Chapter 1

Fundamental Concepts and Basic Results

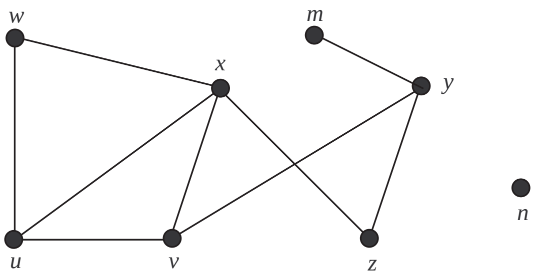
Theorem 1.1 *Let G be a multigraph with $V(G) = \{v_1, v_2, \cdots, v_n\}$. Then*

$$\sum_{i=1}^{n} d(v_i) = 2e(G).$$

Corollary 1.2 *The number of odd vertices in any multigraph is even.*

Exercise 1.2

Problem 1. *Let G be the multigraph representing the following diagram. Determine $V(G)$, $E(G)$, $v(G)$ and $e(G)$. Is G a simple graph?*



Solution. $V(G) = \{m, n, u, v, w, x, y, z\}$,

$$E(G) = \{my, uv, uw, ux, vx, vy, wx, xz, yz\}, \quad v(G) = 8 \text{ and } e(G) = 9.$$

Yes, G is a simple graph. □

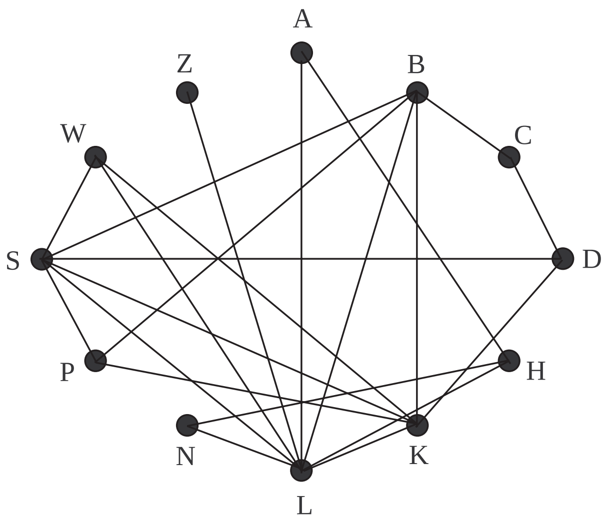
Problem 2. *Draw the graph G modeling the flight connectivity between twelve capital cities with the following vertex set $V(G)$ and edge set $E(G)$.*

$V(G) = \{$*Asuncion, Beijing, Canberra, Dili, Havana, Kuala Lumpur, London, Nairobi, Phnom Penh, Singapore, Wellington, Zagreb*$\}$.

$E(G) = \{$*Asuncion-Havana, Asuncion-London, Beijing-Canberra, Beijing-Kuala Lumpur, Beijing-London, Beijing-Phnom Penh, Beijing-Singapore, Canberra-Dili, Dili-Kuala Lumpur, Dili-Singapore, Havana-London, Havana-Nairobi, Kuala Lumpur-London, Kuala Lumpur-Phnom Penh, Kuala Lumpur-Singapore, Kuala Lumpur-Wellington, London-Nairobi, London-Singapore, London-Wellington, London-Zagreb, Phnom Penh-Singapore, Singapore-Wellington*$\}$.

(Note that you may use 'A' to represent 'Asuncion', 'B' to represent 'Beijing', 'C' to represent 'Canberra', etc.)

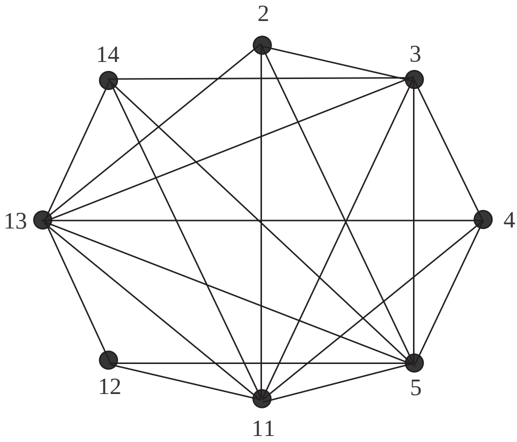
Solution.



□

Problem 3. *Define a graph G such that $V(G) = \{2, 3, 4, 5, 11, 12, 13, 14\}$ and two vertices 's' and 't' are adjacent if and only if $gcd\{s, t\} = 1$. Draw a diagram of G and find its size $e(G)$.*

Solution.



$e(G) = 21$. □

Problem 4. *The diagram below is a map of the road system in a town. Draw a multigraph to model the road system, using a vertex to represent a junction and an edge to represent a road joining two junctions.*

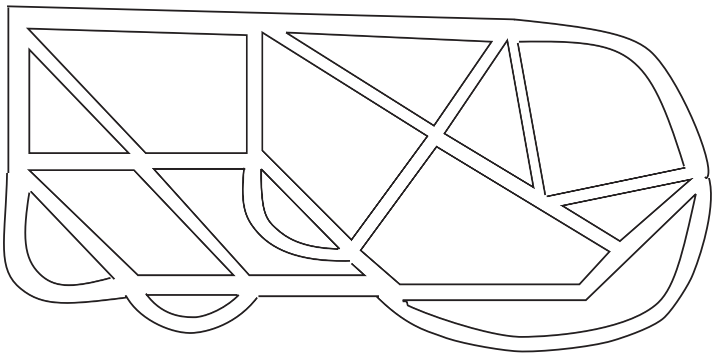
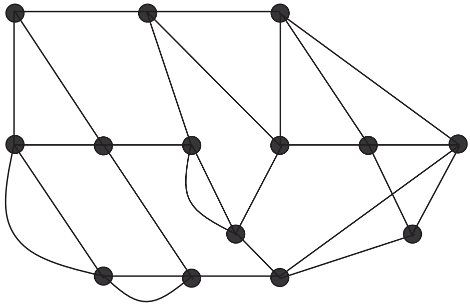


