

# Introduction

It is one of the main goals of modern mathematics to describe a mathematical subject, situation, result by a sequence of honest numbers. We remind e.g. in topology / global analysis of the rank of (co-)homology groups, homotopy groups,  $K$ -(co-)homology, characteristic numbers, topological and analytical index, Novikov–Shubin–invariants, analytical torsion, the eta invariant, all these numbers defined in the compact case. Including bordism and Wall groups, which are also of finite rank, one has an appropriate approach to the classification problem for compact manifolds.

For open manifolds, all these numbers above are not defined in general. The ranks of the group of algebraic topology can be infinite, the integrals to define characteristic numbers can diverge, elliptic operators must no longer be Fredholm, the spectrum of self-adjoint operators must not be purely discrete, etc.

We will prove at the beginning of chapter I that there are no non-trivial number valued invariants which are defined for all oriented (including open) manifolds and which behave additively under connected sum. Moreover, we prove, that for any  $n \geq 2$  there are uncountably many homotopy types of open manifolds. Hence a classification, essentially relying on number valued invariants, probably should not exist.

The main idea of our approach – brought to a point – is as follows. We consider pairs  $(P, P')$ , where  $P$  e.g. stands for a triple (manifold, bundle, differential operator), and we define relative invariants  $i(P, P')$ , where  $P'$  runs through a so called generalized component  $\text{gen comp}(P)$  which consists of all  $P'$  with finite distance from  $P$ . The distance comes from a metrizable uniform structure. To define the corresponding metrizable uniform structure is the content of chapter II and is one of the columns of our approach. Then the classification of the  $P$ s amounts to the classification of the generalized components and the classification of the  $P$ s inside  $\text{gen comp}(P)$ .

This treatise is organized as follows. In chapter I section 1 we

present classes of open manifolds for which the classical characteristic numbers via Chern–Weil construction are defined, study their invariance and meaning. Here we include the important contributions of Cheeger/Gromov from [15], [16]. We call them absolute invariants since they are defined for single objects and not for pairs with one component fixed. Section two is devoted to some index theorems for certain classes of open manifolds and elliptic differential operators. It is visible that all these are very special classes and that the wish for a corresponding theory for all open manifolds requires a new, another approach. For us this is the relative index theory, applied to pairs. Then the main question is, what is an admissible pair of Riemannian manifolds or Clifford bundles with associated generalized Dirac operators? In a local classical language this would mean, what are the admissible perturbations of the coefficients in questions – and of the domains?

We answer these questions in a very general and convenient language, the language of metrizable uniform structures. We define for a pair of objects under consideration a local distance, define by means of this local distance a neighbourhood basis of the diagonal and finally a metrizable uniform structure. In all our cases, the distance contains a certain Sobolev distance. For this reason, we give in chapter II 1 a brief outline of the needed facts and refer to [27] for more. II 2 is devoted to a very brief outline on uniform structures of proper metric spaces. Since diffeomorphisms enter into the definition of our local distances, we collect in II 3 some definitions and facts on completed diffeomorphism groups for open manifolds. In II 4, we introduce those uniform structures of manifolds and Clifford bundles and their generalized components, which are fundamental in the central chapters IV, V, VI. The final section II 5 contain the first steps of our approach to the classification problem for open manifolds. We introduce bordism for open manifolds, several bordism groups, reduce their calculation to that for generalized components, introduce relative characteristic numbers and establish generators for bordism groups of manifolds with non-expanding ends.

Chapter III is the immediate preparation for the chapters IV, V, VI. Section III 1 is devoted to the invariance of the essential spectrum under perturbation inside a generalized component, to heat kernel estimates, and we introduce in III 2 standard facts of scattering theory as wave operators, their completeness and the spectral shift function of Birman / Krein / Yafaev.

As it is clear from our approach and the criteria in section III 2, the absolute central question is the trace class property of  $e^{-tD^2} - e^{-t\tilde{D}'^2}$ ,  $E \in \text{gen comp}(E)$ . Moreover, the expression  $\text{tr}(e^{-tD^2} - e^{-t\tilde{D}'^2})$  enters into the integral for the relative index, relative zeta and eta functions. We establish this trace class property step by step in sections IV 1 – IV 3, admitting larger and larger perturbations. The proof of the trace class property is the heart of the treatise, really rather complicated and the technical basis for what follows.

In section V 1, we prove several relative index theorems and in V 2 properties of the scattering index. We remark that there are other well known relative index theorems e.g. in [8], [9] for exponentially decreasing perturbations. Both these theorems are very special cases of our more general result. In chapter VI, we apply our achievements until now to define relative zeta and eta functions, relative analytic torsion, relative eta invariants and relative determinants, which are in particular important in QFT. Section VI 6 presents numerous examples, special cases and applications of these notions. In particular, we present classes of open manifolds which satisfy the geometric and spectral assumptions which we assumed in the preceding sections. A particular simple case are manifolds with cylindrical ends for which we describe the scattering theory. Here we essentially rely on [3], [49]. An interpretation of our relative determinants in the case of cylindrical ends is given by theorem 6.17 in chapter VI. Until now, we always assumed bounded geometry, i.e. injectivity radius  $> 0$  and bounded curvature together with a certain number of derivatives. Clearly, this restricts the classes of metrics under consideration. In [54], W. Müller and G. Salomonsen established a scattering theory without the assumption injectivity

radius  $> 0$  but they admit other perturbations. The difference  $g - h$  together with certain derivatives must be of so-called moderate decay. We reformulate their approach in our language of uniform structures and generalized components and extend their results to arbitrary vector bundles with bounded curvature.

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