in L_1$, as a very special case if this integrand is bounded and (M^n, g) has finite volume.

We present in chapter II a comprehensive discussion of Sobolev uniform structures. For our purpose here we briefly introduce the notion of a basis $\mathfrak{B} \subset \mathfrak{P}(X \times X)$ for a uniform structure $\mathfrak{U}$ on a set X. $\mathfrak{B}$ is a basis if it satisfies the following conditions.

(B_1) If $V_1, V_2 \in \mathfrak{B}$ then $V_1 \cap V_2$ contains an element of $\mathfrak{B}$.
(U_1') Each $V \in \mathfrak{B}$ contains the diagonal $\Delta \subset X \times X$.
(U_2') For each $V \in \mathfrak{B}$ there exists $V' \in \mathfrak{B}$ s.t. $V' \subseteq V^{-1}$.
(U_3') For each $V \in \mathfrak{B}$ there exists $W \in \mathfrak{B}$ s.t. $W \circ W \subset V$.

A uniform structure $\mathfrak{U}$ is metrizable if and only if $\mathfrak{U}$ has a countable basis. For a tensor field t on a Riemannian manifold (M^n, g) we denote

$$^{b,m}|t| := \sum_{\mu=0}^{m} \sup_x |\nabla^\mu t|_x,$$

where $|\,|_x \equiv |\,|_{g,x}$ denotes the pointwise norm with respect to g and we set $^b|t| \equiv {}^{b,0}|t|$. By $|t|_{p,r}$ we denote the Sobolev norm

$$|t|_{p,r} \equiv |t|_{\nabla,p,r} = \left(\int_M \sum_{i=0}^{r} |\nabla^i t|_x^p \,\mathrm{dvol}_x(g) \right)^{\frac{1}{2}}$$

and set $|\,|_p \equiv |\,|_{p,0}$. The same definitions hold for tensor fields t with values in a Riemannian vector bundle E.

Let (M^n, g) be an open complete manifold, G a compact Lie group with Lie algebra $\mathfrak{G}$, $\varrho : G \longrightarrow U_N$ or $\varrho : G \longrightarrow SO_N$ a faithful representation, $P = P(M, G)$ a principal fibre bundle and $E = P \times_G E_N$ the associated vector bundle which is endowed with a Hermitean or Riemannian metric. According to the faithfulness of ϱ, the connections on P and E are in a one–to–one relation, $\omega \longleftrightarrow \nabla^\omega = \nabla$. Denote by $\mathcal{C}(P, B_0, f, p) = \mathcal{C}(E, B_0, f, p)$ the set of all connections $\omega \longleftrightarrow \nabla^\omega = \nabla$ with bounded curvature, i.e. satisfying (B_0) : $|R| \leq C$, where R denotes the curvature and $|\,|$ the pointwise norm, and having finite p–action

$$\int |R^{\nabla^\omega}|^p \,\mathrm{dvol}_x(g) < \infty.$$

We fix P and E and write therefore simply $\mathcal{C}(B_0, f, p)$. Let $\delta > 0$ and set

$$\begin{aligned} V_\delta &= \{(\nabla, \nabla') \in \mathcal{C}(B_0, f, p)^2 \mid {}^{b,1}|\nabla - \nabla'|_{\nabla,p,1} \\ &= {}^b|\nabla - \nabla'| + {}^b|\nabla(\nabla - \nabla')| \\ &+ |\nabla - \nabla'|_p + |\nabla(\nabla - \nabla')|_p < \delta\} \end{aligned}$$

Lemma 1.1 $\mathfrak{B} = \{V_\delta\}_{\delta>0}$ *is a basis for a metrizable uniform structure* ${}^{b,1}\mathfrak{U}^{p,1}(\mathcal{C}(B_0, f, p))$.

Proof. We start with (U_2'): For each $V \in \mathfrak{B}$ there exists $V' \in \mathfrak{B}$ such that $V' \subseteq V^{-1}$.

$${}^{b,1}|\nabla' - \nabla|_{\nabla',p,1} = {}^b|\nabla' - \nabla| + {}^b|\nabla'(\nabla' - \nabla)| + |\nabla' - \nabla|_p + |\nabla'(\nabla' - \nabla)|_p.$$

Hence we have to estimate only

$$\begin{aligned} {}^b|\nabla'(\nabla' - \nabla)| &\le {}^b|(\nabla' - \nabla)(\nabla' - \nabla)| + {}^b|\nabla(\nabla' - \nabla)| \\ &\le C^{b,1}|\nabla' - \nabla|^2 + {}^{b,1}|\nabla' - \nabla| \end{aligned}$$

and

$$\begin{aligned} |\nabla'(\nabla' - \nabla)|_p &\le |(\nabla' - \nabla)(\nabla' - \nabla)|_p + |\nabla(\nabla' - \nabla)|_p \\ &\le C_2{}^b|\nabla' - \nabla||\nabla' - \nabla|_p + |\nabla(\nabla' - \nabla)|_p, \end{aligned}$$

i.e.

$${}^{b,1}|\nabla' - \nabla|_{\nabla',p,1} \le P_1({}^{b,1}|\nabla' - \nabla|_{\nabla,p,1}),$$

where P_1 is a polynomial without constant term. (U_2') is done. For (U_3') : For each $V \in \mathfrak{B}$ there exists $W \in \mathfrak{B}$ such that $W \circ W \subseteq V$ we have to estimate in

$${}^{b,1}|\nabla_1 - \nabla_2|_{\nabla_1,p,1} \le {}^{b,1}|\nabla_1 - \nabla|_{\nabla_1,p,1} + {}^{b,1}|\nabla - \nabla_2|_{\nabla_1,p,1} \quad (1.3)$$

only the term ${}^{b,1}|\nabla - \nabla_2|_{\nabla_1,p,1}$. But

$$\begin{aligned} {}^{b,1}|\nabla - \nabla_2|_{\nabla_1,p,1} &= {}^b|\nabla - \nabla_2| + {}^b|\nabla_1(\nabla - \nabla_2)| \\ &\quad + |\nabla - \nabla_2|_p + |\nabla_1(\nabla - \nabla_2)|_p \\ &\le {}^b|\nabla - \nabla_2| + {}^b|(\nabla_1 - \nabla)(\nabla - \nabla_2)| \\ &\quad + {}^b|\nabla(\nabla - \nabla_2)| + |\nabla - \nabla_2|_p + \\ &\quad + |(\nabla_1 - \nabla)(\nabla - \nabla_2)|_p + |\nabla(\nabla - \nabla_2)|_p \\ &\le {}^{b,1}|\nabla - \nabla_2|_{\nabla,p,1} \\ &\quad + 2\,{}^{b,1}|\nabla_1 - \nabla|_{\nabla_1,p,1} \cdot {}^{b,1}|\nabla - \nabla_2|_{\nabla,p,1}, \end{aligned}$$

together with (1.3)

$${}^{b,1}|\nabla_1 - \nabla_2|_{\nabla_1,p,1} \le P_2({}^{b,1}|\nabla_1 - \nabla|_{\nabla_1,p,1}, |\nabla - \nabla_2|_{\nabla,p,1}),$$

where P_2 is a polynomial without constant term. (U_3') is done.

□

Denote by ${}^{b,m}\Omega^q(\mathfrak{G}_E)$ or $\Omega^{q,p,r}(\mathfrak{G}_E)$ or ${}^{b,m}\Omega^{q,p,r}(\mathfrak{G}_E)$ the completion of

$$ {}^b_m\Omega^q(\mathfrak{G}_E) = \{\eta \in \Omega^q(\mathfrak{G}_E) \mid {}^{b,m}|\eta| := \sum_{\mu=0}^{m} \sup_x |\nabla^\mu \eta|_x < \infty\} $$

or

$$ \begin{aligned} \Omega^{q,p}_r(\mathfrak{G}_E) &:= \{\eta \in \Omega^q(\mathfrak{G}_E) \mid |\eta|_{p,r} \\ &:= \left(\int \sum_{i=0}^{r} |\nabla^i \eta|^p_x \, \mathrm{dvol}_x(g)\right)^{\frac{1}{p}} < \infty\} \\ {}^b_m\Omega^{q,p}_r(\mathfrak{G}_E) &= {}^b_m\Omega^q(\mathfrak{G}_E) \cap \Omega^{q,p}_r(\mathfrak{G}_E) \end{aligned} $$

with respect to ${}^{b,m}|\ |$ or $|\ |_{p,r}$ or ${}^{b,m}|\ |_{p,r} = {}^{b,m}|\ | + |\ |_{p,r}$, respectively. We obtain $\Omega^{q,p,d}$ etc. by replacing $\nabla \longrightarrow d$ and $\Omega^{\cdot\cdot}(E, D)$ by replacing $\nabla \longrightarrow D$. Here $\Omega^*(\mathfrak{G}_E)$ are the forms with values in $\mathfrak{G}_E = P \times_G \mathfrak{G}$.

Denote by ${}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$ the completion w.r.t. ${}^{b,1}\mathfrak{U}^{p,1}$.

Theorem 1.2 **a)** *${}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$ is locally arcwise connected.*
b) *In ${}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$ coincide components and arc components.*
c) *${}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$ has a decomposition as a topological sum*

$$ {}^{b,1}\mathcal{C}^{p,1}(B_0, f, p) = \sum_{i \in I}^{b,1} \mathrm{comp}^{p,1}(\nabla_i). $$

d)
$$ \begin{aligned} &{}^{b,1}\mathrm{comp}^{p,1}(\nabla) = \{\nabla' \in {}^{b,1}\mathcal{C}^{p,1}(B_0, f, p) \mid {}^{b,1}|\nabla - \nabla'|_{\nabla,p,1} < \infty\} \\ &= \nabla + (\text{completion of } {}^b_1\Omega^1(\mathfrak{G}_E, \nabla) \cap \Omega^{1,p}_1(\mathfrak{G}_E, \nabla) \\ &\text{w.r.t.}\ {}^{b,1}|\ |_{\nabla,p,1}) = \nabla + {}^{b,1}\Omega^{1,p,1}(\mathfrak{G}_E, \nabla). \end{aligned} $$

Proof. The only fact to prove is a). b) and c) are consequences of a) and d) follows from $\nabla' = \nabla + (\nabla' - \nabla)$. Let $\varepsilon > 0$ be so small that in $U_\varepsilon(\nabla)$ ${}^{b,1}|\cdot - \cdot|_{\nabla,p,1}$ and the metric of ${}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$ are equivalent. Put for $\nabla' \in U_\varepsilon(\nabla)$, ${}^{b,1}|\nabla - \nabla'|_{\nabla,p,1} < \varepsilon$, $\nabla_t := (1-t)\nabla + t\nabla' = \nabla + t(\nabla' - \nabla)$. If $\nabla_\nu \in {}^b_1\Omega(\mathfrak{G}_E, \nabla) \cap \Omega_1^{1,p}(\mathfrak{G}_E, \nabla)$ and ${}^{b,1}|\nabla_\nu - \nabla|_{\nabla,p,1} \underset{\nu\to\infty}{\longrightarrow} 0$ then $\nabla_{\nu,t} = \nabla + t(\nabla_\nu - \nabla) \longrightarrow \nabla + t(\nabla' - \nabla) = \nabla_t$, i. e. $\nabla_t \in {}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$. Moreover, ${}^{b,1}|\nabla_{t_1} - \nabla_{t_2}|_{\nabla,p,1} = |t_1 - t_2| \cdot {}^{b,1}|\nabla' - \nabla|_{\nabla,p,1} \underset{t_1\to t_2}{\longrightarrow} 0$. □

Lemma 1.3 *The elements* ∇ *of* ${}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$ *satisfy* (B_0) *and*

$$\int |R^\nabla|_x^p \,\mathrm{dvol}_x(g) < \infty.$$

Proof. By the definition of ${}^{b,1}\mathcal{C}^{p,1}$ its elements are C^1 (since they arise by uniform convergence of 0–th and 1rst derivatives) hence R^∇ is defined. If $\nabla_\nu \longrightarrow \nabla$, $\nabla_\nu \in \mathcal{C}(B_0, f, p)$, $\nabla = \nabla_\nu + (\nabla - \nabla_\nu)$, then, for fixed ν,

$$R^\nabla = R^{\nabla_\nu + (\nabla - \nabla_\nu)} = R^{\nabla_\nu} + d^\nabla(\nabla - \nabla_\nu) + \frac{1}{2}[\nabla - \nabla_\nu, \nabla - \nabla_\nu]. \tag{1.4}$$

Each term of the r. h. s. of (1.4) is bounded, hence R^∇. Moreover $|R^{\nabla_\nu}| \in L_p$, $d^\nabla(\nabla - \nabla_\nu)| \in L_p$ and $[\nabla - \nabla_\nu, \nabla - \nabla_\nu] \le C \cdot^b |\nabla - \nabla_\nu| \cdot |\nabla - \nabla_\nu| \in L_p$. □

Now let $\omega \longleftrightarrow \nabla^\omega = \nabla$ be given. After choice of a bundle chart with local base $s_1, \ldots, s_N : U \longrightarrow E|_U$ the curvature Ω will be described as $\Omega s_i = \sum_j \Omega_{ij} \otimes s_j$, where (Ω_{ij}) is a matrix of 2–forms on U, $\Omega_{ij}(s_k, s_l) = \Omega_{ij,kl} = R_{ij,kl}$. An invariant polynomial $P : \mathrm{Mat}_N \longrightarrow \mathbb{C}$ defines in well known manner a closed graded differential form $P = P(\Omega) = P_0 + P_1 + \cdots$, where P_ν is a homogeneous polynomial, $P_r(\Omega) = 0$ for $2r > n$. The determinant is an example for P. If ω is not smooth then $P(\Omega)$ is closed in the distributional sense. Let $\sigma_r(\Omega)$ be the $2r$–homogeneous part (in the sense of forms) of $\det(1 + \Omega_{ij})$.

Lemma 1.4 *Each invariant polynomial is a polynomial in* $\sigma_1, \ldots, \sigma_N$.

Lemma 1.5 *If* $\omega \in {}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$ *and* $r \geq 1$ *then*

$$\int |\sigma_r(\Omega)|_x^p \,\mathrm{dvol}_x(g) < \infty. \tag{1.5}$$

Proof. For the pointwise norm $|\,|_x$ there holds $|\Omega|_x^2 = \frac{1}{2}\sum\limits_{i,j}\sum\limits_{k<l} |\Omega_{ij,kl}|_x^2$, where $\Omega_{ij,kl} = \Omega_{ij(e_k,e_l)}$ and $e_1, \dots e_n$ is an orthogonal base of T_xM. According to our assumption we have $|\Omega|_p^p = \int |\Omega|_x^p \,\mathrm{dvol} < \infty$ and $|\Omega|_x \leq b$ for all $x \in M$. The proof is done if we could estimate $|\sigma_r(\Omega)|_x$ from above by $|\Omega|_x$. By definition

$$\sigma_r(\Omega) = \frac{1}{r!}\sum \varepsilon^{i_1\dots i_r}_{j_1\dots j_r}\Omega_{i_1j_1}\wedge\dots\wedge\Omega_{i_rj_r}, \tag{1.6}$$

where summation runs over all $1 \leq i_1 < \dots < i_r \leq N$ and all permutations $(i_1\dots i_r) \longrightarrow (j_1\dots j_r)$. ε denotes the sign of this permutation. We perform induction. For $r = 1$ follows $\sigma_1(\Omega) = \sum \Omega_{ii}$. The inequality

$$|\Omega_{ij}|_x^2 \leq \sum_{s,t} |\Omega_{st}|_x^2 = 2|\Omega|_x^2 \tag{1.7}$$

implies in particular $|\sigma_1(\Omega)|_x^2 \leq 2N|\Omega|_x^2$. For arbitrary forms φ, ψ there holds

$$|\varphi \wedge \psi|_x \leq |\varphi|_x \cdot |\psi|_x. \tag{1.8}$$

For forms with values in a vector bundle we have to multiply the r.h.s. of (1.8) with a constant. (1.6), (1.7), (1.8) and the induction assumption thus give

$$|\sigma_r(\Omega)|_x^2 \leq a \cdot |\Omega|_x^{2r}, \tag{1.9}$$

together with $|\Omega|_x^2 \leq b^2$ finally $|\sigma_r(\Omega)|_x = c \cdot |\Omega|_x$. □

Corollary 1.6 *Let* P *be an invariant polynomial,* $\omega \in {}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$, $r \geq 1$. *Then each form* $P_r(\Omega)$ *is an element of* ${}^{b,1}\Omega^{2r,p,1}$.

Proof. This follows form 1.4, 1.5 and (1.8). □

Denote by $H^{*,p,\{d\}} = Z^{*,p,\{d\}}/B^{*,p,\{d\}}$ or ${}^bH^* = {}^bZ^*/{}^bB^*$ the L_p– or bounded cohomology, respectively, where $\{d\}$ refers to the closure of d as coboundary operator.

Corollary 1.7 *Under the assumptions of 1.6, P and ω define well defined classes $[P_\varrho(\Omega^\omega)] \in H^{2\varrho,p,\{d\}}(M)$, $[P_\varrho(\Omega^\omega)] \in^b H^{2\varrho}(M)$.* □

Now arises the natural question, how does $[P_\varrho(\Omega^\omega)]$ depend on ω? We denote $I = [0,1]$, $i_t : M \longrightarrow I \times M$ the imbedding $i_t(x) = (t,x)$ and furnish $I \times M$ with the product metric $\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$. Here we write $H^{q,p,\{d\}} \equiv H^{q,p}$ etc..

Lemma 1.8 *For every $q \geq 0$ there exists a linear bounded mapping $K : \Omega^{q+1,p,d}(I \times M) \longrightarrow \Omega^{q,p,d}(M)$ resp. $K :^b \Omega^{q+1,d}(I \times M) \longrightarrow^b \Omega^{q,d}(M)$ such that $dK + Kd = i_1^* - i^* - 0$.*

Proof. Since $g_{I\times M} = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$ is an isometric imbedding and i_t^* is bounded. i_t^* maps into $\Omega^{q,p,d}(M)$ because $|di^*\varphi|_x = |i^*d\varphi|_x \leq C \cdot |d\varphi|_x$. Denote $X_0 = \frac{\partial}{\partial t}$ and for $\varphi \in \Omega^{q+1,p,d}(I \times M)$ $\varphi_0(X_1, \ldots, X_q) := \varphi(X_0, X_1, \ldots, X_q)$. Then $\varphi_0 \in \Omega^{q,p,d}(I \times M)$, $|\varphi_0|_{(t,x)} \leq |\varphi|_{(t,x)}$, and we define

$$K\varphi(X_1, \ldots, X_n) := \int_0^1 i_t^* \varphi_0(X_1, \ldots, X_q) dt.$$

Thus K is bounded too. The equation $dK + Kd = i_1^* - t_0^*$ is standard. Replacing $\Omega^{q,p,d}$ by ${}^b\Omega^{q,d}$ gives the same conclusions.