Diagram for Problem 4

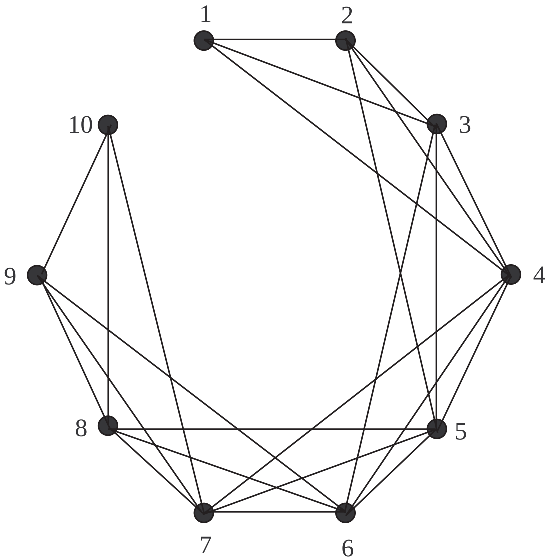
Solution.



□

Problem 5. *Let G be a graph with $V(G) = \{1, 2, \cdots, 10\}$, such that two numbers 'i' and 'j' in $V(G)$ are adjacent if and only if $|i - j| \leq 3$. Draw the graph G and determine $e(G)$.*

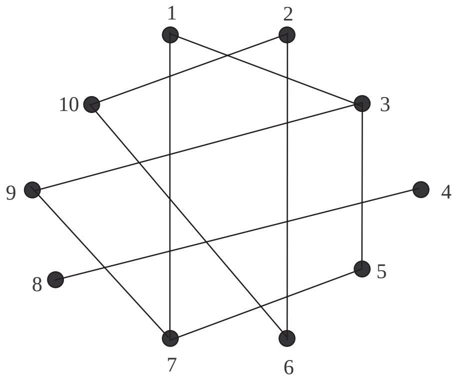
Solution.



$e(G) = 24.$ □

Problem 6. *Let G be a graph with $V(G) = \{1, 2, \cdots, 10\}$, such that two numbers 'i' and 'j' in $V(G)$ are adjacent if and only if $i + j$ is a multiple of 4. Draw the graph G and determine $e(G)$.*

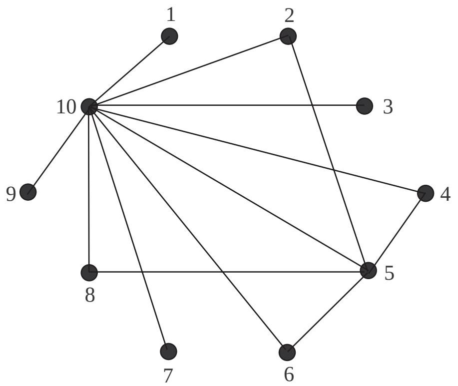
Solution.



$e(G) = 10.$ □

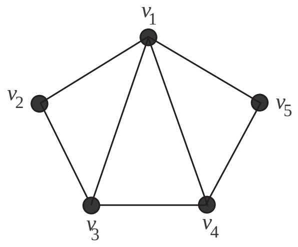
Problem 7. *Let G be a graph with $V(G) = \{1, 2, \cdots, 10\}$, such that two numbers 'i' and 'j' in $V(G)$ are adjacent if and only if $i \times j$ is a multiple of 10. Draw the graph G and determine $e(G)$.*

Solution.



$e(G) = 13.$ □

Problem 8. *Find the adjacency matrix of the following graph G.*



Solution.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

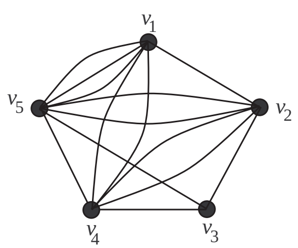
□

Problem 9. *The adjacency matrix of a multigraph G is shown below:*

$$\begin{pmatrix} 0 & 1 & 0 & 2 & 3 \\ 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 2 & 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 & 0 \end{pmatrix}$$

Draw a diagram of G.

Solution.



□

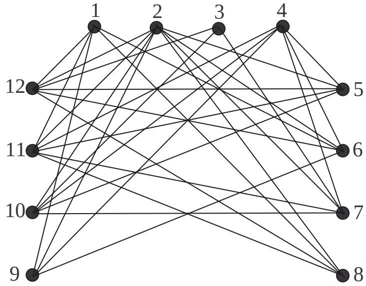
Problem 10. *Four teams of three specialist soldiers each (a scout, a signaler and a sniper) are to be sent into enemy territory. However, some of the soldiers cannot work well with some others. The following table shows the soldiers, their specializations and who they cannot work with.*

Soldier	*Specialization*	*Cannot cooperate with*
1	*Scout*	$5, 7, 10$
2	*Scout*	–
3	*Scout*	$5, 6, 8, 9, 11$
4	*Scout*	$8, 12$
5	*Signaler*	$1, 3, 9$
6	*Signaler*	$3, 10, 11$
7	*Signaler*	$1, 9, 12$
8	*Signaler*	$3, 4, 9, 10$
9	*Sniper*	$3, 5, 7, 8$
10	*Sniper*	$1, 6, 8$
11	*Sniper*	$3, 6$
12	*Sniper*	$4, 7$

(i) *Draw a multigraph to model the situation so that we may see how to form 3-man teams such that each specialization is represented and every member of the team can work with every other. State clearly what the vertices represent and under what condition(s) two vertices are joined by an edge.*

(ii) *Can you form four 3-man teams such that each specialization is represented and all members of the team can work with one another?*

Solution. (i) Vertex i represents soldier i. Two vertices are joined by an edge if the two corresponding soldiers can cooperate with each other and are not of the same specialization.



(ii) From the graph, one possible arrangement is

$$\{1, 6, 9\}, \{2, 8, 12\}, \{3, 7, 10\}, \{4, 5, 11\}.$$

□

Exercise 1.3

Problem 1. *In the following multigraph G, find*

(i) *the size of G,*

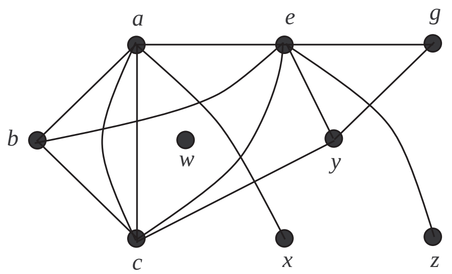
(ii) *the degree of each vertex,*

(iii) *the sum $\sum_{v \in V(G)} d(v)$,*

(iv) *the number of odd vertices,*

(v) $\Delta(G)$, *and*

(vi) $\delta(G)$.



