Chapter 2

Time Dilation

From now on, we will often make use of *two coordinate systems*: The first, called the S-frame or S-system, is a Cartesian coordinate system with *three orthogonal spatial coordinate axes* (x, y and z) and one *time coordinate*, t. The other one, called the S'-frame or S'-system, also has three orthogonal spatial coordinate axes (x', y' and z'), *parallel* to the axes of the S-frame, and a time coordinate, t' (Fig. 2.1). S' moves at the *constant velocity* $\mathbf{v}$ relative to S. For most problems in this book $\mathbf{v} = (v, 0, 0)$, i.e. a *one-dimensional motion*, is sufficiently general. For simplicity, we take $t = t' = 0$ when the origins of both the unprimed frame and the primed frame coincide.

Let us now embark on a thought experiment with a *light clock*. A light clock (Fig. 2.2) is a box 0.15 m tall, at both ends equipped with perfect

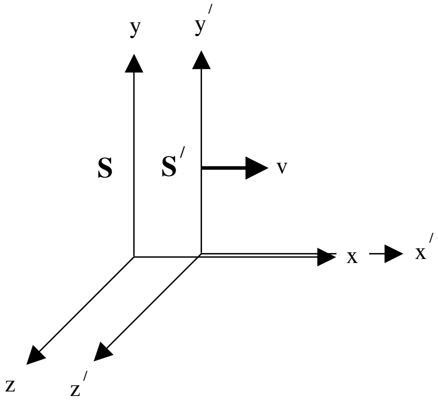

Fig. 2.1. The S-frame and the S'-frame.

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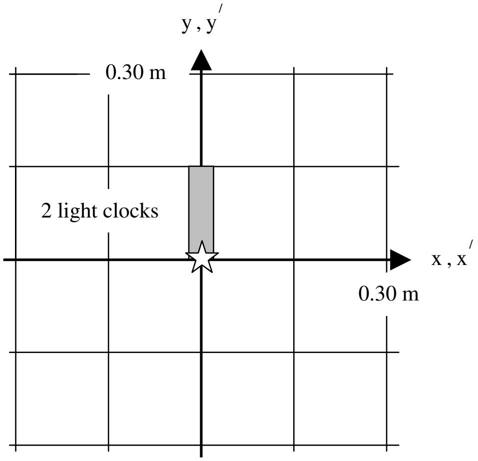

Fig. 2.2. The light flash and the light clocks at $t = 0$ s.

mirrors. At the lower end, a light flash is being ignited which gets reflected very many times between the two mirrors. For one round-trip the light requires one nanosecond (ns $= 10^{-9}$ s). Thus, each time the reflected light returns to the lower mirror, the display of a counting mechanism calibrated in nanoseconds proceeds by one unit.

The origins of both coordinate frames are equipped with one light clock each, stretching from $y_1 = y'_1 = 0.00$ m to $y_2 = y'_2 = 0.15$ m. An unimpaired mutual penetration of the light clocks is presupposed, of course. When both origins coincide, i.e. at $t = t' = 0$, a light flash is being ignited at the origins, henceforth beating time in *both* light clocks.

After 0.5 ns, the light flash has turned into a *spherical wave* with a radius of 0.15 m. When we give the S'-frame a speed of $v = c/\sqrt{2}$, the origin of S' has moved about 0.11 m to the right. The spherical wave is now arriving at the upper mirror of the light clock in the S-frame. According to the above assumptions, a time of $t = 0.5$ ns has elapsed there. But the spherical wave has not yet reached the upper mirror of the light clock in the S'-frame. Therefore, $t' < 0.5$ ns must be true in that frame (Fig. 2.3). It seems that time in the S'-frame passes more slowly than time in the S-frame.