□

Lemma 1.9 *Let $f, g : M \longrightarrow N$ be smooth mappings, $F : I \times M \longrightarrow N$ a smooth homotopy, $f^*, g^* : \Omega^{q,p,d}(N) \longrightarrow \Omega^{q,p,d}(M)$, $F^* : \Omega^{q,p,d}(N) \longrightarrow \Omega^{q,pd}(I \times M)$ resp. $f^*, g^* : {}^b\Omega^{q,d}(N) \longrightarrow {}^b\Omega^{q,d}(M)$, $F^* : {}^b\Omega^{q,d}(N) \longrightarrow {}^b\Omega^{q,d}(I \times M)$ bounded and $\varphi \in \Omega^{q,p,d}(N)$ resp. $\varphi \in {}^b\Omega^{q,d}(N)$ closed, i. e. $\varphi \in Z^{q,p}(N)$ resp. $\varphi \in {}^bZ^q(N)$. Then there holds $(g^* - f^*)\varphi \in B^{q,p}(M)$ resp. $(g^* - f^*)\varphi \in {}^bB^q(M)$.*

Proof. We consider the case $\Omega^{q,p,d}$. According to our assumption, we have $K\varphi := KF^*\varphi \in \Omega^{q-1,p,d}(M)$ and $(g^* - f^*)\varphi = ((F \circ t_1)^* - (F \circ i_0)^*)\varphi = (i_1^*F^* - i_0^*F^*)\varphi = (i_1^* - i_0^*)F^*\varphi = (dK + Kd)F^*\varphi = dKF^*\varphi = dK\varphi$. The case of bounded forms will be treated by the same equation. □

Now we are able to prove one of our main theorems.

Theorem 1.10 *Let $Q : M_N(C) \longrightarrow C$ be an invariant polynomial, $r \geq 1$, $p = 1$ or 2. Then each component U of ${}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$ determines uniquely a cohomology class $[Q_r(\Omega^U)] \in H^{p,2r}(M)$ resp. $[Q_r(\Omega^U)] \in {}^bH^{2r}(M)$.*

Proof. Assume $\omega_0, \omega_1 \in U$. Then, according to theorem 1.2, d) $\eta := \omega_1 - \omega_0 \in {}^{b,1}\Omega^{1,p,1}(\mathfrak{G}_E, \omega_0)$ and $\omega_t = \omega_0 + t\eta$, $-\delta < t < 1+\delta$, is contained in U. We have to show $[Q_r(\Omega^{\omega_0})] = [Q_r(\Omega^{\omega_1})]$. Consider

$$\Omega_t := \Omega^\omega, \quad \Omega_t = \Omega_0 + td^{\omega_0}\eta + \frac{1}{2}t^2[\eta, \eta]. \qquad (1.10)$$

For all $t \in]-\delta, \delta + 1[$ is $\int |\Omega_t|^p \,\mathrm{dvol} < \infty$ and $|\Omega_t|_x$ is bounded at M. This follows from (1.10) and the assumption $\omega_0, \omega_1 \in U$. If $\overline{p} :]-\delta, 1+\delta[\times M \longrightarrow M$ denotes the projection $(t, x) \longrightarrow x$, $P' = \overline{p}^*P$ resp. $E' = \overline{p}^*E$ the liftings of the bundles to $]-\delta, 1+\delta[\times M$, p (which covers $\overline{p}$) the associated mapping of the bundle spaces, then $p^*\omega_0$, $p^*\omega_1$ are connections for the lifted bundles.

$tp^*\omega_1 + (1-t)p^*\omega_0 = p^*\omega_0 + tp^*\eta$ is again a connection ω'. According to (1.10), there holds $\Omega^{\omega'} = p^*\Omega_0 + t\partial^{p^*\omega_0}p^*\eta + \frac{t^2}{2}[p^*\eta, p^*\eta]$. p^*

is bounded. Thus $\Omega^{\omega'}$ is surely p–integrable and bounded if this holds for $d^{p^*\omega_0}p^*\eta$. But this follows from the equation $(p^*\omega_0)_{ij} = p^*(\omega_{0,ij})$ for the connection matrix, $\eta \in {}^{b,1}\Omega^{1,p,d}(\mathfrak{G}_E, \omega_0)$ and from the boundedness of p^*. ω', $\Omega^{\omega'}$ define well determined p–integrable resp. bounded cocycles at $]-\delta, 1+\delta[\times M$. Let i_t again be the mapping $x \longrightarrow (t,x)$. Then $i_0^*(E',\omega')$ resp. $i_1^*(E',\omega')$ can be identified with (E,ω_0) resp. (E,ω_1). i_t, $0 \le t \le 1$, is a smooth bounded homotopy between i_0 and i_1. According to 1.9, $i_0^*\Omega_r(\Omega')$ and $i_1^*\Omega_r(\Omega')$ are in $H^{2r,p}$ resp. ${}^bH^{2r}$ cohomologous, i. e. $Q_r(\Omega_0)$ and $Q_r(\Omega_1)$ are in $H^{2r,p}$ resp. ${}^bH^{2r}$ cohomologous. □

Definition. *For a component U of ${}^{b,1}\mathcal{C}^{p,1}(B_0,f,p)$ we define the r–th Chern class $c_r(PU,p)$ by*

$$c_r(E,U,p) = c_r(P,U,p) := \frac{1}{(2\pi)^r}[\sigma_r(\Omega^U)].$$

Then we have $c_r \in H^{2r,p}$, $c_r \in {}^bH^{2r}$.

Remark 1.11 For $\omega_0, \omega_1 \in {}^{b,1}\mathcal{C}^{p,1}(B_0,f,p)$ the cocycles $\sigma_r(\Omega^{\omega_0}/(2\pi i)^r$, $\sigma_r(\Omega^{\omega_1}/(2\pi i)^r$ are contained in the Chern class $c_r(E)$ and therefore they are cohomologous, but they do not need to be cohomologous in $H^{2r,p}$. Take for example an $\omega \in \Omega^{p,1}(B_0,f,p)$ and apply a gauge transformation g with $\omega - g^*\omega \notin {}^{b,1}\Omega^{p,1,d}(\mathfrak{G}_E,\omega)$. There holds $|\Omega^\omega|_x = |\Omega^{g^*\omega}|_x$. An explicit example is given by $M = R^2$, $P = M \times U_N$, ω the canonical flat connection, the gauge transformation g at the point (x,y) given by $e^{i(x^2+y^2)} \cdot \mathrm{id}$, where id denotes the unit matrix. Then $|\omega - g^*\omega|_{(x,y)} = |g^{-1}dg|_{(x,y)} = |i(x\,dx - y\,dy) \cdot \mathrm{id}|_{(x,y)} = |N(x^2+y^2)|^{\frac{1}{2}}$ is neither bounded nor p–integrable. For this reason our above approach seems to be adequate to the general situation on noncompact Riemannian manifolds. □

Definition. *For $\varrho : G \longrightarrow O_N$, $E = P \times_G \mathbb{R}^N$ denote by E^c or P^c the complexification of E or P, respectively. Any connection*

ω on E resp. P extends in a canonical manner to a connection on E^c resp.P^c and we have an inclusion of the components U of ${}^{b,1}\mathcal{C}^{p,1}(P, B_0, f, p)$ into the components U^c of ${}^{b,1}\mathcal{C}^{p,1}(P^C, B_0, f, p)$. Then we define the k–th Pontrjagin class $p_k(P, U, p)$ by

$$p_k(P, U, p) = p_k(E, U, p) := (-1)^k c_{2k}(P^c, U^c, p).$$

Let Pf be the Pfaff polynomial for skew symmetric $2N$–matrices, $\varrho : G \longrightarrow SO_{2N}$, $E = P \times_G \mathbb{R}^{2N}$. Then we call for a component U of ${}^{b,1}\mathcal{C}^{p,1}(P^C, B_0, f, p)$

$$e(E, U, p) := \frac{1}{(2\pi)^N} Pf(\Omega^U)$$

the Euler class of U. There holds $e \in H^{2N,p}(M)$, $e \in {}^bH^{2N}(M)$.

Now come in characteristic numbers. Consider $\varrho : G \longrightarrow U_N$, let be $\dim M = 2k$ and Q an invariant polynomial, $\omega \in {}^{b,1}\mathcal{C}^{p,1}(P^C, B_0, f, p)$, $Q(\Omega^\omega) = a + Q_1(\Omega) + \cdots + Q_k(\Omega)$. Then $Q_{i_1 \ldots i_k} := Q_{i_1} \wedge \cdots \wedge Q_{i_k}$ with $i_1 + \cdots + i_k = k$ defines a characteristic $2k$–form and a characteristic number $\int Q_{i_1 \ldots i_k} = Q_{i_1 \ldots i_k}(P, \omega)(M)$ if the latter integral exists. In particular we consider classes $c_{i_1 \ldots i_k} := c_{i_1} \wedge \cdots \wedge c_{i_k}$ and have to ensure the existence of the corresponding integral.

Lemma 1.12 a) *If $k = 1$ and $\omega \in {}^{b,1}\mathcal{C}^{1,1}(B_0, f, 1)$, then $\int_M c_1$ converges.*

b) *If $k > 1$ and $\omega \in {}^{b,1}\mathcal{C}^{1,1}(B_0, f, 1)$ or $\omega \in {}^{b,1}\mathcal{C}^{2,1}(B_0, f, 2)$ then $\int_M c_{i_1 \ldots i_k}$ converges.*

Proof. a) is clear. We have to prove b). At each $x \in M$, $c_{i_1 \ldots i_k}$ is a sum of monomials $a \cdot \Omega_{i_1 j_1} \wedge \cdots \wedge \Omega_{i_k j_k}$. There holds according to lemma 1.5 $|\Omega_{i_1 j_1} \wedge \cdots \wedge \Omega_{i_k j_k}|_x \leq D_1 \cdot |\Omega|_x$ resp. $\leq D_2 \cdot |\Omega|_x^2$ if $p = 1$ resp. $p = 2$. □

Corollary 1.13 *Under the assumption 1.12, for any invariant polynomial Q converges $\int\limits_M Q_{i_1 \dots i_k}$.*

This follows from 1.4 and the proof of 1.12. □

Lemma 1.12 b) is also valid in the case $\varrho : G \longrightarrow O_N$, $\dim M = 4k$ for $p_{i_1 \dots i_k}$, $i_1 + \cdots + i_k = k$, $k \geq 1$, resp. in the case $\varrho : G \longrightarrow SO_{2N}$, $\dim M = N$ for the Euler form $e(E, \omega, 1, g)$ 1.12 a) for $N = 1$, 1.12 b) for $N > 1$).

The above characteristic numbers are defined until now only for a chosen connection ω. One would like that the charakteristic numbers are constant at least at the components of ${}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$. This is in fact the case for $p = 1$.

Theorem 1.14 *The characteristic numbers are constant at the components of ${}^{b,1}\mathcal{C}^{1,1}(B_0, f, 1)$.*

Proof. If ω, ω' are contained in the same component U, then according to 1.10, $Q_{i_1 \dots i_k}(\omega)$ and $Q_{i_1 \dots i_k}(\omega')$ define the same cohomology clas in $H^{2k,1}(M)$ resp. $H^{4k,1}(M)$, i. e. there exists an absolutely integrable φ with $d\varphi = Q_{i_1 \dots i_k}(\omega) - Q_{i_1 \dots i_k}(\omega')$. A fundamental result of Gaffney then says $\int\limits_M d\varphi = 0$ for (M, g) complete and $d\varphi$ itself absolutely integrable. □

Thus one gets characteristic numbers $Q_{i_1 \dots i_k}(P, U)(M)$.

Remark 1.15 For $\omega, \omega' \in {}^{b,1}\mathcal{C}^{2,1}(B_0, f, 2)$ and $\deg(Q_{i_1 \dots i_k}) \geq 4$ the characteristic numbers $Q_{i_1 \dots i_k}(\omega)(M)$, $Q_{i_1 \dots i_k}(\omega')(M)$ are defined. If $Q_{i_1 \dots i_k}(\omega)$, $Q_{i_1 \dots i_k}(\omega')$ define the same cohomology class in $H^{2k,1}(M^{2k})$ resp. $H^{4k,1}$ resp. $H^{2N,1}$, then the characteristic numbers coincide. □

A very special but interesting case in our considerations is the case $\mathrm{vol}(M) < \infty$. Consider ${}^{b,1}\mathcal{C}(B_0)$. It is defined by means of $\mathfrak{B} = \{V_\delta\}_{\delta > 0}$, where $V_\delta = \{(\nabla, \nabla') \in \mathcal{C}(B_0)^2 \mid {}^{b,1}|\nabla - \nabla'|_\nabla < \delta\}$.

Theorem 1.16 *If $\mathrm{vol}(M) < \infty$ then characteristic numbers are constant on each component of ${}^{b,1}\mathcal{C}(B_0)$.*

Proof. According to 1.10 each component U of ${}^{b,1}\mathcal{C}(B_0)$ determines uniquely a cohomology class $[Q_{i_1\ldots i_k}]$ ${}^bH^{2k}(M^{2k})$ or ${}^bH^{4k}$ or ${}^bH^{2N}$ respectively. Taking two cocycles of this class, there exists a bounded C^1–form φ such that there difference equals to $d\varphi$. φ, $d\varphi$ are bounded, $\mathrm{vol}(M) < \infty$, thus φ, $d\varphi$ are absolutely integrable and the theorem of Gaffney gives the desired result. □

Remark 1.17 $\mathrm{vol}(M) < \infty$ implies ${}^{b,1}\mathcal{C}(B_0) = {}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$. Thus the conclusion of 1.16 also holds for the components of ${}^{b,1}\mathcal{C}^{p,1}(B_0, f, p)$. □

We call the quasi isometry class of g the uniform structure $US(g)$ generated by g. For all metrics of $US(g)$ the cohomology spaces $H^{*,p}(M^n, g')$ coincide. The same holds for ${}^bH^*(M^n, g')$. This leads immediately to

Theorem 1.18 *The cohomology classes $Q_{i_1\ldots i_k}(\omega)$ resp. characteristic numbers $Q_{i_1\ldots i_k}(\omega)(M)$ in 1.10 respectively 1.14, 1.15 are the same for all metrics $g' \in US(g)$.* □

The situation completely changes if ω itself depends on g. Then it will be wrong in general that for $g' \in US(g)$ $c(\omega(g)) \sim c(\omega(g'))$. The case $\omega = \omega(g)$ is essentially the case $P =$ bundle of orthogonal frames of (M^n, g), $\nabla =$ Levi–Civita connection ∇^g. Therefore we briefly describe the metrics which come into question and describe their admitted variation (for fixed M).

Let

$$\mathcal{M}(B_0, p, f) = \Big\{ g \,|\, g \text{ complete, satisfies } (B_0) \text{ and } \int |R^g|_x^p \,\mathrm{dvol}_x(g) < \infty \Big\},$$

$$\begin{aligned}
&{}^{b,2}|g - g'|_{g,p,2} = {}^{b,2}|g - g'|_g + |g - g'|_{g,p,2} = \\
&= {}^b|g - g'|_g + {}^b|\nabla^g - \nabla^{g'}|_g + {}^b\, |\nabla^g(\nabla^g - \nabla^{g'})| + \\
&+ \left(\int \left(|g - g'|_{g,x}^p + \sum_{i=0}^{1} |(\nabla^g)^i(\nabla^g - \nabla^{g'})|_{g,x}^p \right) \mathrm{dvol}_x(g) \right)^{\frac{1}{p}}
\end{aligned}$$

and set

$$V_\delta = \{(g,g') \in \mathcal{M}(B_0,p,f)^2 \,|\, C(n,\delta)^{-1} g \le g' \le C(n,\delta) g \\ \text{and } {}^{b,2}|g-g'|_{g,p,2} < \delta\}.$$

Here $C(n,\delta) = 1 + \delta + \delta\sqrt{2n(n-1)}$.

Lemma 1.19 $\mathfrak{B} = \{V_\delta\}_{\delta>0}$ *is a basis for a metrizable uniform structure.* □

Denote by ${}^{b,2}\mathcal{M}^{p,2}(B_0,p,f)$ its completion.

Proposition 1.20 a) ${}^{b,2}\mathcal{M}^{p,2}(B_0,p,f)$ *is locally arcwise connected.*

b) *In* ${}^{b,2}\mathcal{M}^{p,2}(B_0,p,f)$ *coincide components with arccomponents.*

c) ${}^{b,2}\mathcal{M}^{p,2}(B_0,p,f)$ *has a representation as a topological sum*

$${}^{b,2}\mathcal{M}^{p,2}(B_0,p,f) = \sum_{i\in I} {}^{b,2}\mathrm{comp}^{p,2}(g_i).$$

d) $\mathrm{comp}(g) = \{g' \in {}^{b,2}\mathcal{M}^{p,2}(B_0,p,f) \,|\, {}^{b,2}|g-g'|_{g,p,2} < \infty\}$. □

Proposition 1.21 *If* $g' \in \mathrm{comp}(g)$ *then* $\nabla^{g'} \in \mathrm{comp}(\nabla^g)$ *is the sense of theorem 1.2, d).* □

Hence we obtain well defined characteristic classes $C(\nabla^g) = C(g)$ and characteristic numbers $C\ldots(\nabla^g)(M) = C\ldots(g)(M)$ as above. The main important cases are the Euler form $e = E(g)$,

$$\chi(M^n,g) := \int_M E(g)$$

and the signature case

$$\sigma(M^n,g) := \int_M L(g),$$

where $L(g)$ is the Hirzebruch genus.

There arise the following natural questions.

1) How does $E(g)$ depend on g?

2) What is the topological meaning of $\chi(M^n, g)$?

3) Under which conditions does there hold $\chi(M^n, g) = \chi(M^n)$, i.e. the Gauß–Bonnet formula?

The same questions should be put for $\sigma(M^n, g)$, $\sigma(M^n)$. To the first question we have a partial answer.