Is your answer for (iii) double your answer for (i)? Is your answer for (iv) an even number?

Solution. (i) $e(G) = 13$.

(ii) $d(a) = 5$, $d(b) = 3$, $d(c) = 5$, $d(e) = 6$, $d(g) = 2$, $d(w) = 0$, $d(x) = 1$, $d(y) = 3$, $d(z) = 1$.

(iii) $\sum_{v \in V(G)} d(v) = 5 + 3 + 5 + 6 + 2 + 0 + 1 + 3 + 1 = 26$.

(iv) There are 6 odd vertices (namely, a, b, c, x, y, z).

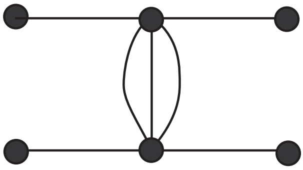
(v) $\Delta(G) = 6$.

(vi) $\delta(G) = 0$.

Yes, the answer for (iii) is double that for (i); and the answer for (iv) is an even number. □

Problem 2. *Construct a multigraph of order* 6 *and size* 7 *in which every vertex is odd.*

Solution. A required multigraph is shown below.



□

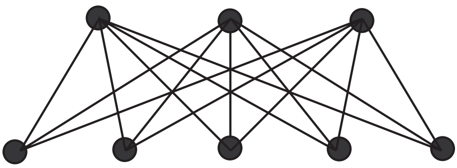
Problem 3. *Let G be a multigraph with $V(G) = \{v_1, v_2, \cdots, v_n\}$. Prove that the sum of all the entries in the ith row of the adjacency matrix $A(G)$ is the degree of the vertex v_i for each $i = 1, 2, \cdots, n$.*

Solution. Given i, where $1 \leq i \leq n$, the sum of the entries in the ith row of $A(G)$ is the sum of the numbers of edges joining v_i to v_j, where $j = 1, 2, \cdots, n$, which is thus the degree of v_i in G. □

Problem 4. *Let G be a graph of order* 8 *and size* 15 *in which each vertex is of degree* 3 *or* 5*. How many vertices of degree* 5 *does G have? Construct one such graph G.*

Solution. Let x and y be the number of vertices in G of degree 3 and 5 respectively. Then $x + y = 8$ and $3x + 5y = 2 \times 15 = 30$. Solving the equations yields $(x, y) = (5, 3)$.

An example of G is shown below.



□

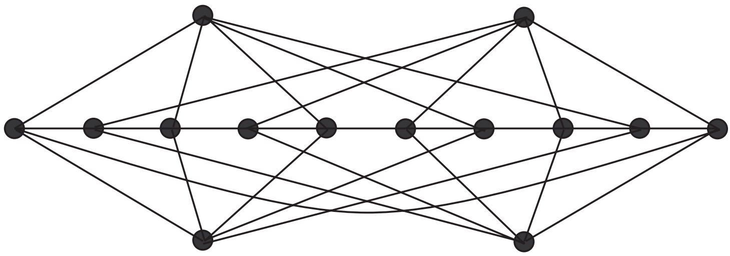
Problem 5. *Let H be a graph of order* 10 *such that* $3 \leq d(v) \leq 5$ *for each vertex v in H. Not every vertex is even. No two odd vertices are of the same degree. What is the size of H?*

Solution. Let x, y and z be the number of vertices in H of degree $3, 4$ and 5 respectively. Since not every vertex is even, $x + z \geq 2$. As no two odd vertices are of the same degree, $x = z = 1$. Thus, $(x, y, z) = (1, 8, 1)$, and so $e(H) = (3 + 4 \times 8 + 5)/2 = 20$. □

Problem 6. *Let G be a graph of order* 14 *and size* 30 *in which every vertex is of degree* 4 *or* 5*. How many vertices of degree* 5 *does G have? Construct one such graph G.*

Solution. Let x and y be the number of vertices in G of degree 4 and 5 respectively. Then $x + y = 14$ and $4x + 5y = 2 \times 30 = 60$. Solving the equations yields $(x, y) = (10, 4)$. Thus, G has 4 vertices of degree 5.

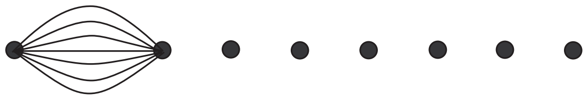
An example of G is shown below.



□

Problem 7. *Does there exist a multigraph G of order* 8 *such that* $\delta(G) = 0$ *while* $\Delta(G) = 7$*? What if 'multigraph G' is replaced by 'graph G'?*

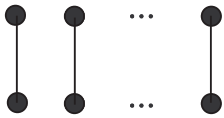
Solution. 'Yes' for multigraph G. An example is shown below.



'No' for graph G. Since if there is a vertex v in G with $d(v) = 7$, then v is adjacent to the remaining 7 vertices in G, and so $\delta(G) \geq 1$. □

Problem 8. *Characterize the* 1*-regular graphs.*

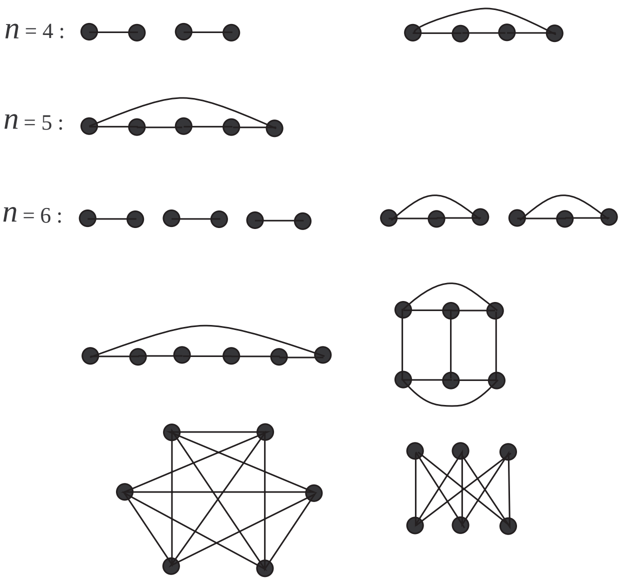
Solution. A graph is 1-regular if and only if it is of even order and is the disjoint union of some K_2's (see below).