After $t = 0.5\sqrt{2}$ ns, the radius of the spherical wave has grown to about 0.21 m. The origin of S' has moved 0.15 m to the right. The spherical wave

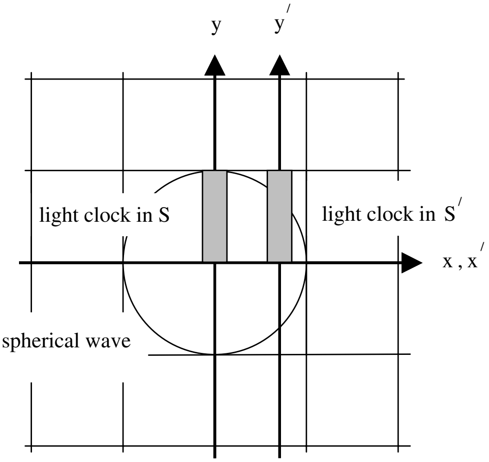

Fig. 2.3. The light clocks at $t = 0.5$ ns.

is now touching the upper mirror of the light clock in the S'-frame, so that the time $t' = 0.5$ ns has elapsed. In the light clock of the S-frame, however, the light is already returning. Therefore, $t > 0.5$ ns must be true (Fig. 2.4). Again, we can clearly see that time runs faster in the S-frame than in the S'-frame.

With Pythagoras's theorem, t and t' are connected by $c^2t^2 = v^2t^2 + c^2t'^2$. Note that ct' rather than $c't$ denotes the length of the vertical leg of the triangle! Nothing shows Einstein's ingenuity better than this choice of variables! Common sense would suggest that the speed of light is reduced in the S'-frame from c to $c' = \sqrt{c^2 - v^2}$, but the same time has elapsed in both frames. Einstein, however, postulates that the speed of light is the same in both frames which necessitates *a slower lapse of time in the primed frame*, thus giving ct' for the length in question. Rearranging the foregoing equation gives

$$t^2(c^2 - v^2) = c^2t'^2$$

$$t^2 = \frac{t'^2}{1 - \left(\frac{v}{c}\right)^2}.$$

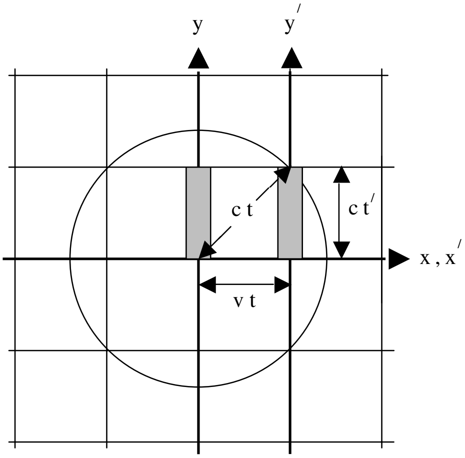

Fig. 2.4. The light clocks at $t = 0.5\sqrt{2}$ ns.

The negative solution has no physical meaning, thus

$$t = \frac{t'}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}. \tag{2.1}$$

The square root expression $\sqrt{1 - \left(\frac{v}{c}\right)^2}$ appears many times in the special theory of relativity. Thus one defines

$$\beta = \frac{v}{c}, \tag{2.2}$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}. \tag{2.3}$$

Obviously, γ is always greater than or equal to unity. So let us summarize the results of our thought experiment:

When a frame S' is in motion at a constant velocity v relative to another frame S, any process (e.g. the tick of a clock) being at rest in S' and requiring the time t' in S' is lengthened for an observer at rest in S to

$$t = \frac{t'}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{t'}{\sqrt{1-\beta^2}} = \gamma t' \geq t'.$$

This phenomenon is called *time dilation*, literally *time stretching*.

These considerations remain unchanged when the origin of the S'-frame is displaced into the y- or z-direction. So the most general case, i.e. the flyby of the S'-frame past the S-frame, is already included.

Let us continue with our thought experiment: For an observer at rest in the S'-frame, the S-frame moves to the left at the velocity $v = -c/\sqrt{2}$. The same considerations show that for an observer at rest in the S'-frame a process at rest in the S-frame requiring the time t in S is lengthened to

$$t' = \gamma t \geq t. \tag{2.4}$$

This has to be so because of the principle of special relativity; failing this, one of the two reference frames would be preferred. This brings us to one of the most important conclusions:

Time is relative. Of all inertial observers, an observer at rest relative to a process measures the shortest time for that process. This time is called the *proper time* of a process, denoted by τ.