Proposition 1.22 *If* $g' \in {}^{b,2}\mathrm{comp}^{1,2}(g)$ *then*

$$\chi(M^n, g) = \chi(M^n, g')$$

and

$$\sigma(M^n, g) = \sigma(M^n, g').$$

□

In the case $g' \notin {}^{b,2}\mathrm{comp}^{1,2}(g)$ we can't say anything. The examples in [3] for $\chi(M^n, g) \neq \chi(M^n, g')$, $\sigma(M^n, g) \neq \sigma(M^n, g')$ are of this kind, i.e. g' does not lie in the component of g.

Concerning the second question, we start with a simple case in dimension two which has been discussed by Cohn–Vossen [21] and Huber [43] and has been endowed with particular short proofs by Rosenberg [64], which we present below for completion.

Theorem 1.23 *Let* (M^n, g) *be a finitely connected complete noncompact Riemannian surface with curvature* K_g.

a) *If* $K \in L_1$ *then* $\chi(M) \geq \int\limits_M K \,\mathrm{dvol}_x(g)$.

b) *If* $\mathrm{vol}(M^2, g) < \infty$ *and* $K \in L_1$ *then*

$$\chi(M) = \int\limits_M K \,\mathrm{dvol}_x(g) = \chi(M, g).$$

Proof. M^2 is diffeomorphic to a compact surface with p points deleted. A neighbourhood of each point is diffeomorphic to $S^1 \times R_+$ and the metric can be put in the form

$g_{11}(\theta, t)d\theta^2 + dt^2$. Set $M_k = M \setminus \bigcup\limits_1^p S^1 \times]k, \infty[$. The Gauß–Bonnet theorem for surfaces with boundary yields $\chi(M_k) = \int\limits_{M_k} K \,\mathrm{dvol}_x(g) + \int\limits_{\partial M_k} \omega_{12}$, where ω_{12} is the connection 1–form associated to an orthonormal frame on M. $\chi(M) = \chi(M_k)$, hence one has to show $\lim_{k\to\infty} \int\limits_{\partial M_k} \omega_{12} \geq 0$ for a) and $\lim_{k\to\infty} \int\limits_{\partial M_k} \omega_{12} = 0$ for b). W.r.t. the orthonormal frame $\theta^1 = \sqrt{g_{11}}d\theta$ and $\theta^2 = dt$ the first structure equation $d\theta^1 = \omega_{12} \cap \theta^2$ gives $\omega_{12} = \frac{d}{dt}(\sqrt{g_{11}})d\theta$ and the second one gives $K \,\mathrm{dvol}_x(g) = \Omega_{12} = d\omega_{12} = \frac{d^2}{dt^2}(\sqrt{g_{11}})d\theta dt$. $\int\limits_M K \,\mathrm{dvol}_x(g) < \infty$ implies $\lim\limits_{k\to\infty} \int\limits_{\partial M_k} \frac{d^2}{dt^2}\sqrt{g_{11}}d\theta = 0$ or $\lim\limits_{k\to\infty} \int\limits_{\partial M_k} \frac{d^2}{dt^2}\sqrt{g_{11}}d\theta = const = C$. In the case b), $\mathrm{vol}(M, g) < \infty$, i.e. $\int\limits_M \sqrt{g_{11}}d\theta dt < \infty$ which implies $\lim\limits_{k\to\infty} \int\limits_{\partial M_k} \sqrt{g_{11}}d\theta = 0$, hence $\lim\limits_{k\to\infty} \int\limits_{\partial M_k} \omega_{12} = \lim\limits_{k\to\infty} \frac{d}{dt} \int \sqrt{g_{11}}d\theta = C$, $C = 0$. In the case a), $\int\limits_{\partial M_k} \sqrt{g_{11}}d\theta \sim C \cdot k + D$ as $k \to \infty$. $C < 0$ would imply $\int\limits_{\partial M_k} \sqrt{g_{11}}d\theta = 0$ for k sufficiently large. But this is impossible for a positive integrand. □

In the case for arbitrary n, there are many approaches to study the equation $\chi(M, g) = \chi(M)$. To have $\chi(M)$ defined, one must require that each homology group over $\mathbb{R}$ is finitely generated. Sufficient for this is that M has finite topological type, i.e. it has only finitely many ends $\varepsilon_1, \ldots, \varepsilon_s$, each of them collared, $U(\varepsilon_i) \cong \partial U_i \times [0, \infty[$. Then M can be given a boundary ∂M to get a compact manifold $\overline{M}$. The case n odd is absolutely trivial.

Proposition 1.24 *Assume (M^{2k+1}, g) is of finite topological type, g arbitrary. Then*

$$\chi(M) = \int\limits_M E(g) = \chi(M, g) \text{ if and only if } \chi(\partial M) = 0.$$

Proof. For $n = 2k + 1$, the Euler form $E(g)$ vanishes since the

Pfaffian of an odd dimensional skew symmetric matrix is zero, $\int E(g) = \chi(M,g) = 0$. On the other hand, $0 = \chi(\overline{M} \underset{\partial M}{\cup} \overline{M}) = 2\chi(\overline{M}) - \chi(\partial M) = 2\chi(M) - \chi(\partial M)$. □

The more interesting case are even dimensional manifolds. We recall some definitions.

For a local orthonormal frame $\theta^1, \ldots, \theta^n$ the connection 1–forms ω_{ij} satisfy the equations

$$d\theta^i = \sum_j \omega_{ij} \wedge \theta^j \text{ and } \omega_{ji} = -\omega_{ij}.$$

They are related with the curvature 2–forms Ω_{ij} by

$$\Omega_{ij} = d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj}.$$

Denote by $S(M)$ the tangent sphere bundle which is a $(2n-1)$–dimensional manifold. For a point $(x,\xi) \in S(M)$ let $\theta^1, \ldots, \theta^n$ be a frame such that θ^1 is dual to ξ. We put

$$\begin{aligned} II(g) \;:=\; & \sum_{0 \le k < n} c_k \sum_\alpha \operatorname{sign}(\alpha) \Omega_{\alpha(2)\alpha(3)} \wedge \cdots \wedge \Omega_{\alpha(2k)\alpha(2k+1)} \\ & \wedge \omega_{\alpha(2k+2)1} \wedge \cdots \wedge \omega_{\alpha(n)1}, \end{aligned} \tag{1.11}$$

where we will not specify the c_k and $\sum\limits_\alpha$ means the sum over all permutations α of $\{2, \ldots, n\}$. $II(g)$ can be understand as pull back on an $(n-1)$–form on M to $S(M)$ by means of $pr : S(M) \longrightarrow M$. If $X \in TS(M)$ at (x,ξ), $X = X_1 + X_2$ with X_1 tangent to M and X_2 tangent to S_x^{n-1} then $\Omega_{ij}(X) = \Omega_{ij}(X_1)$ and similarly for $\omega_{i1}(X)$. If M is compact with boundary ∂M and ϱ is the section of $S(M)$ over ∂M given by the outward normal vector, then $\varrho^* \Omega_{ij}(X) = \Omega_{ij}(X_1)$, the same for ω_{i1}. Then, according to Chern,

$$\chi(M) = \int_M E(g) + \int_{\partial M} \varrho^* II(g) = \int_M E(g) + \int_{\partial M} II(g). \tag{1.12}$$

Assume now that $(M^n, g) = (M^{2m}, g)$ is even–dimensional and of finite topological type. By gradient flow of an appropriate Morse function we can introduce coordinates $(x_1, \ldots, x_{n-1}, x_n = r)$ at each end such that $0 \le r < \infty$, $g_{in} = 0$, $1 \le i \le n-1$, $g_{nn} = 1$. Let as above M_k be characterized by $x_n = r \le k$. Then

$$\chi(M) = \chi(M_k) = \int\limits_{M_k} E(g) + \int\limits_{\partial M_k} II(g). \tag{1.13}$$

At each end ε $TM|_\varepsilon$ splits as $TM = W \oplus \mathbb{R}$. Suppose additionally that W splits as

$$W = W_2 \oplus \cdots \oplus W_{r_\varepsilon}, \quad r_\varepsilon \ge 2, \quad [W_i, W_j] = 0 \quad \text{if } i \ne j, \tag{1.14}$$

and that with respect to this splitting g has the form

$$g = f_2^2(r) g_2 \oplus \cdots \oplus f_{r_\varepsilon}^2(r) g_{r_\varepsilon} + dr^2. \tag{1.15}$$

Then S. Rosenberg calculated in [65] the expression (1.11) at each end can could show if $f_j(r) \underset{r\to\infty}{\longrightarrow} 0$, $f_j'(r) \underset{r\to\infty}{\longrightarrow} 0$, then $\int\limits_{M_k} E(g) \longrightarrow \int\limits_{M} E(g)$ and $\int\limits_{\partial M_k} II(g) \longrightarrow 0$. We will not repeat the really simple calculations but state Rosenberg's

Theorem 1.25 *Let (M^n, g) be open, complete and of finite topological type. Assume that in an open neighbourhood of each end ε M splits as a product manifold $N_2 \times \cdots \times N_{r_\varepsilon} \times \mathbb{R}$ with the metric $f_2^2(r) g_2 \oplus \cdots \oplus f_{r_\varepsilon}^2(r) g_{r_\varepsilon} + dr^2$, where g_j is a metric on N_j. If $f_j(r) \underset{r\to\infty}{\longrightarrow} 0$ and $f_j'(r) \underset{r\to\infty}{\longrightarrow} 0$, then $\chi(M) = \int\limits_M E(g) \equiv \chi(M, g)$. In particular, any evendimensional manifold of finite topological type admits complete warped product metrics satisfying Gauß–Bonnet (setting $N_2 = \partial M$).* □

Corollary 1.26 *Assume the hypothesis of 1.25 and additionally $g \in {}^{b,2}\mathcal{M}^{1,2}(B_0, f, 1)$. If $g' \in {}^{b,2}\mathrm{comp}^{1,2}(g)$ then $\chi(M) = \int\limits_M E(g') \equiv \chi(M, g')$.* □

Remark 1.27 We see in 1.26 a considerable improvement of 1.25 since now the admitted class of metrics is much larger. □

If one gives up the integrability of the Ws in (1.14), i.e. the product structure of the εs, then one must strengthen the conditions to the f_j. This has been done by Rosenberg too.

Theorem 1.28 *Let (M^n, g) be open, complete and of finite topological type. Assume that in an open neighbourhood of each end ε, $TM|_\varepsilon = W_2 \oplus \cdots \oplus W_{r_\varepsilon} \oplus \mathbb{R}$ and the metric is of the form $f_2^2(r)g_2 \oplus \cdots \oplus f_{r_\varepsilon}^2(r)g_{r_\varepsilon} + dr^2$ with g_i a metric on W_i. If $f_i(r) \underset{r\to\infty}{\longrightarrow} 0$, $f_i'(r) \underset{r\to\infty}{\longrightarrow} 0$ and $f_j f_i^{-1}$ and $(f_j f_i^{-1})'$ are bounded for all r, i, j then*

$$\chi(M) = \int E(g).$$

□

Example. Let $M\backslash G/K$ be an arithmetic quotient of an evendimensional split rank–one symmetric space. Then at each component ∂M_i of ∂M, ∂M is the total space of a fibration over a torus T_1 with a torus T_2 as fiber. We have $TM|_{V\times\mathbb{R}} = W_1 \oplus W_2 \oplus \mathbb{R}$ for open $V \subset \partial M$ where the fibration restricted to V is trivial. W_i is the tangent space to the torus T_i. But in general the G–invariant metric g does not respect this splitting. Donnelly has shown in [24] that each end ε has the structure $N \times \mathbb{R}$, N at most two–step nilpotent. The Lie algebra $\mathfrak{n}$ of N splits as $\mathfrak{n} = V_2 \oplus V_3$ of root spaces, $V_3 = Z(\mathfrak{n})$, and the invariant metric at the identity of N has the form

$$e^{-2r}g_2 + e^{-4r}g_3 + dr^2, \tag{1.16}$$

where g_2 is a metric on V_2, g_3 a metric on V_3. $[\mathfrak{n}, \mathfrak{n}] \subset Z(\mathfrak{n})$ and the G–invariant distribution V_2 is not integrable. Hence theorem 1.25 is not applicable in general. In the hyperbolic case $G/K = SO(n,1)/SO(n)$, one has $V_2 = \mathfrak{n}$, which yields Gauß–Bonnet. □

Corollary 1.29 *Assume the hyptheses of theorem 1.28 and additionally* $g \in {}^{b,2}\mathcal{M}^{1,2}(B_0, f, 1)$. *If* $g' \in {}^{b,2}\mathrm{comp}^{1,2}(g)$ *then* $\chi(M) = \int E(g') \equiv \chi(M, g')$. □

There is another Gauß–Bonnet case which does not fall under 1.2 – 1.29.

Proposition 1.30 *Let* (M^{2m}, g) *be open, complete, oriented, of finite topological type and the metric at* ∞ *constant with respect to* r, *i.e. there exists an* $r_0 \geq 0$ *such that* $g(r_1, x) = g(r_2, x)$ *for all* $x \in \partial M$ *and* $r_1, r_2 > r_0$. *Then*

$$\chi(M) = \int_M E(g) \equiv \chi(M, g).$$

Proof. Let $k > r_0 + \delta$. Then $M_k \cup M_k$ yields a smooth closed manifold. Hence

$$\begin{aligned} \chi(M_k \cup M_k) &= \int_{M_k \cup M_k} E(g_{M_k \cup M_k}) = 2 \int_{M_k} E(g_{M_k}), \\ \chi(M_k \cup M_k) &= 2\chi(M_k) - \chi(\partial M_k) = 2\chi(M) \\ \chi(M) &= \int_{M_k} E(g_{M_k}). \end{aligned} \tag{1.17}$$

Forming $\lim_{k \to \infty}$ in (1.17) gives the desired result. □

A special case of 1.28 would be a metric cylinder at infinity, $g|_{U(\infty)} = g_{\partial M} \otimes +dr^2$. This is simultaneously a warped product with warping function $f(r) = 1$. $f(r) = 1$ does not satisfy $f(r) \underset{r \to \infty}{\longrightarrow} 0$, 1.25 is not applicable. Clearly, such an (M^{2m}, g) satifies (B_0) but either $\int_{U(\infty)} |R|^p \,\mathrm{dvol}_x(g) = 0$ or $\int_{U(\infty)} |R|^p \,\mathrm{dvol}_x(g) = \infty$, similarly either $\int_{U(\infty)} |E(g)| \,\mathrm{dvol}_x(g) = 0$ or $\int_{U(\infty)} |E(g)| \,\mathrm{dvol}_x(g) = \infty$. In the second case $\int E(g)$ exists but $|E(g)| \notin L_p$, $p \geq 1$.

Another class of examples which submits very useful insights are surfaces of revolution. We state from [65] without proof

Proposition 1.31 *Let* $f\ :]0,\infty[\longrightarrow \mathbb{R}$ *be smooth,* $f(0) = f'(0) = 0$ *and* ($M^2 = \{z = f(x^2+y^2)\}$, *induced metric from* $\mathbb{R}^3$) *be the associated surface of revolution. Then*

$$\chi(M) = \frac{1}{2\pi}\int\limits_M K\,\mathrm{dvol}_x(g) = \chi(M,g) \tag{1.18}$$

if and only if

$$r^{\frac{1}{2}} f'(r) \underset{r\to\infty}{\longrightarrow} \pm\infty.$$

□

Hence, if f is for all $r > 0$ strongly convex or concave, (1.18) holds. In both cases M has for $r > 0$ positive curvature and infinite volume. On the other hand, we have 1.15 in the case of 1.23 b) in the finite volume case, i.e. one can have $\chi(M) = \chi(M,g)$ as in the finite volume case. For this reason we should find additional conditions which assure in the finite volume case or the infinite volume case, respectively, that

1) $\chi(M,g)$ is a (proper) homotopy invariant,

2) $\chi(M,g) = \chi(M)$ if M has finite topological type.

We start with $\mathrm{vol}(M^n,g) < \infty$ and $|K| \le 1$ where the letter (after rescaling) is equivalent to (B_0). Then

$$\chi(M,g) = \int\limits_M E(g)$$

is well defined and for $g' \in {}^{b,2}\mathrm{comp}^{1,2}(g)$

$$\chi(M,g) = \chi(M,g'). \tag{1.19}$$

Lemma 1.32 *Let* (M^n,g) *be complete,* $\mathrm{vol}(M,g) < \infty$ *and* $|K| \le 1$. *Then* M^n *admits an exhaustion by compact manifolds with smooth boundary,* $M_1^n \subset M_2^n \subset \cdots$, $\bigcup\limits_k M_k^n = M$, *such that* $\mathrm{vol}(\partial M_k^n) \longrightarrow 0$ *and for which the second fundamental forms* $II(\partial M_k^n)$ *are uniformly bounded.*

This is just a corollary of theorem 1.33 below. □

If we take such an exhaustion as just described then

$$\chi(M_k^n) = \chi(M_k^n, g) + \int\limits_{\partial M_k^n} II(\partial M_k^n). \tag{1.20}$$

$\int\limits_{\partial M_k^n} II(\partial M, g) \underset{k\to\infty}{\longrightarrow} 0$, $\chi(M_k^n) \in \mathbb{Z}$, hence for k sufficiently large $\chi(M_k^n, g) \in \mathbb{Z}$, but we are far from a certain convergence of $(\chi(M_k^n, g))_k$ and don't know anything about the topological properties of such a limit if it exists. To obtain more insight and definite results we follow [3] and consider the following additional hypothesis.

For some neighbourhood $U(\infty) \subset M$, some profinite or normal covering space $\tilde{U}(\infty)$ has the injectivity radius at least (say) 1 for the pull back metric,

$$r_{\mathrm{inj}}(\tilde{U}_\infty)) \geq 1. \tag{1.21}$$

Together with $|K| \leq 1$ on $\tilde{U}(\infty)$ we write $\mathrm{geo}_{\tilde{\infty}}(M) \leq 1$. If $U = M$ then we denote $\mathrm{geo}(\tilde{M}) \leq 1$. In any case we assume in this hypothesis that $\tilde{U}$ or $\tilde{M}$ are profinite or normal coverings.

Here $\tilde{M} \longrightarrow M$ is profinite if there exists a decreasing sequence $\{\Gamma_j\}_j$ of subgroups of finite index, $\Gamma_j \subset \pi_1(M)$, such that $\bigcap \Gamma_j = \pi_1(\tilde{M})$.

The key for everything is the following very general theorem which assures the existence of sufficiently "smooth" exhaustions and which yields 1.32 in the case of $\mathrm{vol}(M, g) < \infty$.