□

Problem 9. *Draw all regular graphs of order* n*, where* $2 \leq n \leq 6$*.*

Solution. All null graphs N_n and complete graphs K_n, where $2 \leq n \leq 6$, are candidates. The remaining ones are shown below.



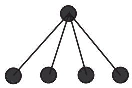
□

Problem 10.

(i) *Does there exist a graph G of order 5 such that $\delta(G) = 1$ and $\Delta(G) = 4$?*

(ii) *Does there exist a graph G of order 5 which has two vertices of degree 4 and $\delta(G) = 1$?*

Solution. (i) Yes. An example is shown below.



(ii) No. Suppose G were such a graph having the vertices u and v of degree 4. As $v(G) = 5$, each of the other vertices must be adjacent to both u and v, and so $\delta(G) \geq 2$. □

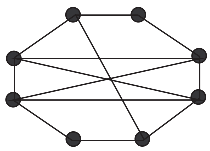
Problem 11. *Let H be a graph of order 8 and size 13 with $\delta(H) = 2$ and $\Delta(H) = 4$. Denote by n_i the number of vertices in H of degree i, where $i = 2, 3, 4$. Assume that $n_3 \geq 1$. Find all possible answers for (n_2, n_3, n_4). For each of your answers, construct a corresponding graph.*

Solution. We have $n_2 + n_3 + n_4 = 8$ and by Theorem 1.1,

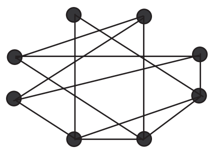
$$2n_2 + 3n_3 + 4n_4 = 26.$$

It follows from the above that $n_3 + 2n_4 = 10$. As $n_3 \geq 1$, by Corollary 1.2, $n_3 = 2, 4$ or 6.

When $n_3 = 2$, we have $(n_2, n_3, n_4) = (2, 2, 4)$, and a corresponding graph is shown below:



When $n_3 = 4$, we have $(n_2, n_3, n_4) = (1, 4, 3)$, and a corresponding graph is shown below:



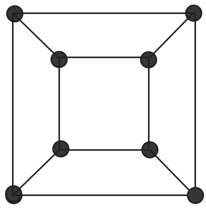
When $n_3 = 6$, we have $n_4 = 2$, and so $n_2 = 0$, which is not possible as $\delta(H) = 2$. □

Problem 12. *Suppose G is a k-regular graph of order n and size m, where $k \geq 0$, $m \geq 0$ and $n \geq 1$. Find a relation linking k, n and m. Justify your answer.*

Solution. By Theorem 1.1, $kn = \sum_{x \in V(G)} d(x) = 2m$. □

Problem 13. *Does there exist a 3-regular graph with eight vertices? Does there exist a 3-regular graph with nine vertices?*

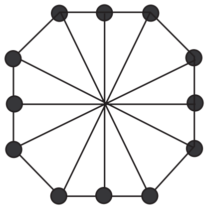
Solution. Yes, a 3-regular graph of order 8 is shown below.



No, there does not exist any 3-regular graph of order 9 by Corollary 1.2 (or the result of Problem 12). □

Problem 14. *Construct a cubic (i.e., 3-regular) graph of order 12. What is its size? Does there exist a cubic graph of order 11? Why?*

Solution. A cubic graph of order 12 is shown below.



Its size is $(3 \times 12)/2 = 18$ (see Problem 12).

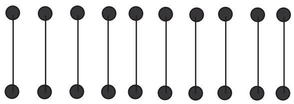
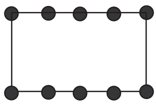
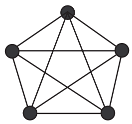
By Corollary 1.2 (or the result of Problem 12), there does not exist any cubic graph of order 11. □

Problem 15. *Let H be a k-regular graph of order n. If $e(H) = 10$, find all possible values for k and n; and for each case, construct one such graph H.*

Solution. By the result of Problem 12, $kn = 20$. As $k \le n - 1$,

$$(k, n) = (1, 20), (2, 10) \text{ or } (4, 5).$$

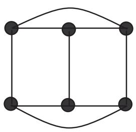
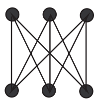
An example of H for each case is shown below:

□

Problem 16. (+) *Let G be a 3-regular graph with $e(G) = 2v(G) - 3$. Determine the values of $v(G)$ and $e(G)$. Construct all such graphs G.*

Solution. Let $n = v(G)$ and $m = e(G)$. By the assumption and Theorem 1.1, we have: $3n = 2m = 2(2n - 3)$, which implies that $(n, m) = (6, 9)$. There are only two such G as shown below.

□

Problem 17. *Find all integers n such that $100 \leq e(K_n) \leq 200$.*

Solution. As $e(K_n) = n(n-1)/2$, we have $200 \leq n(n-1) \leq 400$. It follows that $15 \leq n \leq 20$. □

Problem 18. (+) *Let G be a multigraph of order 13 in which each vertex is of degree 7 or 8. Show that G contains* **at least eight** *vertices of degree 7 or* **at least seven vertices** *of degree 8.*

Solution. Suppose that the conclusion is **false**. Then G contains **at most seven** vertices of degree 7 **and at most six** vertices of degree 8. Since $v(G) = 13$, G contains **exactly seven** vertices of degree 7. This is, however, impossible by Corollary 1.2. □

Problem 19. (+) *Let G be a graph of order n in which there exist* **no** *three vertices u, v and w such that uv, vw and wu are all edges in G. Show that $n \geq \delta(G) + \Delta(G)$.*

Solution. Let x be a vertex in G such that $d(x) = \Delta(G)$. Pick a vertex y in $N(x)$. By assumption,

$$n \geq 1 + \Delta(G) + (d(y) - 1) = \Delta(G) + d(y) \geq \Delta(G) + \delta(G).$$

□

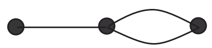
Problem 20. (+) *There were n (≥ 2) persons at a party and, as usually happens, some shook hands with others. No one shook hands with the same person more than once. Show that there are at least two persons in the party who had the same number of handshakes.*

Solution. Model the situation as a graph G of order n, where the vertices are the persons, and two vertices are adjacent if and only if the two corresponding persons shook hands. By assumption, G is a simple graph. The problem is equivalent to showing that there exist two vertices u, v in G such that $d(u) = d(v)$.