At this point, we would like to draw the reader's attention to the coexistence of two seemingly contradictory equations that may give us a headache, namely, $t = \gamma t'$ and $t' = \gamma t$. These equations have the following meaning: The time variables on the *right-hand sides* stand for the *proper time* of a process, i.e. for the time that an observer measures who is at rest relative to that process; the left-hand sides denote the time an observer measures who is in motion relative to that process. In the equation $t = \gamma t'$, the time variable t' is the proper time of a process, the S'-frame is at rest relative to it; in the equation $t' = \gamma t$, the time variable t is the proper time of a process, the S'-frame is in motion relative to it. When the S-frame and the S'-frame are in motion relative to each other, there can only be *one* frame where a process is at rest. Therefore, one has no choice between two equations.

The question remains whether the above considerations are applicable for light clocks only, or whether they are equally true for mechanical, electric, or other clocks, or even for an aperiodic course of events, such as radioactive decay or a biological process. The answer is a clear *yes*. If there were a difference between the time scales of a light clock and another clock at rest relative to it, these differences would have to vary from one inertial reference frame to another one in order to become detectable. Then a reference frame with a minimal difference would exist. Such an inertial reference frame would be a distinguished reference frame and thus a violation of the principle of special relativity, i.e. a violation of one of the two premises of the whole theory of special relativity.

A famous experimental proof of time dilation is linked to the *lifetime of muons* which come into existence through *cosmic rays* in the *upper atmosphere*. According to the *law of radioactive decay*, $N = N_0(1/2)^{t/T_{1/2}}$, muons have a *half-life*, $T_{1/2}$, of 1.5 μs before they decay into an electron or a positron, respectively, and two neutrinos. In $t = T_{1/2} = 1.5\,\mu\text{s}$, muons, travelling at almost the speed of light, cover a distance of 450 m. Therefore, their intensity should double with every height increase of 450 m. In reality, however, intensity increases more gradually. The reason is time dilation: For an observer on earth the rapidly moving reference frame of the muons makes time elapse more slowly, the muons' half-life is lengthened to $\gamma T_{1/2}$. The exact number is: $v = 0.994c$, or $\gamma = 9$. Thus, for an observer on earth, the muons' half-life is 14 μs within which they cover a distance of 4 km.

A rather practical problem in the age of very precise atomic clocks is the question of whether *two clocks, after having been synchronized in one place, can be moved apart without losing their synchronization*. Let us imagine that one clock, remaining at rest during the experiment, shows the time t. A second clock, being transported at the constant velocity v

over the distance s, shows the time τ. Obviously, for an observer at rest,

$$t = \frac{\tau}{\sqrt{1-\left(\frac{v}{c}\right)^2}}. \tag{2.5}$$

Since in reality $v \ll c$ is always true, a Taylor expansion of the square root can be made, giving

$$t \approx \left[1 + \frac{1}{2}\left(\frac{v}{c}\right)^2\right]\tau,$$

or, for the time difference,

$$t - \tau \approx \frac{1}{2}\left(\frac{v}{c}\right)^2 \tau.$$

But, on the other hand, from (2.5),

$$\tau \approx \left[1 - \frac{1}{2}\left(\frac{v}{c}\right)^2\right] t$$

is also true, so the time difference between the clocks becomes

$$t - \tau \approx \frac{1}{2}\left(\frac{v}{c}\right)^2 \left[1 - \frac{1}{2}\left(\frac{v}{c}\right)^2\right] t,$$

or, to second order in v/c, simply

$$t - \tau \approx \frac{1}{2}\left(\frac{v}{c}\right)^2 t.$$

With $v = s/t$, we obtain

$$t - \tau \approx \frac{s^2}{2c^2 t}. \tag{2.6}$$

So *synchronization is lost!* However, the more time one allows for the transport over the given distance s, i.e. the more slowly the second clock is being transported, the smaller is the time difference between the two clocks.