Theorem 1.33 *(Neighborhoods of bounded geometry).*
Let (M^n, g) be complete, $X \subset M^n$ a closed subset and $0 < r \leq 1$. Then there is a submanifold U^n with smooth boundary ∂U^n such that for some constant $c(n)$ depending only on n

a) $X \subset U \subset T_r(x) = r -$ *tubular neighbourhood of* X,

b) $\mathrm{vol}(\partial U) \leq c(n) \cdot \mathrm{vol}(T_r(X) \setminus X) \cdot r^{-1}$, (1.22)

c) $|II(\partial U)| \leq c(n) \cdot r^{-1}$. (1.23)

We refer to [17] for the proof. □

Now we will discuss $\chi(M, g)$ in the profinite or normal case, $\text{geo}(\tilde{M}) \le 1$. Here we follow [16]. Put for $j : A_1 \subset A_2$ and real coefficients $\beta^i(A_1, A_2) = \dim\{j^*(H^i(A_2)) \subset H^i(A_1)\}$ and $\beta^i(A) = \dim\{j^*(H^i(A, \partial A)) \subset H^i(A)\}$. b^i shall denote the usual Betti number. Then for $A_1 \subset A_2 \subset A_3 \subset A_4$ and $A \subset Y$ a finite closed and $f : Y \longrightarrow Z$, $g : Z \longrightarrow Y$ simplicial, determining a homotopy equivalence,

$$\beta^i(A_1) \subseteq \beta^i(A_2) \le \beta^i(A_2, A_4) \le \beta^i(A_3, A_4) \tag{1.24}$$

and

$$\beta^i(A, Y) \le \beta^i(f(A), Z) \le \beta^i(g \circ f(A), Y). \tag{1.25}$$

Put for $p : \tilde{Y}^n \longrightarrow Y^n$ profinite with $\text{ind}(\Gamma_j) = d_j$ and corresponding covering spaces $p_j : \tilde{Y}_j^n \longrightarrow Y^n$

$$\sup \tilde{\chi}(Y^n) := \overline{\lim_{A\to\infty}}\, \overline{\lim_{j\to\infty}} \sum_{i=1}^{n} (-1)^i \frac{1}{d_j} \beta^i(P_j^{-1}(A), \tilde{Y}_j^n) \le \infty \tag{1.26}$$

and define $\inf \tilde{\chi}(Y^n)$ similarly. $A \longrightarrow \infty$ is defined by partial ordering of finite subcomplexes induced by inclusion. Using (1.24) and a diagonal argument, there are subsequences $S = \tilde{Y}_{j(e)}^n$ s.t.

$$\begin{aligned} \infty \ge \tilde{\beta}^i(Y^n, S) &:= \lim_{A\to\infty} \overline{\lim_{e\to\infty}} \frac{1}{d_{j(e)}} \beta^i(P_{j(e)}^{-1}(A), \tilde{Y}_{j(e)}^n) \\ &= \lim_{A\to\infty} \underline{\lim_{e\to\infty}} \frac{1}{d_{j(e)}} \beta^i(P_{j(e)}^{-1}(A), \tilde{Y}_{j(e)}^n) \end{aligned} \tag{1.27}$$

exists. From (1.25) we infer immediately that $\tilde{\beta}^i(Y^n, S)$ is a homotopy invariant. Suppose $\tilde{\beta}^i(Y^n, S) < \infty$, $i = 0, \ldots, n$ and $\sup \tilde{\chi}(Y^n) = \inf \tilde{\chi}(Y^n)$, then the latter number is also a homotopy invariant.

Theorem 1.34 *Suppose* (M^n, g) *complete,* $\text{vol}(M^n, g) < \infty$, $\tilde{M}$ *either profinite or normal and* $\text{geo}(\tilde{M}) \le 1$.

a) *Then* $\chi(M^n, g)$ *is a proper homotopy invariant,*

b) *in the case* $\tilde{M}$ *profinite*

$$\chi(M, g) = \sup \tilde{\chi}(M) = \inf \tilde{\chi}(M),$$

c) *if additionally* M *has finite topological type,*

$$\chi(M, g) = \chi(M).$$

Proof. Assume $\tilde{M} \longrightarrow M$ profinite, let $M_1 \subset M_2 \subset \cdots$, $\bigcup_k M_k = M$ be an exhaustion of M by compact submanifolds with boundary and denote $M_k - R = \{x \in M_k | \mathrm{dist}(x, \partial M_k) = R\}$. For j sufficiently large, theorem 1.33 is applicable and we apply it to $p_j^{-1}(M_{k-1})$, $p_j^{-1}(M_k)$ with $\varepsilon = \frac{1}{2}$. This yields submanifolds $A_{jk} \subset p_j^{-1}(M_k) \subset B_{jk}$. Given $\varepsilon > 0$ arbitrary, there exist k_0, $N(k)$ such that for $k > k_0$, $j > N(k)$

$$\begin{aligned} \left|\chi(M^n, g) - \frac{1}{d_j}\chi(B_{jk})\right| &\leq \left|\chi(M^n, g) - \frac{1}{d_j}\int\limits_{B_{jk}} E(g)\right| \\ &+ \left|\frac{1}{d_j}\int\limits_{B_{jk}} E(g) - \frac{1}{d_j}\chi(B_{jk})\right| < \varepsilon. \end{aligned} \tag{1.28}$$

We see this immediately from (1.12) and (1.22), (1.23): $\chi(M^n, g) = \chi(M_k^n, g) + \chi(M^n \setminus M_k^n, g)$, here $|\chi(M^n \setminus M_k^n, g)|$ becomes arbitrarily small for k sufficiently large.

$$\begin{aligned} \left|\chi(M^n, g) - \frac{1}{d_j}\int\limits_{B_{jk}} E(g)\right| &\leq |\chi(M^n, g) - \chi(M_k^n, g)| \\ &+ \left|\chi(M_k^n, g) - \frac{1}{d_j}\int\limits_{B_{jk}} E(g)\right|, \end{aligned}$$

$$\left|\chi(M_k^n, g) - \frac{1}{d_j}\int\limits_{B_{jk}} E(g)\right| \leq \left|\chi(M_k^n, g) - \frac{1}{d_j}\int\limits_{p_j^{-1}(M_k^n)} E(g)\right|$$
$$+ \left|\frac{1}{d_j}\int\limits_{p_j^{-1}(M_k^n)} E(g) - \frac{1}{d_j}\int\limits_{B_{jk}} E(g)\right|$$
$$= \left|\frac{1}{d_j}\int\limits_{B_{jk}\setminus p_j^{-1}(M_k^n)} E(g)\right|,$$

but this becomes arbitrarily small for j and k sufficiently large. Finally

$$\left|\frac{1}{d_j}\int\limits_{B_{jk}} E(g) - \frac{1}{d_j}\chi(B_{jk})\right| = \left|\frac{1}{d_j}\int\limits_{\partial B_{jk}} II(\partial B_{jk})\right| \underset{j,k\to\infty}{\longrightarrow} 0$$

according to (1.22). (1.28) is proven.
We obtain from (1.24)

$$\beta^i(A_{jk}) \leq \beta^i(p_j^{-1}(M_k)) \leq \beta^i(p_j^{-1}(M_k), \tilde{M}_j) \leq b^i(B_{jk}) \quad (1.29)$$

and from the exact cohomology sequence of the pair $(B_{jk}, \overline{B_{jk} \setminus A_{jk}})$ together with the excision property

$$|\beta^i(A_{jk}) - b^i(B_{jk})| \leq b^{i-1}(\overline{B_{jk} \setminus A_{jk}}) + b^i(\overline{B_{jk} \setminus A_{jk}}) :$$

$$\cdots \longrightarrow H^{i-1}(\overline{B_{jk} \setminus A_{jk}}) \longrightarrow H^i(B_{jk}, \overline{B_{jk} \setminus A_{jk}}) \cong H^i(A_{jk}, \partial A_{jk})$$
$$\longrightarrow H^i(B_{jk}) \longrightarrow H^i(\overline{B_{jk} \setminus A_{jk}}) \longrightarrow \cdots$$

The manifold $\overline{B_{jk} \setminus A_{jk}}$ satisfies (B_0), (I) for $j > N(k)$ and for k sufficiently large,

$$\text{vol}(\overline{B_{jk} \setminus A_{jk}}) \leq d_j\varepsilon. \quad (1.30)$$

According to a theorem of Gromov,

$$\sum_i \beta^i(\overline{B_{jk} \setminus A_{jk}}) \leq c(n) \cdot \mathrm{vol}(\overline{B_{jk} \setminus A_{jk}}). \qquad (1.31)$$

We infer from (1.29) – (1.32) that we can replace in (1.28) $\chi(B_{jk})$ by $\chi(p_j^{-1}M_k, \tilde{M}_j)$, hence

$$\left| \chi(M^n, g) - \frac{1}{d_j}\chi(p_k^{-1}(M_k), \tilde{M}_j) \right|$$

becomes arbitrarily small, any proper homotopy equivalence preserves a subsequence of $\left(\frac{1}{d_j}\chi(p_j^{-1}(M_k), \tilde{M}_j\right)_{j,k}$, $\chi(M^n, g)$ is a proper homotopy invariant. By the same argument we conclude in the profinite case assertion b). If M has finite topological type then for k sufficiently large $\beta^i(p_j^{-1}(M_k), \tilde{M}_j) = \beta^i(\tilde{M}_j)$ and

$$\chi(p_j^{-1}(M_k), \tilde{M}_j) = \chi(\tilde{M}_j) \cdot \frac{1}{d_j}\chi(\tilde{M}_j) = \chi(M_j) = \chi(M)$$

yields assertion c). □

The case of a normal covering $\tilde{M} \longrightarrow M$ will be discussed in theorem 1.38.

The second characteristic number of particular importance is given by $\sigma(M, g) = \int\limits_M L(M, g)$, where $L(M, g)$ is the Hirzebruch genus. For closed M it is the topological index of the signature operator, i.e. it coincides with the topological signature. For simple open manifolds this equality does not longer hold in general. Nevertheless, we could ask for $\sigma(M, g)$ the same questions as for $\chi(M, g)$, the question for the invariance properties and the topological significance of $\sigma(M, g)$. Concerning the invariance, a first answer is given by proposition 1.22.

But we consider also other variations of g. A key role plays again the formula for the compact case with boundary, $\partial M = N$,

$$\sigma(M, g) + \eta(N, g) + \int\limits_N II_\sigma(N, g) = \sigma(M), \qquad (1.32)$$

where $II_\sigma(N,g)$ essentially involves the second fundamental form and $\eta(N,g)$ is the eta invariant. If M^n is open and $M_1 \subset M_2 \subset \cdots$, $\bigcup_k M_k = M$, an appropriate exhaustion such that $\int II_\sigma(\partial M_k) \longrightarrow 0$ and $\eta(\partial M_k) \longrightarrow 0$ then we would have in fact $\sigma(M_k, g) \longrightarrow \sigma(M)$. Hence we should ask for conditions which assure $\eta(\partial M_k) \longrightarrow 0$. There is a clear (and for our case complete) answer.

Theorem 1.35 *Let* (N^{4l-1}, g) *be compact satisfying* $\mathrm{geo}(N) \leq 1$. *Then there is a constant* $c = c(4l-1)$ *such that*

$$|\eta(N^{4l-1})| \leq c(4l-1) \cdot \mathrm{vol}(N^{4l-1}, g). \qquad (1.33)$$

We refer to [16], [27] for the proof. □

Now we define $\sup \tilde{\sigma}(M)$, $\inf \tilde{\sigma}(M)$ quite analogous to the Euler characteristic as follows. Let M^{4l} be complete, $\tilde{M}^{4l} \longrightarrow M$ profinite and $M_k^{4l} \subset M^{4l}$ a compact submanifold with boundary. Put

$$\begin{aligned} \sup \tilde{\sigma}(M_k) &:= \lim_j \sup \frac{1}{d_j} \sigma(P_j^{-1}(M_k)), \\ \sup \tilde{\sigma}(M) &:= \lim_{M_k} \sup \sup \tilde{\sigma}(M_k)) \end{aligned}$$

and similarly $\inf \tilde{\sigma}(M_k)$, $\inf \tilde{\sigma}(M)$. Here as always $\sigma(M_k)$ is defined as the signature of the cup product pairing on $j^* H^{2l}(M_k^{4l}, \partial M_k^{4l}) \subset H^{2l}(M_k^{4l})$.

Theorem 1.36 *Let* (M^{4l}, g) *be complete,* $\mathrm{vol}(M,g) < \infty$ *and suppose* $\tilde{M}$ *either profinite or normal and* $\mathrm{geo}(\tilde{M}) \leq 1$. *Then there holds*

a) *Assume* $\tilde{M}$ *normal. Then* $\sigma(M,g)$ *is a proper homotopy invariant of* M.

b) *In the case* $\tilde{M} \longrightarrow M$ *profinite, for any exhaustion* $M_1 \subset M_2 \subset \cdots$, $\bigcup_k M_k = M$, *by compact manifolds,*

$$\sigma(M,g) = \sup \tilde{\sigma}(M) = \inf \tilde{\sigma}(M).$$

c) *If, additionally, M has finite topological type,*

$$\sigma(M,g) = \lim_{j\to\infty} \frac{1}{d_j}\sigma(\tilde{M}_j).$$

Proof. In the normal case $\tilde{M} \longrightarrow M$ below a) follows from theorem 1.38. The proof of b) is quite analogous to that of theorem 1.34 b), using a chopping of M according to theorem 1.33, (1.32) and theorem 1.35. c) then follows from b) and the fact that for sufficiently large k, $\frac{1}{d_j}\sigma(p_j^{-1}(M_k)) = \frac{1}{d_j}\sigma(\tilde{M}_j)$. □

We now turn to the normal case $\tilde{M} \longrightarrow M$, being even more explicit than in the profinite case. The first key here is the extension of Atiyah's L_2–index theorem for normal coverings $\tilde{M} \longrightarrow M$ of closed M to normal coverings $\tilde{M} \longrightarrow M$, $M = \tilde{M}/\Gamma$, $r_{\mathrm{inj}}(\tilde{M}) \geq 1$, (M^n, g) complete, $\mathrm{vol}(M^n, g) < \infty$, $|K| \leq 1$. We denote by $\mathcal{H}^{q,2}(\tilde{M})$ the space of L_2–harmonic q–forms, by $P_{\mathcal{H}^{q,2}} : L_2(\Lambda^q T^* A) = \Omega^{q,2} \longrightarrow \mathcal{H}^{q,2}$ the orthogonal projection. $P_{\mathcal{H}}$ has Schwartz kernel $\tilde{h}^q(x,y)$ which is a symmetric C^∞ double form whose pointwise norm satisfies

$$|h^q(x,y)| \leq c(n). \tag{1.34}$$

(1.34) comes from $\mathrm{geo}(\tilde{M}) \leq 1$ and the elliptic estimate for the Laplacian. $\tilde{h}^q(x,y)$ is invariant under the isomtries Γ, hence the pointwise trace $\mathrm{tr}\tilde{h}^q(x,x)$ can be understood as function on M and we put as usual

$$\tilde{b}^{q,2}(M) := \mathrm{tr}_\Gamma P_{\mathcal{H}^{q,2}(\tilde{M})} = \int_M \mathrm{tr}\tilde{h}^q(x,x)\,\mathrm{dvol}_x(g) < \infty.$$

$\tilde{b}^{q,2}(M)$ is just the von Neumann dimension $\dim_\Gamma \overline{H}^{q,2}(\tilde{M})$ of the Γ–module $\overline{H}^{q,2}(\tilde{M})$. We define the L_2–Euler characteristic and L_2–signature by

$$\tilde{\chi}_{(2)}(M) := \sum_{q=0}^{n} (-1)^q \tilde{b}^{q,2}(M)$$

and

$$\tilde{\sigma}_{(2)}(M) := \mathrm{tr}_\Gamma(*P_{\mathcal{H}^{2k,2}(\tilde{M}^{4k})}).$$

Now we state the L_2–index theorem for open manifolds with finite volume and bounded curvature.