It is clear that $0 \leq d(x) \leq n - 1$ for each vertex x in G. If the above statement is false, then there exist two vertices y and z in G such that $d(y) = 0$ and $d(z) = n - 1$, which however is impossible. □

Problem 21. *The preceding problem says that in any graph of order $n \geq 2$, there exist two vertices having the same degree. Is the result still valid for multigraphs?*

Solution. No! A multigraph in which no two vertices have the same degree is shown below.

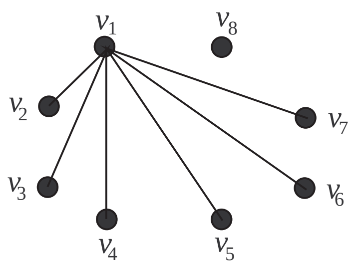


□

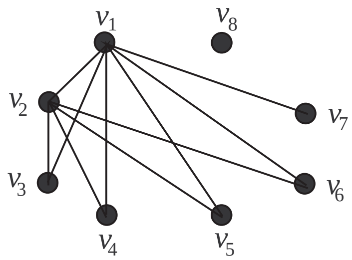
Problem 22. (+) *Mr. and Mrs. Samy attended an exclusive party where in addition to themselves, there were only another 3 couples. As usually happens, some shook hands with others. No one shook hands with the same person more than once and no one shook hands with his/her spouse. After all the handshakes had been done, Mr. Samy asked each person, including his wife, how many hands he/she had shaken. To everyone's amusement, each one gave a different answer. How many hands did Mrs. Samy shake?*

Solution. Model the situation by a graph G with 8 vertices for 8 persons, and defining 'adjacency' for 'handshaking'. By assumption, $0 \leq d(v) \leq 6$ for each v in G, and each of '$0, 1, 2, 3, 4, 5, 6$' is the *degree* of some vertex.

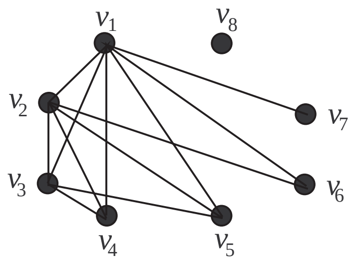
Let v_1 be such that $d(v_1) = 6$ and $N(v_1) = \{v_2, v_3, \cdots, v_7\}$, say. Then $d(v_8) = 0$, and v_1 and v_8 are spouses (see below).



Let v_2 be such that $d(v_2) = 5$ and $N(v_2) = \{v_1, v_3, v_4, v_5, v_6\}$, say. Then $d(v_7) = 1$ and v_2 and v_7 are spouses (see below).



Let v_3 be such that $d(v_3) = 4$ and $N(v_3) = \{v_1, v_2, v_4, v_5\}$, say. Then $d(v_6) = 2$ and v_3 and v_6 are spouses (see below).



It follows that $d(v_4) = d(v_5) = 3$, and v_4 and v_5 are spouses.

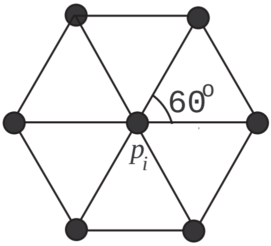
As Mr Samy received different answers, either v_4 or v_5 represents Mrs Samy. Thus Mrs Samy shook hands with three others. □

Problem 23. (+) *In the preceding problem, there were four couples altogether in a party. Solve the general problem where 'four couples' is replaced by '$n(\geq 2)$ couples'.*

Solution. Using a similar argument as shown in the solution of Problem 22, it can be shown that the answer is '$n - 1$' for this general problem. □

Problem 24. ($*$) *There are $n \geq 2$ distinct points in the plane such that the distance between any 2 points is at least one. Prove that there are at most $3n$ pairs of these points at distance exactly one.*

Solution. Let $p_1, p_2, \cdots, p_n$ be the n given points in the plane. Form a graph G with $V(G) = \{p_1, p_2, \cdots, p_n\}$ in which two vertices are adjacent if their distance in the plane is '1'. What is the largest possible value that each $d(p_i)$ can have? By the assumption that the distance between any 2 points is at least one, it follows that $d(p_i) \leq 6$ (see the figure below).



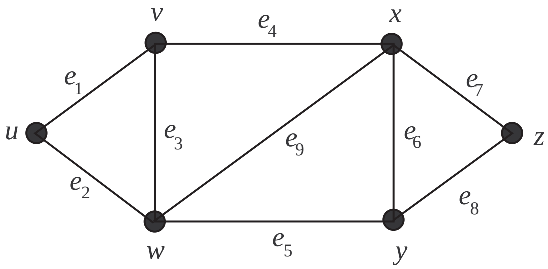
Thus, by Theorem 1.1,

$$2e(G) = \sum_{i=1}^{n} d(p_i) \leq 6n,$$

and so $e(G) \leq 3n$, as was to be shown. □

Exercise 1.4

Problem 1. *Consider the following graph H.*



(a) *Which of the following sequences represents a* $u - z$ *walk in* H*?*
   (i) $ue_2we_5xe_7z$
   (ii) $ue_1ve_5ye_8z$
   (iii) $ue_1ve_3we_3ve_4xe_7z$

(b) *Find a* $u - z$ *trail in* H *that is not a path.*

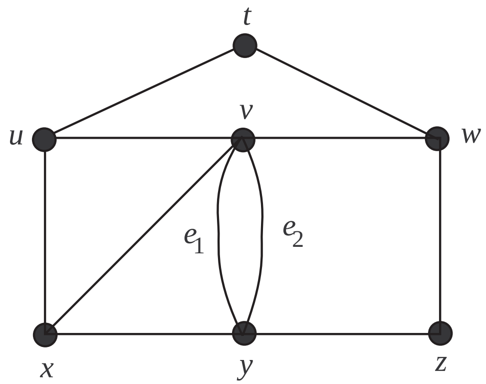
(c) *Find all* $u - z$ *paths in* H *which pass through* e_9*.*

Solution. (a) Only the sequence (iii) represents a $u - z$ walk in H.