Theorem 1.37 *Suppose* (M, g) *complete with* $\mathrm{vol}(M^n, g) < \infty$, $|K| \leq 1$ *and* $\tilde{M} \longrightarrow M$ *normal with* $\mathrm{geo}(\tilde{M}) \leq 1$. *Then*

$$\chi(M, g) = \tilde{\chi}_{(2)}(M) \tag{1.35}$$

and

$$\sigma(M, g) = \tilde{\sigma}_{(2)}(M). \tag{1.36}$$

We refer to [15], [27] for the proof. □

We recall the existence of good chopping sequences $M_1 \subset M_2 \subset \cdots$, $\bigcup_1^\infty M_k = M$, $\mathrm{vol}(\partial M_k) \longrightarrow 0$, $|II(\partial M_k)| \leq c$, $|\tilde{h}_k^q(x, y) \leq c(n)|$, where $\tilde{h}_k^q$ denotes the kernel corresponding to projection on the harmonic q–forms for $p^{-1}(M_k) \subset \tilde{M}$. Then we obtain

$$\lim_{k\to\infty} \tilde{b}^{q,2}(\partial M_k) = 0 \tag{1.37}$$

and

$$\lim_{k\to\infty} \tilde{b}^{q,2}(M \setminus M_k, \partial(M \setminus M_k)) = 0. \tag{1.38}$$

Define $\tilde{\beta}^{q,2}(B)$ by

$$\tilde{\beta}^{q,2}(B) := \dim_\Gamma \mathrm{im}\ (H^{q,2}(p^{-1}(B), p^{-1}(\partial B)) \subset \overline{H}^{q,2}(p^{-1}(B))) \tag{1.39}$$

and for $A \subset B$

$$\tilde{\beta}^{q,2}(A, B) := \dim_\Gamma \mathrm{im}\ (\overline{H}^{q,2}(p^{-1}(B)) \subset \overline{H}^{q,2}(p^{-1}(A))). \tag{1.40}$$

It follows from the properties of $\dim_\Gamma$ that

$$\tilde{\beta}^{q,2}(A) \leq \tilde{\beta}^{q,2}(B) \tag{1.41}$$

and

$$\tilde{\beta}^{q,2}(A) \leq \tilde{\beta}^{q,2}(A, B) \leq \tilde{b}^{2,q}(A). \tag{1.42}$$

We remark that (1.41) and (1.42) are the adequate reformulation of (1.24), (1.29) in the language of $\dim_\Gamma$. We established in theorem 1.37 the equations $\chi(M,g) = \tilde{\chi}_{(2)}(M)$, $\sigma(M,g) = \tilde{\sigma}_2(M)$. Now we discuss the invariance properties of the right hand sides. This is the content of

Theorem 1.38 *Let* (M^n, g) *be complete,* $|K| \leq 1$, $\mathrm{vol}(M,g) < \infty$ *and assume for some normal covering* $\mathrm{geo}(\tilde{M}) \leq 1$.
a) *If* $M_1 \subset M_2 \subset \cdots$, $\bigcup M_k = M$ *is an exhaustion then*

$$\lim_{k\to\infty} \tilde{\beta}^{q,2}(M_k) = \lim_{k\to\infty}\lim_{l\to\infty} \tilde{\beta}^{q,2}(M_k, M_l) = \tilde{b}^{q,2}(M). \qquad (1.43)$$

This implies the homotopy invariance of the $\tilde{b}^{q,2}(M)$.
b) $\chi(M,g)$ *resp.* $\sigma(M,g)$ *is a homotopy invariant resp. proper homotopy invariant of* M.
c) *If* M *has the topological type of some* $M_k \subset M$, *then*

$$\tilde{b}^{q,2}(M_k) = \tilde{b}^{q,2}(M) \qquad (1.44)$$

and

$$\chi(M,g) = \chi(M_k). \qquad (1.45)$$

Proof. b) follows immediately from theorem 1.37 and a). For c) suppose that M has finite topological type. Then there exists an exhaustion $M_1 \subset M_2 \subset \cdots$ s.t. each inclusion $M_k \longrightarrow M$ is a homotopy equivalence. This implies

$$\tilde{\beta}^{q,2}(M_k, M_k) = \tilde{b}^{q,2}(M_k)$$

and we obtain (1.44) from (1.43) and moreover $\chi(M,g) = \chi(M_k)$. Hence there remains to show a). For this we must refer to [15]. □

We apply these results on characteristic numbers to 4–manifolds. Let (M^4, g) be open, complete and oriented, $* : \Lambda^2 M \longrightarrow \Lambda^2 M$ the Hodge operator, $*^2 = 1$, $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. The special orthogonal group acts on the space of algebraic curvature tensors $\mathcal{C}^2_b$

(cf. [57]). Let $\mathcal{C}_b^2 = \mathcal{U} + \mathcal{S} + \mathcal{W}$ be the corresponding (fiberwise) decomposition into irreducible subspaces. Then this induces for the curvature tensor $R = R^g$ a decomposition $R = U + S + W$. For $R = R^g = R_+ + R_-$, we denote by $\mathrm{Ric} = \mathrm{Ric}^g$ the Ricci tensor, by $\tau = \tau^g$ the scalar curvature, by $K = K^g$ the sectional curvature and by $W = W^g = W_+ + W_-$ the Weyl tensor. There are decompositions for the pointwise norms $|\ |_x$ as follows

$$\begin{aligned} |R|^2 &= |R_+|^2 + |R_-|^2 = |U|^2 + |S|^2 + |W|^2 \\ &= 4|W_+|^2 + |W_-|^2 + 2|\mathrm{Ric}\,|^2 - \frac{1}{3}\tau^2, \end{aligned} \tag{1.46}$$

$$|\mathrm{Ric}\,|^2 = 6|U|^2 + 2|S|^2, \tag{1.47}$$

$$\tau^2 = 24|U|^2. \tag{1.48}$$

We obtain still other decompositions if we consider the curvature operator R as acting from $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ to $\Lambda^2_+ \oplus \Lambda^2_-$, for an orthonormal basis e_1, e_2, e_3, e_4

$$R(e_i \wedge e_j) = \frac{1}{2} \sum R_{ijkl} e_k \wedge e_l = \Omega_{ij},$$

$\Omega = (\Omega_{ij})$ = matrix of curvature forms, $\Omega_{ij}(e_k, e_l) = R_{ijkl}$. We can write R with respect to the orthogonal basis $e_1 \wedge e_2 + e_3 \wedge e_4$, $e_1 \wedge e_4 + e_2 \wedge e_3$, $e_1 \wedge e_3 + e_2 \wedge e_4$ in Λ^2_+, $e_1 \wedge e_3 + e_2 \wedge e_4$, $e_1 \wedge e_2 - e_3 \wedge e_4$, $e_1 \wedge e_4 - e_2 \wedge e_3$ in Λ^2_-, as

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $A = A^*$, $C = B^*$, $D = D^*$, $\mathrm{tr}A = \mathrm{tr}D = \frac{\tau}{4}$, $B = \mathrm{Ric} - \frac{1}{4}\tau g$ and $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} - \frac{\tau}{12} = W$, $W^+ = A - \frac{\tau}{12}$, $W^- = D - \frac{\tau}{12}$. We obtain for the first Pontrjagin form p_1

$$\begin{aligned} p_1 &= -\frac{1}{8\pi^2}\mathrm{tr}(R \wedge R) = -\frac{1}{8\pi^2}\mathrm{tr}(A \wedge A) + \mathrm{tr}(D \wedge D) \\ &= -\frac{1}{8\pi^2}(-2)(|W_+|^2 - |W_-|^2)\,\mathrm{dvol} \\ &= \frac{1}{4\pi^2}(|W_+|^2 - |W_-|^2)\,\mathrm{dvol} \\ &= \frac{1}{12\pi^2}(|R_+|^2 - |R_-|^2)\,\mathrm{dvol} \end{aligned}$$

and for $\sigma(M^4,g)=\int L(g)=\int \frac{1}{3}p_1=\frac{1}{12\pi^2}\int(|W_+|^2-|W_-|^2)\,\mathrm{dvol}$. Assuming $g\in {}^{b,2}\mathcal{M}^{1,2}(B_0,1,f)$, $\sigma(M^4,g)$ is well defined. The Euler form $E(g)$ has the representation

$$\begin{aligned} E(g) &= \frac{1}{8\pi^2}\mathrm{tr}(*R)^2\,\mathrm{dvol} \\ &= \frac{1}{8\pi^2}(|U|^2-|S|^2+|W|^2)\,\mathrm{dvol} \\ &= \frac{1}{8\pi^2}\mathrm{tr}(A^2-2BB^*+D^2)\,\mathrm{dvol} \\ &= \frac{1}{32\pi^2}(|R|^2-4|\mathrm{Ric}\,|^2+\tau^2)\,\mathrm{dvol}. \end{aligned}$$

For $g\in {}^{b,2}\mathcal{M}^{1,2}(B_0,1,f)$, $\int E(g)=\chi(M,g)$ is well defined. Hence we obtain

Proposition 1.39 *Let (M^4,g) be open, complete, oriented and $g\in {}^{b,2}\mathcal{M}^{1,2}(B_0,1,f)$. Then $\sigma(M,g)$ and $\chi(M,g)$ are well defined and an invariant of* comp(g). □

Remark 1.40 According to (1.46) – (1.48), $\int|R^g|^2\,\mathrm{dvol}<\infty$ would be sufficient for the existence of $\sigma(M,g)$ and $\chi(M,g)$. But this condition would not establish a uniform strucutre, we would not have components and invariance properties (where we used in particular Gaffney's theorem). Moreover, we need the bounded curvature property for the connection with the theorems 1.37, 1.38. □

We obtain from proposition 1.39 and its proof the simple

Corollary 1.41 *If (M^4,g) is additionally Einstein then $\chi(M,g)\geq 0$ and $|\sigma(M^4,g)|\leq \frac{2}{3}\chi(M^4,g)$. Moreover, $\chi(M^4,g)=0$ if and only if (M^4,g) is flat.*

Proof. If (M^4,g) is Einstein then $S\equiv 0$, $B\equiv 0$ and $\frac{1}{12\pi^2}(|W_+|^2-|W_-|^2)\leq \frac{2}{3}\frac{1}{8\pi^2}(|U|^2+|W_+|^2-|W_-|^2)$. Hence $\sigma(M^4,g)\leq\frac{2}{3}\chi(M,g)$. Changing the orientation replaces

$\sigma(M^4, g)$ by $-\sigma(M^4, g)$ and we get altogether $|\sigma(M^4, g)| \leq \frac{2}{3}\chi(M^4, g)$. □

The same estimate holds for $\frac{2}{3}$–pinched Ricci curvature.

Proposition 1.42 *Suppose the hypotheses of 1.39 and additionally that the Ricci curvature of (M^4, g) is negative and $\frac{2}{3}$-pinched, i.e. there exists $A > 0$ s.t.*

$$-Ag \leq \text{Ric} \leq -\frac{2}{3}Ag. \tag{1.49}$$

Then there holds for all $g' \in \text{comp}(g) \subset {}^{b,2}\mathcal{M}^{1,2}(B_0, 1, f)$

$$|\sigma(M^4, g')| \leq \frac{2}{3}\chi(M^4, g'). \tag{1.50}$$

Proof. We have

$$\begin{aligned} |\sigma(M^4, g)| &= |\int L(g)| \leq \int |L(g)| \,\text{dvol} \\ &= \frac{1}{12\pi^2}\int (|W_+|^2 + |W_-|^2) \,\text{dvol} \end{aligned}$$

and

$$\chi(M^4, g) = \int E(g) = \frac{1}{8\pi^2}\int (|U|^2 - |S|^2 + |W|^2) \,\text{dvol}.$$

Sufficient for (1.50) would be $|S|^2 \leq |U|^2$ and sufficient for this is (1.49) as pointed out by [57].

□

Examples 1.43 **1)** Examples for 1.39 with infinite volume are e.g. manifolds M^4 of the smooth type $M^4 = M_0^4 \cup \partial M_0^4 \times [0, \infty[$ where the curvature at the cylinder $\partial M_0^4 \times [0, \infty[$ is bounded and asymptotically flat in the sense $\int_{\partial M_0^4 \times [0,\infty[} |R| \,\text{dvol} < \infty$. This can be easily realized by warped product metrics.

2) Examples for 1.39, 1.41, 1.42 with finite volume are given by hyperbolic 4–manifolds of finite volume.

3) Generalizations of these examples are given by variation of g inside comp(g). □

Theorem 1.44 *Let* (M^4, g) *be open, complete,* $\mathrm{vol}(M^4, g) < \infty$, $|K| \le 1$ *and suppose that* (M^4, g) *admits a normal covering* $(\tilde{M}, g)$ *satisfying* $\mathrm{geo}(\tilde{M}) \le 1$.

a) *If* $\chi(M^4, g) < 0$ *then* M^4 *does not admit a complete Einstein metric* g' *satisfying* $\mathrm{vol}(M^4, g') < \infty$, $|K_{g'}| \le 1$, $\mathrm{geo}(\tilde{M^4}, g') \le 1$ *for some normal covering.*

b) *If* $\chi(M^4, g) > 0$ *and* $|\sigma(M^4)| > \frac{2}{3}\chi(M^4, g)$ *then* M^4 *does not admit a complete Einstein metric* g', *s.t.* $\mathrm{vol}(M^4, g') < \infty$, $|K_{g'}| \le 1$, $\mathrm{geo}(\tilde{M^4}, g') \le 1$. *Moreover, there does not exist a complete metric* g' *satisfying*

$$-Ag' \le \mathrm{Ric}\,(g') \le -\frac{2}{3}Ag'$$

and $|K_{g'}| \le 1$, $\mathrm{vol}(M^4, g') < \infty$ *and* $\mathrm{geo}(\tilde{M^4}, g') \le 1$ *for some normal covering.*

Proof. a) Suppose the existence of an Einstein metric g' with the required properties. Then $\chi(M^4, g)$, $\chi(M^4, g')$ are well defined. $\chi(M^4, g) = \chi(M^4, g')$, according to theorem 1.38 b). But this contradicts $\chi(M^4, g') = \frac{1}{8\pi^2}\int(|U|^2 + |W|^2)\,\mathrm{dvol} \ge 0$. b) and c): Quite analogously we derive by means of theorem 1.38 b), corollary 1.41 and proposition 1.42 a contradiction. □

Until now we defined characteristic numbers in the following cases

1) $R \in L_1$ and bounded, $\mathrm{vol}(M)$ arbitrary,

2) R bounded, $\mathrm{vol}(M) < \infty$.

There remains the case R bounded, $\mathrm{vol}(M) = \infty$. It is clear that in this case we will not get characteristic numbers by integration. (M^n, g) is called closed at infinity if for any $\varphi \in C(M)$, $0 <$

$A^{-1} < \varphi < A$, $A > 0$ some constant, the form $\varphi \cdot$ dvol generates a nontrivial cohomology class in ${}^bH^n(M^n, g)$. A fundamental class for M is a positive continuous linear function $\mathfrak{m} : {}^b\Omega^n(M) \longrightarrow \mathbb{R}$ such that $\langle \mathfrak{m}, \text{dvol}\rangle \neq 0$ and $\langle \mathfrak{m}, d\psi\rangle = 0$.

Proposition 1.45 *M has a fundamental class if and only if M is closed at infinity.*

Proof. Denote $\mathcal{L}(\text{dvol})$ for the linear hull of dvol, let $0 \notin [\text{dvol}] \in {}^b\overline{H}^n(M)$ and set $\langle \mathfrak{m}, \text{dvol}\rangle = 1$, $\mathfrak{m}|_{{}^b\overline{B}^n} \equiv 0$. Then we obtain by linear extension $\mathfrak{m}$ on $\mathcal{L}(\text{dvol}) \oplus {}^b\overline{B}^n$ as positive continuous linear functional. The Hahn–Banach theorem for the extension of such functionals yields the desired $\mathfrak{m}$. The other direction is absolutely trivial. □

Define the penumbra for $K \subset M$.

$$\begin{aligned}
\text{Pen}^+(K, r) &= \text{CL}\Big(\bigcup_{x\in K} B_r(x)\Big), \\
\text{Pen}^-(K, r) &= \text{CL}(M \setminus \text{Pen}^+(M \setminus K, r)).
\end{aligned}$$

We call an exhaustion $M_1 \subset M_2 \subset \cdots$, $\bigcup_i M_i = M$, by compact submanifolds a regular exhaustion if for each $r \geq 0$

$$\lim_{i\to\infty} \text{vol}(\text{Pen}^+(M_i, r))/\text{vol}(\text{Pen}^-(M_i, r)) = 1.$$

It is clear that then automatically

$$\begin{aligned}
\lim_{i\to\infty} \text{vol}(\text{Pen}^+(M_i, r))/\text{vol}(M_i) &= 1, \\
\lim_{i\to\infty} \text{vol}(M_i)/\text{vol}(\text{Pen}^-(M_i, r)) &= 1.
\end{aligned}$$

Examples 1.46 **1)** $(M^n, g) = (\mathbb{R}^n, g_{standard})$ admits a regular exhaustion.

2) Any (M^n, g) with subexponential growth admits a regular exhaustion.

3) The hyperbolic space admits no regular exhaustion. □

Let $\{M_i\}_{i\geq 1}$ be a regular exhaustion and set for $\omega \in {}^b\Omega^n$

$$\langle \mathfrak{m}_i, \omega\rangle := \frac{1}{\operatorname{vol}(M_i)} \int\limits_{M_i} \omega.$$

Then $|\langle \mathfrak{m}_i, \omega\rangle| \leq \sup\limits_x |\omega|_x = {}^b|\omega|$, i.e. $|\mathfrak{m}_i| \leq 1$, the $\mathfrak{m}_i$ belong to the unit ball in $({}^b\Omega^n)^*$. This unit ball is compact in the weak star topology, according to the Banach–Alaoglu theorem, hence the sequence $\{\mathfrak{m}_i\}_i$ has a weak star limit point $\mathfrak{m}$. $\mathfrak{m}$ is then called associated to the regular exhaustion $\{M_i\}_i$.

Proposition 1.47 *Let* $\mathfrak{m}$ *be associated to a regular exhaustion* $\{M_i\}_i$. *Then* $\mathfrak{m}$ *is a fundamental class for* M.

Proof. There remains only to show $\langle \mathfrak{m}, d\psi\rangle = 0$. Let $\Phi_i \in C^\infty(M)$ such that $0 \leq \Phi_i(x) \leq 1$, $\Phi_i = 1$ on M_i, $\Phi_i = 0$ outside $\operatorname{Pen}^+(M_i, 1)$, $|\nabla \Phi_i| \leq 2$. We obtain for $\omega \in {}^b\Omega^n$

$$\left| \int\limits_{M_i} \omega - \int\limits_M \Phi_i \omega \right| \leq (\operatorname{vol}(\operatorname{Pen}^+(M_i, 1)) - \operatorname{vol}(M_i))^b|\omega|,$$

hence

$$\lim_{i\to\infty} \frac{1}{\operatorname{vol}(M_i)} \left(\int\limits_{M_i} \omega - \int\limits_M \Phi_i \omega \right) = 0.$$

Therefore we would be done if we could show

$$\lim_{i\to\infty} \frac{1}{\operatorname{vol}(M_i)} \int\limits_M \Phi_i d\psi = 0.$$

Integration by parts yields

$$\begin{aligned} \int \Phi_i d\psi &= -\int d\Phi_i \wedge \psi, \\ \left| \int \Phi_i d\psi \right| &= \left| \int d\Phi_i \wedge \psi \right| \leq 2(\operatorname{vol}(\operatorname{Pen}^+(M_i, 1)) - \operatorname{vol}(M_i))^b|\psi|, \end{aligned}$$

which implies the assertion. □

Define for $\omega \in {}^{b,1}\mathcal{C}_p(B_0)$, $[Q_{i_1\ldots i_k}(\omega)] \in {}^bH^n(M)$ a (bounded) characteristic class and a regular exhaustion $\{M_i\}_i$ with associated fundamental class $\mathfrak{m}$ the characteristic number

$$Q_{i_1\ldots i_k}(P, \mathrm{comp}(\omega))[\mathfrak{m}] := \langle \mathfrak{m}, [Q_{i_1\ldots i_k}]\rangle := \lim_{i\to\infty} \frac{1}{\mathrm{vol}(M_i)}(Q_{i_1\ldots i_k}).$$

Then, according to proposition 1.47, $Q_{i_1\ldots i_k}(P, \mathrm{comp}(\omega))[\mathfrak{m}]$ is well defined. In particular we obtain in this case avarage Euler numbers, avarage signatures, which are special cases of Roe's (avarage) topological index (cf. [60]). Average characteristic numbers are also considered in [44], [45], [42]. Some simple geometric examples are calculated in [44].