(b) The sequence "$uvxywxz$" is a $u - z$ trail that is not a path.

(c) All such paths are: $uvwxz, uvwxyz, uwxz, uwxyz, uvxwyz$. □

Problem 2. *Consider the following multigraph G:*



(a) Find $d(t, v)$, $d(t, y)$, $d(x, w)$ and $d(u, z)$.

(b) For $k = 2, 3, 4, 5, 6, 7$, find a cycle of length k in G.

(c) Find a circuit of length 6 in G that is not a cycle.

(d) Find a circuit of length 8 in G that does not contain t.

(e) Find a circuit of length 9 in G that contains t and v.

Solution.

(a) $d(t, v) = 2, d(t, y) = 3, d(x, w) = 2$ and $d(u, z) = 3$.

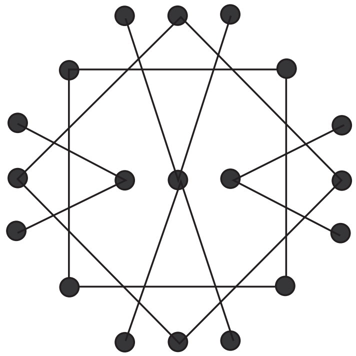
(b) ve_1ye_2v is a cycle of length 2,
$uvxu$ is a cycle of length 3,
$utwvu$ is a cycle of length 4,
$xvwzyx$ is a cycle of length 5,
$uvwzyxu$ is a cycle of length 6 and
$uvxyzwtu$ is a cycle of length 7.

(c) $tuve_1ye_2vwt$ is a circuit of length 6 that is not a cycle.

(d) $uvwzye_1ve_2yxu$ is a circuit of length 8 that does not contain 't'.

(e) $utwzye_1ve_2yxvu$ is a circuit of length 9 that contains both 't' and 'v'. □

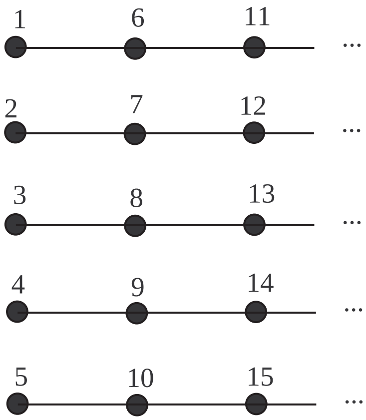
Problem 3. *Is the following graph H disconnected? If it is so, find its number of components.*



Solution. Yes, H is disconnected, and it has 5 components. □

Problem 4. *Let G be a graph with $V(G) = \{1, 2, \cdots, n\}$, where $n \geq 5$, such that two numbers i and j in $V(G)$ are adjacent if and only if $|i - j| = 5$. How many components does G have?*

Solution. By definition, the graph G is depicted as follows:



Thus, G has 5 components. □

Problem 5. (+) *Show that any $u-v$ walk in a graph contains a $u-v$ path.*

Solution. Let P be a $u-v$ walk. We may assume that $u \neq v$. If no vertex in P is repeated, then P is a path, and we are through. Assume that a vertex x is repeated in P as shown below (it is possible that $x = u$ or $x = v$):

$$P : \underbrace{u \cdots}_{(a)} \underbrace{x \cdots x}_{(b)} \underbrace{\cdots v}_{(c)}.$$

Then P can be cut short by deleting the section (b) internally resulting in a shorter $u-v$ walk P' as shown below:

$$P' : \underbrace{u \cdots}_{(a)} x \underbrace{\cdots v}_{(c)}.$$

This procedure is repeatedly applied until no vertex in the resulting $u-v$ walk is repeated, and in this case, the resulting $u-v$ walk is a desired path. □

Problem 6. (+) *Show that any circuit in a graph contains a cycle.*

Solution. Let Q be a circuit of length at least 2. If no vertex in Q is repeated, then Q is a cycle, and we are through. Assume that a vertex x is repeated in Q as shown below:

$$P : \underbrace{u \cdots}_{(a)} \underbrace{x \cdots x}_{(b)} \underbrace{\cdots u}_{(c)}.$$

Then Q can be cut short by deleting the section (b) internally resulting in a shorter circuit Q' as shown below:

$$Q' : \underbrace{u \cdots}_{(a)} x \underbrace{\cdots u}_{(c)}.$$

This procedure is repeatedly applied until no vertex in the resulting circuit is repeated, and in this case, the resulting circuit is a desired cycle. □

Problem 7. (+) *Show that any graph G with $\delta(G) \geq k$ contains a path of length k.*

Solution. Let $P = v_0 v_1 \cdots v_r$ be a longest path (of length r) in G. By assumption, $d(v_0) \geq k$, and so v_0 has at least k neighbors. Note that all these neighbors must be contained in P; for if there is a neighbor (say, w) of v_0 not in P, then we would have a path of the form: $wv_0v_1 \cdots v_r$, which is of length $r+1$, contradicting the fact that P is a longest path. Thus, $N(v_0) \subseteq \{v_1, \cdots, v_r\}$, and so $r \geq |N(v_0)| = d(v_0) \geq k$, as required. □

Problem 8. (+) *Let G be a graph of order $n \geq 2$ such that $\delta(G) \geq \frac{1}{2}(n-1)$. Show that $d(u, v) \leq 2$ for any two vertices u, v in G.*

Solution. Let u, v be any two distinct vertices u, v in G. If u and v are adjacent, then $d(u, v) = 1$. Assume that u and v are not adjacent. Consider $N(u)$ and $N(v)$. We claim that $N(u) \cap N(v) \neq \emptyset$.