In all cases discussed until now, we restricted to the case of connections (or metrics) with finite p–action or bounded curvature or both. The next proposition shows that this is in fact a restriction.

Proposition 1.48 *Let (M^n, g) be open, complete, satisfying (I), G a compact Lie group, $P = P(M, G)$ a G–principal fibre bundle, $\varrho : G \longrightarrow U(N)$ resp. $O(N)$ a faithful representation, E the associated vector bundle, $p \leq 1$. Then there exist G–connections ω such that their p–action is infinite or the curvature is unbounded or both, respectively.*

Proof. Consider the closed unit ball $\overline{B}_1(0) \subset \mathbb{R}^n$ and set up in $\overline{B}_1(0)$ constant 1–forms ω_{ij}, $\omega_{ij} = -\overline{\omega}_{ji}$ or $\omega_{ij} = -\omega_{ji}$, $1 \leq i, j \leq N$, respectively, such that some $\Omega_{ij} = d\omega_{ij} - \sum\limits_k \omega_{ik} \wedge \omega_{kj}$ are $\neq 0$. Now consider an infinite sequence $U_\nu = U_{\varepsilon_\nu}(x_\nu)$ of closed geodesic balls with pairwise distance $\geq d > 0$, introduce in each geodesic ball normal coordinates $u^1, \ldots, u^n$, $\sum\limits_i (u^i)^2 \leq \varepsilon_\nu$, choose over U_ν orthonormal bases $e_{1,\nu}, \ldots e_{N,\nu}$ and define with respect to these bases local connection matrices $\omega'_{ij,\nu}$ by $\omega'_{ij,\nu}(u_1, \ldots, u_n) := \omega_{ij}$. If $\int\limits_{U_\nu} |\Omega'_{ij,\nu}|^p_x \,\mathrm{dvol}_x(g) = a_\nu \neq 0$, set

$\omega''_{ij,\nu} = (a_\nu + \frac{1}{a_\nu})^{\frac{1}{2}p}\omega'_{ij,\nu}$. This connection over $\bigcup_\nu U_\nu$ is smoothly extendable over the whole of M and gives a connection with $\int_M |\Omega''|_x^p \operatorname{dvol}_x(g) \geq \sum_\nu \int_{U_\nu} |\Omega''|_x^p \operatorname{dvol}_x(g) \geq \sum_\nu 1 = \infty$. Setting $\omega''_{ij,\nu} = \nu \cdot (a_\nu + \frac{1}{a_\nu})^{\frac{1}{2}p} \cdot \omega''_{ij,\nu}$ yields examples for the other cases.
□

The conditions of finite p–action or boundedness can be reformulated in the language of classifying spaces and classifying mappings.

We start with $G = U(N)$. Let $V_{N,k} \xrightarrow{U(k)} G_{N,k}$ be the Stiefel bundle over the complex Grassmann manifold $G_{N,k}$ of all k–subspaces $\subset \mathbb{C}^N$ and S the matrix valued function on $V_{N,k}$ defined by $S(v_1, \dots, v_k) = a_{ij} := (b_{ij})^t$, where $v_1, \dots, v_k$ is a unitrary k–frame, $e_1, \dots, e_N$ the standard base in $\mathbb{C}^N$ and $v_i = \sum_{j=1}^N b_{ij} e_j$.

Proposition 1.49 a) *$\gamma_U = S^* dS$ is a $U(N)$–invariant connection form at $V_{N,k}$.*
b) *Let be $m = (n+1)(2n+1)k^3$. If P is a $U(k)$–principal fibre bundle over a manifold of dimension $\leq n$ and ω a connection form for P, then there exists a smooth bundle morphism $f_P : P \longrightarrow V_{m,k} = P_{n,U(k)}$ such that $f_P^* \gamma = \omega$.*

We refer to [55], p. 564, 568 for the proof. □

γ_0 is called a n–universal connection for $U(k)$. In a similar manner one defines on the real Stiefel bundle $V^r_{m,k} \xrightarrow{O(k)} G^r_{m,r}$ an n–universal $O(k)$–connection γ^r_0.

For an arbitrary compact Lie group G one constructs by means of a faithful representation $G \longrightarrow O(k)$ an n–universal connection γ_G on the n–universal bundle $P_{n,G} \longrightarrow B_{n,G}$ (cf. [20], p. 570).

According to proposition 1.49, we refine the bundle concept and consider instead of a bundle P pairs (P, f_P), $f_P : P \longrightarrow P_{n,G}$ a

C^1–classifying bundle map.

(P, f_P) is called a (p, f)–bundle if $f_p^* \gamma_G \in C^1\mathcal{C}_p(f, p) = \{\omega$ a C^1–connection$| \int |\Omega^\omega|_x^p \,\mathrm{dvol}_x(g) < \infty\}$, i.e. $\int |\Omega^{f_p^* \gamma_G}|_x^p \,\mathrm{dvol}_x(g) < \infty$. In the same manner we define (P, f_P) to be a b–bundle if $f_p^* \gamma_G \in C^1\mathcal{C}_p(B_0)$, i.e. ${}^b|\Omega^{f_p^* \gamma_G}| < \infty$.

The for the applications most interesting case is the case assuming (B_0) and finite p–action.

Hence we assume (B_0) for (M^n, g). (P, f_P) is a (b, p, f)–bundle, if $f_p^* \gamma_G \in {}^{b,1}\mathcal{C}_p^{p,1}(B_0, f, p)$. Two (b, p, f)–bundles (P, f_P), (P, f'_P) are called equivalent if $f_P^* \gamma_G$, ${f'}_P^* \gamma_G$ are contained in the same component of ${}^{b,1}\mathcal{C}_P^{p,1}(B_0, f, p)$.

Assume G to be a subgroup of $U(N)$, $\dim M^n = 2k$. At the level of base spaces we consider classifying maps $f_M : M \longrightarrow B_{n,G}$. A pair (M, f_M) is called a (p, c)–bundle if all classes $f_M^* c_{i_1 \dots i_k}$, $i_1 + \cdots i_k = k$, are elements of $H^{2k,p}(M)$. (M, f_M) is called a (b, c)–bundle if all classes $f_M^* c_{i_1 \dots i_k}$ are elements of ${}^bH^{2k}(M)$. (M, f_M) is called a (b, p, c)–bundle if all classes $f_M^* c_{i_1 \dots i_k}$ are elements of ${}^bH^{2k,p}(M)$. It is clear that a given $f_P : P \longrightarrow P_{n,G}$ uniquely determines $f_M : M \longrightarrow B_{n,G}$.

The case $G \subseteq O(N)$, $\dim M = 4k$, is quite parallel. Then we consider the $p_{i_1 \dots i_k}$, $i_1 + \cdots i_k = k$ and define (M, f_M) to be a (p, po)–bundle if all classes $f_M^* p_{i_1 \dots i_k}$, $i_1 + \cdots i_k = k$ are elements of $H^{4k,p}(M)$. Analogously for (b, po)– and (b, p, po)–bundles (M, f_M).

If we replace $p_{i_1 \dots i_k}$ by the class of Hirzebruch genuss L_k then we get the notion of a (p, L_k)–, (b, L_k)– or (b, p, L_k)–bundle (M, f_M), respectively.

Theorem 1.50 a) *Suppose $G \subset U(N)$, $\dim M = 2k$. (M, g) satisfying (B_0), $p \geq 1$. A (b, p, f)–bundle (P, f_P) defines a unique (b, p)–bundle (M, f_M). If (P, f_P), (P, f'_P) are equivalent then $f_M^* c_{i_1 \dots i_k} = {f'}_M^* c_{i_1 \dots i_k}$ for all $c_{i_1 \dots i_k}$, $i_1 + \cdots + i_k = k$. If additionally $p = 1$ and (M, g) is complete then even the corresponding characteristic numbers coincide.*

b) *Suppose $G \subseteq O(N)$, $\dim M = 4k$, (M, g) satisfying (B_0), $p \geq 1$. A (b, p, f)–bundle (P, f_P) defines a unique (b, p, po)–*

bundle (M, f_M) which is simultaneously a (b, p, L_k)–bundle. If (P, f_P), (P, f'_P) are equivalent then $f_M^* p_{i_1 \ldots i_k} = f'^*_M p_{i_1 \ldots i_k}$ and $f_M^* L_k = f'^*_M L_k$. If additionally $p = 1$ and (M, g) is complete then the corresponding characteristic numbers coincide.

The proof follows immediately from the definitions and theorem 1.14. □

Example 1.51 It it possible that ${}^{b,1}\mathcal{C}_p^{1,1}(B_0, 1, f) = \emptyset$. Let (M^2, g) be an infinitely connected open complete Riemannian manifold with bounded sectional curvature K, $K = K_+ - K_-$. $K_+ = \begin{cases} K, & K \geq 0 \\ 0, & K < 0 \end{cases}$, $K_- = \begin{cases} -K, & K \leq 0 \\ 0, & K > 0 \end{cases}$, Then there holds $\int K_- \,\mathrm{dvol} = \infty$ (cf. [43], theorem 13). In particular $\int |K| \,\mathrm{dvol} = \infty$ which implies $\int |\Omega^\omega(g)| \,\mathrm{dvol} = \infty$. The proof essentially relies on the Gauß–Bonnet theorem (as one would expect) for compact surfaces. But this theorem holds for any metrizable connection in the orthogonal 2–frame bundle $P(M^2, O(2))$ over M^2 ([47], p. 305/306). The sectional curvature K is defined by $\Omega_{1,2} = K \,\mathrm{dvol}$. As conclusion we obtain ${}^{b,1}\mathcal{C}_p(B_0, 1, f) = \emptyset$. □

We conclude this section with some remarks concerning the Novikov conjecture for open manifolds. As very well known, the Novikov conjecture for closed manifolds stimulated many outstanding topologists to prove this and on this road deep results in C^* algebraic topology, C^* K–theory and geometric group theory have been achieved. Hence, the Novikov conjecture has not only its own meaning but even more meaning as a stimulating question.

If M^n is open and we consider the classifying diagram

$$\begin{array}{c} \tilde{M} \\ \downarrow \\ M \xrightarrow[f]{} B_\pi \end{array}$$

and $a \in H^*(B_\pi)$ then

$$\langle L(M) \cdot f^*a, [M] \rangle$$

will not be defined in general. For this reason, Gromov proposes to consider

$$\sigma_a(M) = \langle L(M) \cdot f^*a, [M] \rangle$$

for $a \in H_c^*(B_\pi)$.

Then the NC for open manifolds would mean the "invariance of $\sigma_a(M)$ under proper homotopy equivalences". Probably much more appropriate would be an approach in the sense of our "open category", i.e.

1) everything is uniformly metrized, we have (I), (B_k), uniform triangulations etc.,

2) maps are bounded and uniformly proper, in particular this holds for homotopy equivalences,

3) one works within functional algebraic topology.

Hence one should consider

$$\langle L(M) \cdot f^*a, [M] \rangle \quad \text{with} \quad L(M) \in L_p, f^*a \in L_q.$$

Of particular meaning would be the cases

$$L(M) \in {}^bH^*(M) \quad \text{and} \quad a \in H^{*,1}(B_\pi) \tag{1.51}$$

or

$$L(M) \in H^{*,2}(M) \quad \text{and} \quad a \in H^{*,2}(B_\pi), \tag{1.52}$$

respectively. If we suppose (M, g) satisfying (B_0) then automatically $L(M) \in {}^bH^*(M)$. (B_0) does not restrict to topological type since any open manifold admits a metric g satisfying even (B_∞) and (I).

In the second case one should additionally assume

$$\inf \sigma_e(\Delta_*(M, g)|_{(\ker \Delta_*)^\perp}) > 0, \tag{1.53}$$

i.e. there is a spectral gap of Δ_* above zero. In this case $H^{*,2} = \mathcal{H}^{*,2} = L_2$–harmonic forms, $C^{*,2}$, $C_{*,2}$ are L_2–complexes and form an L_2–Poincare complex. Every L_2–(co–)homology class can be represented by an L_2–harmonic (co–)cycle. Bordism of L_2–Poincare complexes can be defined easily.

We proved in [34] that (1.53) is invariant under bounded uniformly proper homotopy equivalences. W.l.o.g., classifying maps can be assumed to be bounded and uniformly proper,

$$M^n \longrightarrow B_\pi = M^n \cup \text{cells}.$$

We present now 3 versions of NC (for open manifolds).

1. Version. In the class of open oriented manifolds (M^n, g), $g \in {}^{b,2}\mathcal{M}^{2,2}(B_0, 2, f)$ with $\inf \sigma_e(\Delta_*(g)|_{(\ker \Delta_*)^\perp}) > 0$ is

$$\begin{aligned} &\langle L(M) f^* a, [M]\rangle, \quad a \in H^{*,2}(B_\pi), \quad f \text{ bounded and} \\ &\text{uniformly proper classifying map, invariant under} \\ &\text{bounded and uniformly proper homotopy equivalences.} \end{aligned} \tag{NCO1}$$

Criticism. This version should hold only in very restricted cases. Starting point in the compact case is the equality

$$\sigma(M^{4k}) = \int L_k(M) \tag{1.54}$$

where the l.h.s. is a priori a homotopy invariant and the r.h.s. is a certain characteristic number. The L_2–version of (1.54) is already wrong in simple open cases. Let (M^{4k}, g) be an open manifold with cylindrical ends, i.e. $(M^{4k}, g) = (M'^{4k} \cup \partial M'^{4k} \times [0, \infty[, g)$ with $g|_{\partial M'^{4k} \times [0,\infty[} \cong g|_{\partial M'} + dt^2$. Then it is well known that

$$\sigma(M^{4k}) = \sigma_{L_2}(M^{4k}) = \int L_k(M) - \eta(\partial M'^{4k}),$$

i.e. already the starting point which guarantees the invariance of $L(M)$ in the simplest case is wrong. Hence the first version of NC for open manifolds makes sense only for that classes of open manifolds for which

$$\sigma_{L_2}(M^n) = \int L(M)$$

in the case $n = 4k$ holds.

2. Version of NC, relative version. Fix (M^n, g) and suppose $M_1, M_2 \in \text{gen}^b\text{comp}_{L,iso,rel}(M, g)$

$$\begin{aligned} M_1 \setminus K_1 &\cong M \setminus K \\ M_2 \setminus K_2 &\cong M \setminus K \end{aligned}$$

with a Riemannian collar at ∂K_1, ∂K_2, ∂K. Then we define

$$\begin{aligned} &\sigma(M_i, M) := \int_{K_i} L(M_i) - \int_K L(M) \\ &\sigma(M_1, M_2) := \sigma(M_1, M) - \sigma(M_2, M) \\ &= \int_{K_1} L(M_1) - \int_{K_2} L(M_2) \\ &= \sigma(K_1 \cup K_2) \\ &= \int_{K_1} L(M_1) - \eta(\partial K_1) - \left(\int_{K_2} L(M_2) - \eta(\partial K_2) \right) \\ &= \sigma(K_1) - \sigma(K_2). \end{aligned}$$

The relative NC becomes

$$\int_{K_1} L(M_1) f_1^* a = \int_{K_2} L(M_2) f_2^* a \qquad \text{(NCO2)}$$

if there exist $\Phi_{12} : M_1 \longrightarrow M_2$, $\Phi_{21} : M_2 \longrightarrow M_1$, bounded, uniformly proper, $\Phi_{21}\Phi_{12} \sim \text{id}M_1$, $\Phi_{12}\Phi_{21} \sim \text{id}M_2$ bounded and u.p. and $\Phi_{21}\Phi_{12} = \text{id}$ outside $\tilde{K}_1 \subset M_1$, $\Phi_{12}\Phi_{21} = \text{id}$ outside $\tilde{K}_2 \subset M_2$ and $f_i : M_i \longrightarrow B_\pi$ are bounded and u.p. classifying maps, $a \in H^*(B_\pi)$.

This relative version has the advantage that we require no conditions for (M^n, g) and NC splits to NC for the generalized Lipschitz components (cf. [27], [33]).

3. Version of NC. Consider (M^n, g) open, oriented with (B_0), $r_{\text{inj}} > 0$, embeddings $N^{4k} \hookrightarrow M^n \times \mathbb{R}^j$ with trivial normal bundle and bounded second fundamental form such that $PD[N] = f^*a$, $a \in H^{n-4k,1}(B_\pi)$, $f : M^n \longrightarrow B_\pi$ bounded and uniformly proper classifying map and such that $\sigma_{L_2}(N^{4k})$ is defined (i.e. $\dim \mathcal{H}^{2k,2}(N) < \infty$).

Then the number $\sigma_a(M) := \sigma_{L_2}(N^{4k})$ is invariant under bounded and uniformly proper homotopy invariants.

(NCO3)

How to attack these conjectures will be the content of a forthcoming investigation.

2 Index theorems for open manifolds

Let (M^n, g) be closed, oriented, $(E, h_E), (F, h_F) \longrightarrow M^n$ smooth vector bundles, $D : C^\infty(E) \longrightarrow C^\infty(F)$ an elliptic differential operator. Then $L_2(E) \supset \mathcal{D}_{\overline{D}} \xrightarrow{\overline{D}} L_2(F)$ is Fredholm, i.e. there exists $P : L_2(F) \longrightarrow L_2(E)$ s.t. $PD - \mathrm{id} = K_1$, $DP - \mathrm{id} = K_2$, K_i integral operators with C^∞ kernel $\mathcal{K}_i$ and hence compact. It follows $\dim\ker D, \dim\operatorname{coker} D < \infty$, $\mathrm{ind}_a D = \dim\ker D - \dim\operatorname{coker} D$ is well defined and there arises the question to calculate $\mathrm{ind}_a D$. The answer is given by the seminal Atiyah–Singer index theorem

Theorem 2.1

$$\mathrm{ind}_a D = \mathrm{ind}_t D,$$

where

$$\mathrm{ind}_t D = \langle \mathrm{ch}\ \sigma(D)\mathcal{T}(M), [M] \rangle.$$

□

Assume now (M^n, g) open, E, F, D as above. K_1, K_2 still exist as operators with a smooth kernel where in good cases one can achieve that the support of $\mathcal{K}_i$ is located near the diagonal.

But there arise several troubles.

1) If K_i bounded is achieved then K_i must not be compact.

2) If K_i would be compact then $\text{ind}_a D$ would be defined.

3) If $\text{ind}_a D$ would be defined then $\text{ind}_t D$ must not be defined.

4) If $\text{ind}_a D$, $\text{ind}_t d$ (as above) would be defined then they must not coincide. There are definite counterexamples.

There are 3 ways out from this difficult situation.

1) One could ask for special conditions in the open case under which an elliptic D is still Fredholm, then try to establish an index formula and finally present applications. These conditions could be conditions on D, on M and E or a combination of both. In [2] the author formulates an abstract (and very natural) condition for the Fredholmness of D and assumes nothing on the geometry. But in all substantial applications this condition can be assured by conditions on the geometry. The other extreme case is that discussed in [22], [50], [48], where the authors consider the L_2–index theorem for locally symmetric spaces. Under relatively restricting conditions concerning the geometry and topology at infinity the Fredholmness and an index theorem are proved in [11] and [12].

2) One could generalize the notion of Fredholmness (using other operator algebras) and then establish a meaningful index theory with applications. The discussion of these both approaches will be the content of this section.

3) Another approach will be relative index theory which is less restrictive concerning the geometrical situation (compared with the absolute case) but its outcome are only statements on the relative index, i.e. how much the analytical properties of D differ from those of D'. This approach will be discussed in detail in chapter V.

4) For open coverings $(\tilde{M}, \tilde{g})$ of closed manifolds (M^n, g) and lifted D there is an approach which goes back to Atiyah, (cf. [4]). This has been further elaborated by Cheeger, Gromov and others. The main point is that all considered (Hilbert–) modules are modules over a von Neumann algebra and one replaces the usual trace by a von Neumann trace. We will not dwell on this approach since there is a well established highly elaborated theory. Moreover special features of openess come not

into. The openess is reflected by the fact that all modules under consideration are modules over the von Neumann algebra $\mathcal{N}(\pi)$, $\pi = \text{Deck}(\tilde{M} \longrightarrow M)$. We refer to the very comprehensive representation [46].

This section is a brief review of absolute index theorems under additional strong assumptions. It shows that these approaches are successful only in special situations. In chapter V we will establish very general relative index theorems.

We start with the first approach and with the question which elliptic operators over open manifolds are Fredholm in the classical sense above. Let (M^n, g) be open, oriented, complete, $(E, h) \longrightarrow (M^n, g)$ be a Hermitean vector bundle with involution $\tau \in \text{End}\,(E)$, $E = E^+ \oplus E^-$, $D : C^\infty(E) \longrightarrow C^\infty(E)$ an essentially self–adjoint first order elliptic operator satisfying $D\tau + \tau D = 0$. We denote $D^\pm = D|_{C^\infty(E^\pm)}$. Then we can write as usual

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} : \begin{array}{c} C^\infty(E^+) \\ \oplus \\ C^\infty(E^-) \end{array} \longrightarrow \begin{array}{c} C^\infty(E^+) \\ \oplus \\ C^\infty(E^-) \end{array} . \tag{2.1}$$

The index $\text{ind}_a D$ is defined as

$$\begin{aligned} \text{ind}_a D \;&:=\; \text{ind}_a D^+ := \dim \ker D^+ - \dim \text{coker}\, D^+ \\ &=\; \dim \ker D^+ - \dim \ker D^- \end{aligned} \tag{2.2}$$

if these numbers would be defined. Denote by $\Omega^{2,i}(E, D)$ the Sobolev space of order i of sections of E with D as generating differential operator. We essentially follow [2].

Proposition 2.2 *The following statements are equivalent*

a) *D is Fredholm.*

b) $\dim \ker D < \infty$ *and there is a constant $c > 0$ such that*

$$|D\varphi|_{L_2} \geq c \cdot |\varphi|_{L_2}, \quad \varphi \in (\ker D)^\perp \cap \Omega^{2,1}(E, D), \tag{2.3}$$

where $(\ker D)^\perp \equiv \mathcal{H}^\perp$ *is the orthogonal complement of* $\mathcal{H} = \ker D$ *in* $L_2(E)$.

c) *There exists a bounded non–negative operator* $P : \Omega^{2,2}(E,D) \longrightarrow L_2(E)$ *and bundle morphism* $R \in C^\infty(\mathrm{End}\, E)$, *R positive at infinity (i.e. there exists a compact* $K \subset M$ *and a* $k > 0$ *s. t. pointwise on* $E|_{M\setminus K}$, $R \geq k$*), such that on* $\Omega^{2,2}(E,D)$

$$D^2 = P + R. \tag{2.4}$$

d) *There exist a constant* $c > 0$ *and compact* $K \subset M$ *such that*

$$|D\varphi|_{L_2} \geq c \cdot |\varphi|, \quad \varphi \in \Omega^{2,1}(E,D), \quad \mathrm{supp}\,(\varphi) \cap K = \emptyset. \tag{2.5}$$

□

The main task now is to establish a meaningful index theorem. This has been performed in [2].

Theorem 2.3 *Let* (M^n, g) *be open, complete, oriented,* $(E,h,\tau) = (E^+ \oplus E^-, h) \longrightarrow (M^n, g)$ *a* $\mathbb{Z}_2$*–graded Hermitean vector bundle and* $D : C_c^\infty(E) \longrightarrow C_c^\infty(E)$ *first order elliptic, essentially self–adjoint, compatible with the* $\mathbb{Z}_2$*–grading (i.e. supersymmetric),* $D\tau + \tau D = 0$*. Let* $K \subset M$ *be a compact subset such that 2.2 a) for* K *is satisfied, and let* $f \in C^\infty(M, \mathbb{R})$ *be such that* $f = 0$ *on* $U(K)$ *and* $f = 1$ *outside a compact subset. Then there exists a volume density* ω *and a contribution* I_ω *such that*

$$\mathrm{ind}_a \overline{D}^+ = \int_M (\omega(1 - f(x))\, \mathrm{dvol}_x(g) + I_\omega, \tag{2.6}$$

where ω *has an expression locally depending on* D *and* I_ω *depends on* D *and* f *restricted to* $\Omega = M \setminus K$. □

Until now the differential form $\omega\, \mathrm{dvol}_x(g)$ is mystery. One would like to express it by well known canonical terms coming e.g. from the Atiyah–Singer index form $\mathrm{ch}\, \sigma(D^+) \cup \mathcal{T}(M)$, where $\mathcal{T}(M)$ denotes the Todd genus of M. In fact this can be done.

Index Theorem 2.4 *Let* (M^n, g) *be open, oriented, complete,* $(E,h,\tau) \longrightarrow (M^n, g)$ *a* $\mathbb{Z}_2$*–graded Hermitean vector bundle,* $D :$

$C_c^\infty(E) \longrightarrow C_c^\infty(E)$ a first order elliptic essentially self-adjoint supersymmetric differential operator, $D\tau + \tau D = 0$, which shall be assumed to be Fredholm. Let $K \subset M$ compact such that 2.2 d) is satisfied. Then

$$\mathrm{ind}_a D^+ = \int\limits_K \mathrm{ch}\ \sigma(D^+) \cup \mathcal{T}(M) + I_\Omega, \tag{2.7}$$

where $\mathrm{ch}\ \sigma(D^+) \cup \mathcal{T}(M)$ is the Atiyah–Singer index form and I_Ω is a bounded contribution depending only on $D|_\Omega$, $\Omega = M \setminus K$.
□

Remarks 2.5 a) As we already mentioned, $\mathbb{Z}_2$–graded Clifford bundles and associated generalized Dirac operators D such that in $D^2 = \Delta^E + \mathcal{R}$, $\mathcal{R} \geq c \cdot \mathrm{id}$, $c > 0$, outside some compact $K \subset M$, yield examples for theorem 2.3. A special case is the Dirac operator over a Riemannian spin manifold with scalar curvature $\geq c > 0$ outside $K \subset M$.

b) Much more general perturbations than compact ones will be considered in section V 1. □

The other case of a very special class of open manifolds are coverings $(\tilde{M}, \tilde{g})$ of a closed manifold (M^n, g). Let $E, F \longrightarrow (M^n, g)$ be Hermitean vector bundles over the closed manifold (M^n, g). $D : C^\infty(E) \longrightarrow C^\infty(F)$ be an elliptic operator, $(\tilde{M}, \tilde{g}) \longrightarrow (M, g)$ a Riemannian covering, $\tilde{D} : C_c^\infty(\tilde{E}) \longrightarrow C_c^\infty(\tilde{F})$ the corresponding lifting and $\Gamma = \mathrm{Deck}\ (\tilde{M}^n, \tilde{g}) \longrightarrow (M^n, g)$. The actions of Γ and $\tilde{D}$ commute. If $P : L_2(\tilde{M}, \tilde{E}) \longrightarrow \mathcal{H}$ is the orthogonal projection onto a closed subspace $\mathcal{H} \subset L_2(\tilde{M}, \tilde{E})$ then one defines the Γ–dimension $\dim_\Gamma \mathcal{H}$ of $\mathcal{H}$ as

$$\dim_\Gamma \mathcal{H} := \mathrm{tr}_\Gamma P,$$

where tr_Γ denotes the von Neumann trace and $\mathrm{tr}_\Gamma P$ can be any real number ≥ 0 or $= \infty$.

If one takes $\mathcal{H} = \mathcal{H}(\tilde{D}) = \ker \tilde{D} \subset L_2(\tilde{E})$, $\mathcal{H}^* = \mathcal{H}(\tilde{D}^*) = \ker(\tilde{D}^*) \subset L_2(\tilde{F})$ then one defines the Γ–index $\mathrm{ind}_\Gamma \tilde{D}$ as

$$\mathrm{ind}_\Gamma \tilde{D} := \dim_\Gamma \mathcal{H}(\tilde{D}) - \dim_\Gamma \mathcal{H}(\tilde{D}^*).$$

Atiyah proves in [4] the following main

Theorem 2.6 *Under the assumptions above there holds*

$$\mathrm{ind}_a D = \mathrm{ind}_\Gamma \tilde{D}.$$

□

It was this theorem which was the orign of the von Neumann analysis as a fastly growing area in geometry, topology and analysis. Moreover, the proof of theorem 2.3 is strongly modeled by that of 2.6. Another very important special case which is related to the case above of coverings are locally symmetric spaces of finite volume. There is a vast number of profound contributions, e. g. [7], [22], [48], [50], [51]. We do not intend here to give a complete overview for reasons of space. But we will sketch the main features and main achievements of these approaches.

Let G be semisimple, noncompact, with finite center, $K \subset G$ maximal compact, $\tilde{X} = G/K$ a symmetric space of noncompact type, $\Gamma \subset G$ discrete, torsion free and $\mathrm{vol}(\Gamma\backslash G) < \infty$. Then $X = \Gamma\backslash\tilde{X} = \Gamma\backslash G/K$ is a locally symmetric space of finite volume. If V_E, V_F are unitary K–modules then we obtain homogeneous vector bundles $\tilde{E} = G/K \times_K V_E \longrightarrow G/K = \tilde{X}$, $\tilde{F} = G/K \times_K V_F \longrightarrow G/K = \tilde{X}$, over $\tilde{X}$ and corresponding bundles $E, F \longrightarrow X$ over X. A G–invariant elliptic differential operator $\tilde{D} : C^\infty(\tilde{E}) \longrightarrow C^\infty(\tilde{F})$ descends to an elliptic operator $D : C^\infty(E) \longrightarrow C^\infty(F)$. There arise the following natural questions: to describe the $\tilde{D}$ in question, to establish a formula for the analytical index, to calculate the index via a topological index and an index theorem. We indicate (partial) answers given by Barbasch, Connes, Moscovici and Müller.

Denote by $R(k)$ the right regular representation $R(k)f(g) = f(gk)$, $\tau_E : K \longrightarrow U(V_E)$. Then $k \longrightarrow R(k) \otimes \tau_E(k)$ acts on $C^\infty(G) \otimes V_E$. We identify $C^\infty(\tilde{E})$ with $(C^\infty(G) \otimes V_E)^K$, similarly $L_2(\tilde{E})$ with $(L_2(G) \otimes V_E)^K$. If $\mathfrak{G}$ is the Lie algebra of G, $\mathfrak{G}_c$ its complexification, $\mathfrak{U}(\mathfrak{G})$ the universal enveloping algebra of $\mathfrak{G}$, $\tau_E : K \longrightarrow U(V_E)$, $\tau_F : K \longrightarrow U(V_F)$ are unitary representations then $(\mathfrak{U}(\mathfrak{G}) \otimes \mathrm{Hom}\,(V_E, V_F))^K$ shall denote the subspace of all elements in $\mathfrak{U}(\mathfrak{G}) \otimes \mathrm{Hom}\,(V_E, V_F)$ which are fixed under $k \longrightarrow Ad_G(k) \otimes \tau_E(k^{-1})^t \otimes \tau_F(k)$. Let $d = \sum_i X_i \otimes A_i \in (\mathfrak{A}(\mathfrak{G}) \otimes \mathrm{Hom}\,(V_E, V_F))^K$. Then $\tilde{D} = \sum_i R(X_i) \otimes A_i$ defines a differential operator $\tilde{D} : C^\infty(\tilde{E}) \longrightarrow C^\infty(\tilde{F})$ commuting with the action of G. We state without proof the simple

Lemma 2.7 a) *Any G–invariant differential operator $\tilde{D} : C^\infty(\tilde{E}) \longrightarrow C^\infty(\tilde{F})$ is of the form*

$$\tilde{D} = \sum_i R(X_i) \otimes A_i \tag{2.8}$$

above.

b) *The formal adjoint $\tilde{D}^*$ corresponds to*

$$d^* = \sum_i X_i^* \otimes A_i^* \in (\mathfrak{U}(\mathfrak{G}) \otimes \mathrm{Hom}\,(E, F))^K,$$

where $x \longrightarrow x^*$ denotes the conjugate–linear anti–automorphisms of $\mathfrak{U}(\mathfrak{G})$ such that $x^* = -\overline{x}$, $x \in \mathfrak{G}_c$. □

For a unitary representation $\pi : G \longrightarrow U(\mathcal{H}(\pi))$ and $d = \sum_i X_i \otimes A_i \in (\mathfrak{U}(\mathfrak{G}) \otimes \mathrm{Hom}\,(V_E, V_F))^K$ define $\pi(d) : \mathcal{H}(\pi)_\infty \otimes V_E \longrightarrow \mathcal{H}(\pi)_\infty \otimes V_F$ by

$$\pi(d) := \sum_i \pi(X_i) \otimes A_i.$$

Here $\mathcal{H}(\pi)_\infty$ denotes the space of C^∞–vectors of π. $\pi(d)$ induces an operator $d_\pi : (\mathcal{H}(\pi) \otimes V_E)^K \longrightarrow (\mathcal{H}(\pi) \otimes V_F)^K$.

Proposition 2.8 *Suppose that d is elliptic. Then*

$$\ker d_\pi = \{u \in (\mathrm{Hom}\,(\pi)_\infty \otimes V_E)^K \mid d_\pi u = 0\}$$

coincides with the orthogonal complement of

$$\mathrm{im}\; d_\pi^* = \{d_\pi^* v \mid v \in (\mathcal{H}(\pi)_\infty \otimes V_F)^K\}$$

in $(\mathcal{H}(\pi) \otimes V_E)^K$. □

Corollary 2.9 a) $\ker d_\pi$ *is closed in* $(\mathcal{H}(\pi) \otimes E)^K$.
b) *The closure of* d_π^* *coincides with the Hilbert space adjoint of* d_π. □

Corollary 2.10 *Suppose that d is elliptic and*

$$\pi = \int_\Lambda^\ominus \pi_\lambda \, d\lambda, \quad \mathcal{H}(\pi) = \int_\Lambda^\ominus \mathcal{H}(\pi_\lambda) \, d\lambda$$

is an integral decomposition of π. *Then*

$$\ker d_\pi = \int_\Lambda^\ominus \ker d_{\pi_\lambda} \, d\lambda. \tag{2.9}$$

□

Now we come to the main part of our present discussions, the locally symmetric case. Identifying $L_2(E)$ with $(L_2(\Gamma\backslash G) \otimes V_E)^K$, and taking into consideration the decompositions

$$R^\Gamma = R_d^\Gamma \oplus R_c^\Gamma, \quad L_2(\Gamma\backslash G) = L_{2,d}(\Gamma\backslash G) \oplus L_{2,c}(\Gamma\backslash G)$$

of the right quasi–regular representation R^Γ of G on $L_2(\Gamma\backslash G)$, we obtain the decomposition

$$\begin{aligned} L_2(E) &= L_{2,d}(E) \oplus L_{2,c}(E), \\ L_{2,d}(E) &= (L_{2,d}(\Gamma\backslash G) \otimes V_E)^K, \\ L_{2,c}(E) &= (L_{2,c}(\Gamma\backslash G) \otimes V_E)^K, \end{aligned}$$

similarly for $F = \Gamma\backslash\tilde{F}$.
Consider now the operators $D = d_{R^\Gamma}$ and $D_d = d_{R_d^\Gamma} : C_c^\infty(E) \longrightarrow C_c^\infty(F)$.

Theorem 2.11 *Under the assumptions above (on* G, K, Γ*),*

$$\ker D = \ker D_d \tag{2.10}$$

and

$$\dim \ker D < \infty. \tag{2.11}$$

Denote by $\tilde{G}_d^\Gamma$ the set of all equivalence classes of irreducible unitary representation π of G whose multiplicity $m_\Gamma(\pi)$ in R_d^Γ is nonzero. In particular $L_{2,d}(\Gamma\backslash G) = \sum_{\pi\in\tilde{G}_d^\Gamma} m_\Gamma(\pi)\mathcal{H}(\pi)$.