Suppose that $N(u) \cap N(v) = \emptyset$. Then, as $\{u, v\} \cup N(u) \cup N(v) \subseteq V(G)$ and $\delta(G) \geq (n-1)/2$, we have:

$$n \geq 2 + |N(u)| + |N(v)| = 2 + d(u) + d(v) \geq 2 + 2\delta(G) \geq n + 1,$$

which is impossible.

Thus, $N(u) \cap N(v) \neq \emptyset$, as claimed. Let $w \in N(u) \cap N(v)$. Then uwv is a $u-v$ path of length 2, and so $d(u, v) = 2$.

We thus conclude that $d(u, v) \leq 2$, for any two vertices u, v in G. □

Problem 9. (+) *Let G be a graph of order n and size m such that $m > \binom{n-1}{2}$. Show that G is connected.*

Solution. Suppose on the contrary that G is disconnected. Let G_1 be a component of order $k (1 \leq k \leq n-1)$ in G, and let G_2 be the remaining part of G, which is of order $n-k$. Then

$$\binom{n-1}{2} < m = e(G_1) + e(G_2) \leq \binom{k}{2} + \binom{n-k}{2},$$

which implies that

$$(n-1)(n-2) < k(k-1) + (n-k)(n-k-1)$$

or $(k-(n-1))(k-1) > 0$. As $k \geq 1$, it follows that $k-(n-1) > 0$, i.e. $k > n-1$, a contradiction.

We thus conclude that G is connected if the condition holds. □

Problem 10. *For $n \geq 2$, construct a disconnected graph of order n and size $\binom{n-1}{2}$.*

Solution. The graph with 2 components, namely, K_1 and K_{n-1}, is the candidate. □

Problem 11. *Let G be a disconnected graph of order 5. What is the largest possible value for $e(G)$? If G is a disconnected graph of order $n \geq 2$, what is the largest possible value for $e(G)$? Construct one such extremal graph of order n.*

Solution. If G is a disconnected graph of order $n \geq 2$, the largest possible value for $e(G)$ is $\binom{n-1}{2}$.

To justify this, we note that the disconnected graph $K_1 \cup K_{n-1}$ has order n and its size equal to $\binom{n-1}{2}$, and by Problem 9 above, there is no disconnected graph of order n having its size greater than $\binom{n-1}{2}$. □

Problem 12. (+) *Let G be a graph of order $n \geq 2$ and u, v be two non-adjacent vertices in G such that $d(u) + d(v) \geq n + r - 2$. Show that u and v have at least r common neighbours.*

Solution. Our aim is to show that $|N(u) \cap N(v)| \geq r$. By the Principle of Inclusion and Exclusion, we have

$$\begin{aligned} |N(u) \cap N(v)| &= |N(u)| + |N(v)| - |N(u) \cup N(v)| \\ &= d(u) + d(v) - |N(u) \cup N(v)|. \end{aligned}$$

As u and v are non-adjacent, $N(u) \cup N(v) \subseteq V(G) \backslash \{u, v\}$. Thus,

$$\begin{aligned} |N(u) \cap N(v)| &= d(u) + d(v) - |N(u) \cup N(v)| \\ &\geq d(u) + d(v) - |V(G) \backslash \{u, v\}| \\ &\geq n + r - 2 - (n - 2) \quad \text{(by assumption)} \\ &= r, \end{aligned}$$

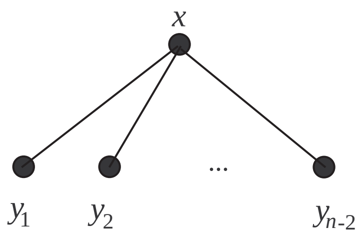
as was shown. □

Problem 13. (+) *Let G be a connected graph that is not complete. Show that there exist three vertices x, y, z in G such that x and y, y and z are adjacent, but x and z are not adjacent in G.*

Solution. As G is not complete, let u and v be two non-adjacent vertices in G. As G is connected, let $uab\cdots v$ be a **shortest** $u-v$ path in G (note that it is possible that $b=v$). Now let $x=u, y=a$ and $z=b$. As $uab\cdots v$ is a **shortest** $u-v$ path, it follows that x and z are not adjacent in G. □

Problem 14. (+) *Let G be a graph of order n and size m such that $\Delta(G) = n-2$ and $d(u,v) \le 2$ for any two vertices u,v in G. Show that $m \ge 2n-4$.*

Solution. Let x be a vertex in G such that $d(x) = \Delta(G) = n-2$ with $N(x) = \{y_1, y_2, \cdots, y_{n-2}\}$ as shown below:



Clearly, the $(n-2)$ xy_i's are edges in G. As G is of order n, let z be the remaining vertex in G. Note that z and x are not adjacent.

Since $d(z,x)=2$ by assumption, z must be adjacent to some y_i's. Without loss of generality, we may assume that z is adjacent to $y_1, y_2, \cdots, y_k$, where $1 \le k \le n-2$ and k is the largest index such that z is adjacent to y_k.

Now, for each j (if any) with $k+1 \le j \le n-2$, as $d(z,y_j)=2$ (z and y_j are not adjacent now), there must be a new edge joining y_j with some y_i in $\{y_1, y_2, \cdots, y_k\}$.

Summing up, the number of edges in G is at least

$$(n-2) + k + ((n-2) - k) = 2n-4.$$

That is, $m \ge 2(n-2)$, as required. □

Problem 15. *Let G be a graph such that $N(x) \cup N(y) = V(G)$ for every pair of vertices x, y in G. What can be said of G?*

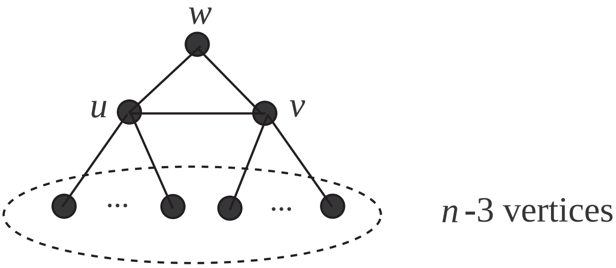
Solution. The graph G must be a complete graph. We justify it as follows. Suppose that G is not complete. Then there exist two non-adjacent vertices u and v in G. In this case, u is not contained in $N(u) \cup N(v)$; which, however, contradicts the assumption that $N(u) \cup N(v) = V(G)$. □

Problem 16. (+) *Let H be a graph of order $n \geq 2$. Suppose that H contains two distinct vertices u, v such that (i) $N(u) \cup N(v) = V(H)$ and (ii) $N(u) \cap N(v)$ is non-empty.*

What is the least possible value of $e(H)$?