Theorem 2.12 *Let $K \subset G$ be maximal compact, $\Gamma \in G$ discrete and torsion free, $\tau_E : K \longrightarrow V_E$, $\tau_F : K \longrightarrow V_F$ unitary representations, $\tilde{E} = G/K \times_K V_E$, $\tilde{F} = G/K \times_K V_F$, $E = \Gamma\backslash\tilde{E}$, $F = \Gamma\backslash\tilde{F}$ and $D = d_{R^\Gamma}$ a corresponding locally invariant elliptic differential operator acting between $L_2(E)$ and $L_2(F)$. Then*

$$\mathrm{ind}_a D = \dim \ker D - \dim \ker D^*$$

is well defined and

$$\mathrm{ind}_a D = \sum_{\pi\in\tilde{G}_d^\Gamma} m_\Gamma(\pi)(\dim(\mathcal{H}(\pi)\otimes E)^K - \dim(\mathcal{H}(\pi)\otimes F)^K). \tag{2.12}$$

□

Corollary 2.13 *Let $X = \Gamma\backslash G/K$ be a locally symmetric space of negative curvature with finite volume and $L_2(E) \supset \mathcal{D}_D \xrightarrow{D} L_2(F)$ a locally symmetric elliptic differential operator then* ind D *is defined and depends only on the K–modules $K \longrightarrow U(V_E), U(V_F)$ which define $\tilde{E}$, $\tilde{F}$, $E = \Gamma\backslash\tilde{E}$, $F = \Gamma\backslash\tilde{F}$.* □

The value of the formula in theorem 2.12 is very limited since in general the $m_\Gamma(\pi)$ are not known. Hence there arises the task to find a meaningful expression for it. This has been done with great success e. g. in [22] and [51], [52] where they essentially restrict to generalized Dirac operators. To be more precise, we must briefly recall what is a manifold with cusps. Here we densely follow [50]. Let G be a semisimple Lie group with finite center, $K \subset G$ a maximal compact subgroup. P_a split rank one parabolic subgroup of G with split component A, $P = UAM$ the corresponding Langlands decomposition, where U is the unipotent radical of P, A a $\mathbb{R}$–split torus of dimension one and M centralizes A. Set $S = UM$ and let Γ be a discrete uniform torsion free subgroup of S. Then $Y = \Gamma\backslash\tilde{Y} = \Gamma\backslash G/K$ is called a complete cusp of rank one. Put $K_M = M \cap K$, K_M is a maximal compact subgroup of M. If $X_M = M/K_M$ there is a canonical diffeomorphism $\tilde{\xi}$: $\mathbb{R}_+ \times U \times X_M \longrightarrow \tilde{Y}$. Set for $t \geq 0$ $\tilde{Y}_t = \tilde{\xi}([t, \infty[\times U \times X_M)$ and call $Y_t = \Gamma\backslash\tilde{Y}_t$ a cusp of rank one. Another, even more explicit description is given as follows. Let $\Gamma_M = M \cap (U\Gamma)$, $Z = S/S \cap K$. Then there is a canonical fibration $P : \Gamma\backslash Z \longrightarrow \Gamma_M\backslash X_M$ with fibre $\Gamma \cap U\backslash U$ a compact nilmanifold and a canonical diffeomorphism $\xi : [t, \infty[\times\Gamma\backslash Z \xrightarrow{\cong} Y_t$. The induced metric on $[t, \infty[\times\Gamma_2\backslash Z$ looks locally as $ds^2 = dr^2 + dx^2 + e^{-br} du_\lambda^2(x) + e^{-4br} du_{2\lambda}^2(x)$, where $|b| = \lambda$, dx^2 is the invariant metric on X_M induced by restriction of the Killing form.

Now a complete Riemannian manifold is called a manifold with cusps of rank one if X has a decomposition $X = X_0 \cup X_1 \cup \cdots \cup X_s$ such that X_0 is a compact manifold with boundary, for $i, j \geq 1$, $i \neq j$ holds $X_i \cap X_j = \emptyset$ and each X_j, $j \geq 1$, is a cusp of rank one. The first general statement for generalized Dirac operators on rank one cusps manifold is

Theorem 2.14 *Let X be a rank one cusp manifold, $(E, h, \nabla, \cdot) \longrightarrow (X, g_X)$ a Clifford bundle and D its corresponding generalized Dirac operator. Then D is essentially self–adjoint and*

$$\dim(\ker \overline{D}) < \infty. \tag{2.13}$$

The spectrum of $H = \overline{D}^2$ consists of a point spectrum and an absolutely continuous spectrum. If $L_2(E) = L_{2,d}(E) \oplus L_{2,c}(E)$ is the corresponding decomposition of $L_2(E)$ and $H_d = H|_{L_{2,d}(E)}$ then for $t > 0$

$$e^{-zH_d} \textit{ is of trace class.} \tag{2.14}$$

□

As we mentioned after corollary 2.13, the main task, main objective consists in the case of a $\mathbb{Z}_2$–grading to get an expression for $\mathrm{ind}_a \overline{D}$. For the sake of simlicity we restrict to spaces

$X = X_0 \cup Y_1$ as above with one cusp Y_1, $Y_0 \cup Y_1 = Y = \Gamma\backslash G/K$. Let $(E = E^+ \oplus E^-, h, \nabla, \cdot) \longrightarrow (Y, g)$ be a Z_2-graded Clifford bundle such that $E^{\pm}|_{Y_1} = \Gamma\backslash\tilde{E}^{\pm}$, where $\tilde{E}^{\pm}$ are homogeneous vector bundles over G/K and let $D^+ : C^\infty(Y, E^+_+) \longrightarrow C^\infty(Y, E^-)$ the corresponding generalized Dirac operator. We recall $K_M = M \cap K$, $X_M = M/K_M$. D^+ induces an elliptic differential operator $D_0^+ : C^\infty(\mathbb{R}_+ \times \Gamma_M\backslash X_M, E_M^+) \longrightarrow C^\infty(\mathbb{R}_+ \times \Gamma_M\backslash X_M, E_M^-)$, where $E_M^{\pm}$ are locally homogenous vector bundles over $\Gamma_M\backslash X_M$. From this come a self-adjoint differential operator $D_M : C^\infty(\Gamma_M\backslash X_M, E_M^+) \longrightarrow C^\infty(\Gamma_M\backslash X_M, E_M^-)$ and a bundle isomorphism $\beta : E_M^+ \longrightarrow E_M^-$ such that $D_0^+ = \beta\left(r\frac{\partial}{\partial r} + D_M\right)$. We set $\tilde{D}_M = D_M + \frac{m}{2}\mathrm{id}$, $m = \dim u_\lambda|\lambda| + 2\dim u_{2\lambda}|\lambda|$, λ the unique simple root of the pair (P, A).

W. Müller then established in [50] the following general index theorem for a locally symmetric graded Dirac operator.

Theorem 2.15 *Assume* $\ker \tilde{D}_M = \{0\}$, *let* $\eta(0)$ *be the eta invariant of* $\tilde{D}_M$ *and* ω_{D^+} *the index form of* D^+. *Then*

$$\mathrm{ind}_a D^+ = \int_X \omega_{D^+} + \mathcal{U} + \frac{1}{2}\eta(0), \tag{2.15}$$

where the term $\mathcal{U}$ *is essentially given by the value of an L–series at zero and an expression in the scattering matrix at zero.* □

Finally, application of an elaborated version of theorem 2.15 allows to prove the famous Hirzebruch conjecture for Hilbert modular varieties. This has been done by W. Müller in [51].

There is another approach to Fredholmness by Gilles Carron, which relies on an inequality quite similar to 2.2 d).

Let $(E, h, \nabla, \cdot) \longrightarrow (M^n, g)$ be a Clifford bundle over the complete Riemannian manifold (M^n, g) and $D : C^\infty(E) \longrightarrow C^\infty(E)$ the associated generalized Dirac operator. D is called non-parabolic at infinity if there exists a compact set $K \subset M$ such that for any open and relative compact $U \subset M \setminus K$ there exists a constant $C(U) > 0$ such that

$$C(U)|\varphi|_W \le |D\varphi|_{L_2(E|_{M\setminus K})} \text{ for all } \varphi \in C_c^\infty(E|_{M\setminus K}). \tag{2.16}$$

To exhibit the consequences of this inequality, we establish another characterization of it.

Proposition 2.16 *Let $(E, h, \nabla, \cdot) \longrightarrow (M^n, g)$ and D as above and let $W(E)$ be a Hilbert space of sections such that*
a) *$C_c^\infty(E)$ is dense in $W(E)$ and*
b) *the injection $C_c^\infty(E) \hookrightarrow \Omega^{2,1}_{\text{loc}}(E, D)$ extends continuously to $W(E) \longrightarrow \Omega^{2,1}_{\text{loc}}(E, D)$.*
Then $D : W(E) \longrightarrow L_2(E)$ is Fredholm if and only if there exist a compact $K \subset M$ and a constant $C(K) > 0$ such that

$$C(K) \cdot |\varphi|_W \leq |D\varphi|_{L_2(E|_{M\setminus K})} \text{ for all } \varphi \in C_c^\infty(E|_{M\setminus K}). \quad (2.17)$$

□

Remark 2.17 The norm $\varphi \longrightarrow |D\varphi|_{L_2} = \mathcal{N}(\varphi)$ above is equivalent to the norm

$$\mathcal{N}_{\overline{U}(K)}(\cdot), \mathcal{N}_{\overline{U}(K)}(\varphi)^2 = |\varphi|^2_{L_2(E|_{\overline{U}(K)})} + |D\varphi|^2_{L_2(E)}. \quad (2.18)$$

Corollary 2.18 *$D : C^\infty(E) \longrightarrow C^\infty(E)$ is non–parabolic at infinity if and only if there exists a compact $K \subset M$ such that the completion of $C_c^\infty(E)$ w. r. t. $\mathcal{N}_K(\cdot)$,*

$$\mathcal{N}_K(\varphi)^2 = |\varphi|^2_{L_2(E|_K)} + |D\varphi|^2_{L_2} \quad (2.19)$$

yields a space $W(E)$ such that the injection $C_c^\infty(E) \longrightarrow \Omega^{2,1}_{\text{loc}}(E, D)$ continuously extends to $W(E)$. □

The point now is that we know if D is non–parabolic at infinity then $D : W(E) \longrightarrow L_2(E)$ is Fredholm. We emphasize, this does not mean $L_2(E) \supset \mathcal{D}_D \longrightarrow L_2(E)$ is Fredholm. We get a weaker Fredholmness, not the desired one. But in certain cases this can be helpful too.

Suppose again a $\mathbb{Z}_2$–grading of E and D, $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$, $L_2(E) = L_2(E^+) \oplus L_2(E^-)$, $W(E) = W(E^+) \oplus W(E^-)$. Following Gilles Carron, we now define the extended index $\text{ind}_e D^+$

as

$$\begin{aligned}\text{ind}_e D^+ &:= \dim\ker{}_W D^+ - \dim\ker{}_{L_2} D^- \\ &= \dim\{\varphi \in W(E^+)\,|\,D^+\varphi = 0\} - \\ &\quad - \dim\{\varphi \in L_2(E^-)\,|\,D^-\varphi = 0\}. \qquad (2.20)\end{aligned}$$

If we denote $h_\infty(D^+) := \dim(\ker_W D^+ / \ker_{L_2} D^+)$ then we can (2.20) rewrite as

$$\begin{aligned}\text{ind}_e D^+ &= h_\infty(D^+) + \text{ind}_{L_2} D^+ \\ &= h_\infty(D^+) + \dim\ker{}_{L_2} D^+ - \dim\ker{}_{L_2} D^-. \qquad (2.21)\end{aligned}$$

The most interesting question now are applications and examples. For D = Gauß–Bonnet operator, there are in fact good examples (cf. [12]). For the general case it is not definitely clear, is non–parabolicity really a practical sufficient criterion for Fredholmness since in concrete cases it will be very difficult it to establish. In some well known standard cases which have been presented by Carron and which we will discuss now it is of great use.

Proposition 2.19 *Let $D : C^\infty(E) \longrightarrow C^\infty(E)$ be a generalized Dirac operator and assume that outside a compact $K \subset M$ the smallest eigenvalue $\lambda_{\min}(x)$ of $\mathcal{R}_x$ in $D^2 = \nabla^*\nabla + \mathcal{R}$ is ≥ 0. Then D is non–parabolic at infinity.* □

We obtain from proposition 2.18

Corollary 2.20 *Assume the hypothesis of 2.18. Then $D : W_0(E) \longrightarrow L_2(E)$ is Fredholm.* □

Under certain additional assumptions the pointwise condition on $\lambda_{\min}(x)$ of $\mathcal{R}_x$ can be replaced by a (weaker) integral condition. Denote $\mathcal{R}_-(x) = \max\{0, -\lambda_{\min}(x)\}$, where $\lambda_{\min}(x)$ is the smallest eigenvalue of $\mathcal{R}_x$.

Theorem 2.21 *Suppose that for a* $p > 2$ (M^n, g) *satisfies the Sobolev inequality*

$$c_P(M)\left(\int_M |u|^{\frac{2p}{p-2}}(x)\,\mathrm{dvol}_x(g)\right)^{\frac{p-2}{2}} \le \int_M |du|^2(x)\,\mathrm{dvol}_x(g) \text{ for all } u \in C_c^\infty(M) \qquad (2.22)$$

and

$$\int_M |\mathcal{R}_-|^{\frac{p}{2}}(x)\,\mathrm{dvol}_x(g) < \infty.$$

Then $D : W_0(E) \longrightarrow L_2(E)$ *is Fredholm.* □

Another important example are manifolds with a cylindri-ccal end which we already mentioned. In this case, there is a compact submanifold with boundary $K \subset M$ such that $(M \setminus K, g)$ is isometric to $(]0,\infty[\times \partial K, dr^2 + g_{\partial K})$. One assumes that $(E,h)|_{]0,\infty[\times\partial K}$ also has product structure and $D|_{M\setminus K} = \nu \cdot \left(\frac{\partial}{\partial r} + A\right)$, where $\nu\cdot$ is the Clifford multiplication with the exterior normal at $\{\gamma\} \times \partial K$ and A is first order elliptic and self-adjoint on $E|_{\partial K}$.

Proposition 2.22 *D is non-parabolic at infinity.*

Proof. There are two proofs. The first one refers to [5]. According to proposition 2.5 of [5], there exists on $M \setminus K$ a parametrix $Q : L_2(E|_{M\setminus K}) \longrightarrow \Omega^{2,1}_{\mathrm{loc}}E|_{M\setminus K}, D)$ such that $QD\varphi = \varphi$ for all $\varphi \in C_c^\infty(E|_{M\setminus K})$. Hence for $C_c^\infty(E|_{M\setminus K})$, $U \supset M \setminus K$ bounded,

$$|\varphi|_{L_2(E|_U)} = |QD\varphi|_{L_2(E|_U)} \le |Q|_{L_2\to\Omega^{2,1}} \cdot |D\varphi|_{L_2}.$$

The other proof is really elementary calculus. For $\varphi \in C_c^\infty(E|_{M\setminus K})$, $|\varphi(r,y)| = \left|\int_0^r \frac{\partial\varphi}{\partial r}\,dr\right| \le \sqrt{r} \cdot \left|\frac{\partial\varphi}{\partial r}\right|_{L_2}$. Hence

$$|\varphi|^2_{L_2(E|_{]0,T[\times\partial K})} \le \frac{T^2}{2}\left|\frac{\partial\varphi}{\partial r}\right|^2_{L_2} \le \frac{T^2}{2}\left(\left|\frac{\partial\varphi}{\partial r}\right|^2_{L_2} + |A\varphi|^2_{L_2}\right) = \frac{T^2}{2}|D\varphi|^2_{L_2}.$$

□

The authors of [5] define extended L_2–sections of $E|_{]0,\infty[\times\partial K}$ as sections $\varphi \in L_{2,\mathrm{loc}}$, $\varphi(r,y) = \varphi_0(r,y) + \varphi_\infty(y)$, $\varphi_0 \in L_2$, $\varphi_\infty \in \ker A$.

Proposition 2.23 *The extended solutions of $D\varphi = 0$ are exactly the solutions of $D\varphi = 0$ in W.*

Proof. Let $\{\varphi_\lambda\}_{\lambda\in\sigma(A)}$ be a complete orthonormal system in $L_2(E|_{\partial K})$ consisting of the eigensections of A. Then we can a solution φ of $D\varphi = 0$ on $]0,\infty[\times\partial K$ decompose as

$$\varphi(r,y) = \sum_{\lambda\in\sigma(A)} c_\lambda e^{-\lambda r}\varphi_\lambda(y) \qquad (2.23)$$

and $\varphi \in W$ if and only if $c_\lambda = 0$ for $\lambda < 0$. In this case

$$\varphi_0(r,y) = \sum_{\substack{\lambda\in\sigma(A)\\ \lambda>0}} c_\lambda e^{-\lambda r}\varphi_\lambda(y), \quad \varphi_\infty(y) = \sum_{\lambda\in\sigma(A)} c_{0,i}\varphi_{0,i}(y).$$

□

This proposition can also be reformulated as

Proposition 2.24 *Denote by $P_{\le 0}$ or $P_{<0}$ the spectral projection of A onto the sum of eigenspaces belonging to eigenvalues ≤ 0 or < 0, respectively. Then*

a) *φ is a solution in W of $D\varphi = 0$ if and only if*

$$D\varphi = 0 \text{ on } K$$

and

$$P_{<0}\varphi = 0 \text{ on } \partial K.$$

b) *φ is an L_2–solution of $D\varphi = 0$ if and only if*

$$D\varphi = 0 \text{ on } K$$

and

$$P_{\leq 0}\varphi = 0 \ \textit{on} \ \partial K.$$

□

There is a very general approach to index theory as established by Connes, Roe and others. The initial data are as follows: D an elliptic differential operator as above, $\mathfrak{B}$ an operator algebra, the K–theory $K_i(\mathfrak{B})$ of $\mathfrak{B}$, the cyclic cohomology $HC^*(\mathfrak{B})$ of $\mathfrak{B}$. Then one constructs the diagram

$$\begin{array}{ccc} D & \longrightarrow & \operatorname{Ind} D \in K_i(\mathfrak{B}) \\ \downarrow & & \downarrow \\ I_D \longrightarrow \langle I_D, m\rangle = \operatorname{ind}_t D & \stackrel{?}{=} & \operatorname{ind}_a D = \langle \operatorname{Ind} D, \zeta\rangle \end{array}$$

Here I_D is of cohomological nature, m a fundamental class, $\langle I_D, m\rangle$ a pairing, $\operatorname{Ind} D$ comes from ellipticity and the 6 term exact sequence of K–theory, $\zeta \in HC^*(\mathfrak{B})$ and $\langle \operatorname{Ind} D, \zeta\rangle$ is the Connes' pairing.

Choice of $\mathfrak{B}, i, \zeta, m, \operatorname{Ind}_D$ yields a concrete index theory. We refer to [60], [61], [62], [74] for details. The classical index theory on closed manifolds is given by the choice $i = 0$, $\mathfrak{B} =$ ideal K of compact operators, $\operatorname{Ind} D \in K_0(K) =$ projectors - projectors, $HC^0 \ni \zeta =$ trace, $\operatorname{trInd} D = \operatorname{ind}_a D$, $I_D =$ classical index from, $m = [M]$. The lack of all these (absolute) index theories for open manifolds is that they either refer to very special cases or there are not enough serious applications. This was one of the motivations for us to establish a general relative index theory as in chapter V.