Solution. The least possible size of H is 'n' (note that the given conditions imply that $n \geq 3$). The justification is as follows.

Firstly, the following graph of order n satisfying the given conditions contains exactly 'n' edges:



Now we show that every graph H satisfying the given conditions must have at least n edges.

The vertices u and v must be adjacent; for if not, then u is not in $N(u) \cup N(v)$, and so $N(u) \cup N(v) \neq V(H)$, violating the condition (i).

By (ii), there exists a vertex, say w, in $N(u) \cap N(v)$, and so w is adjacent to both u and v.

By (i) again, each of the $(n-3)$ vertices other than u, v and w must be adjacent to either u or v.

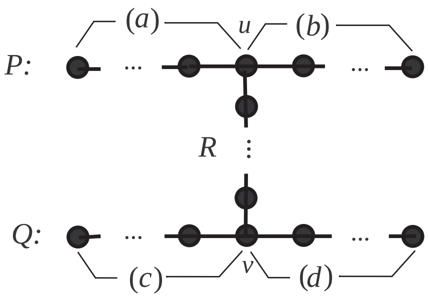
Summing up, H contains at least $1+2+(n-3)$ edges; that is, $e(H) \geq n$, as required. □

Problem 17. *Suppose G is a disconnected graph which contains exactly two odd vertices u and v. Must u and v be in the same component of G? Why?*

Solution. Yes, the two odd vertices u and v must be in the same component of G. Otherwise, let H be the component of G containing u but not v; then H is a graph containing exactly one odd vertex, contradicting Corollary 1.2. □

Problem 18. $(*)$ *Show that any two longest paths in a connected graph have a vertex in common.*

Solution. Let P and Q be two longest paths (of length k each, say) in a connected graph G, and suppose on the contrary that they have no vertex in common. As G is connected, there exist a vertex u in P and a vertex v in Q which are joined by a path R, say. Without loss of generality, we may assume (see the figure below) that (i) this $u-v$ path R contains no vertex in P or Q other than u and v, (ii) the length of the subpath (a) in P is greater than or equal to that of (b) and (iii) the length of the subpath (d) in Q is greater than or equal to that of (c).



With this, however, we observe that the path consisting of the subpath (a) in P, the $u-v$ path R and the subpath (d) in Q is of length greater than k, a contradiction. □

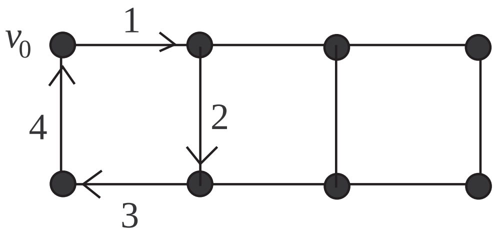
Problem 19. $(+)$ *Show that a graph G is connected if and only if for any partition of $V(G)$ into two non-empty sets A and B, there is an edge in G joining a vertex in A and a vertex in B.*

Solution. [Necessity] Suppose on the contrary that there is a partition (A, B) of $V(G)$ for which there is no edge in G joining a vertex in A and a vertex in B. It is then clear that no vertex in A can be joined to any vertex in B by a path. Thus G is disconnected, a contradiction.

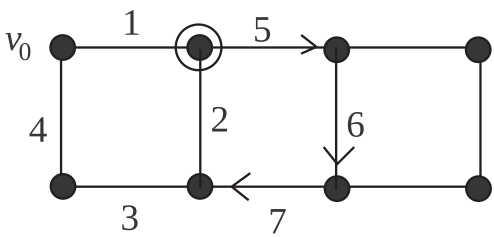
[Sufficiency] Suppose on the contrary that G is disconnected. Let H be a component of G and R the remaining part of G. Then $(V(H), V(R))$ forms a partition of $V(G)$ for which there is no edge in G joining a vertex in $V(H)$ and a vertex in $V(R)$ (note that both $V(H)$ and $V(R)$ are non-empty), a contradiction. □

Problem 20. (∗) *Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \cdots, k$ in such a way that at each vertex which belongs to two or more edges (i.e. which is of degree at least two), the greatest common divisor of the integers labeling those edges is* 1 *(32nd IMO, 1991/4). (Recall that the greatest common divisor of the positive integers $x_1, x_2, \cdots, x_n$ is the maximum positive integer that divides each of $x_1, x_2, \cdots, x_n$.)*

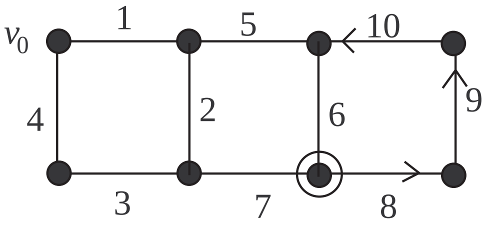
Proof. Starting at an arbitrary vertex, say v_0, in G, we walk along distinct edges in G to produce a maximal trail (no edge is repeated), say of length s, and label the edges along the trail $1, 2, \cdots, s$ (see the example below).



If there are edges not yet labeled, as G is connected, one of them is incident with a vertex, say v_r, which has been visited. Starting at v_r, we walk along distinct unlabelled edges in G to produce another maximal trail, say of length p, and label the edges along the trail $s + 1, s + 2, \cdots, s + p$ (see the diagram below).



We repeat the above procedure until all edges in G are labeled (see the diagram below).



We now show that for each vertex v with $d(v) \geq 2$ in G, the *gcd* of the labels of the edges incident with v is 1. If $v = v_0$, the situation is clear as the first edge incident with it is labeled 1. Assume that $v \neq v_0$. Let e be the edge with which we first visit v via a trail. As $d(v) \geq 2$, along the same trail, we leave v with a new edge, say f. By the above procedure, the labels of e and f are consecutive numbers, say t and $t+1$, and so the corresponding *gcd* (that is, $gcd\{t, t+1, \cdots\}$) is 1. □