

Chapter 1

Introduction

1.1 Advanced Mechatronics Systems

1.1.1 *Monitoring and Control of Distributed Parameters Systems*

Most engineered systems are composed of mixed mechanical-electrical-electronic-thermal subsystems and have fewer sensors (under-sensing) than states needed for monitoring and control and, moreover, have fewer actuators than degrees of freedom (under-actuated). Some of these systems can be modeled in a first approximation as lumped parameters systems but, in general, require more complex approaches for proper design and operation.

The focus in this Advanced Mechatronics text is on the computer based -integration, -monitoring and -control of mixed systems that can be described as distributed parameters systems. The illustrations for distributed parameters systems will be acoustic fields, thermo-dynamic fields, magnetic fields, vibrations in flexible structures, *etc.* The following topics will be presented:

- overview of advanced mechatronic systems: signals versus power transmission, local sensing and actuation in continuous systems, centralized versus local control
- modeling and control issues for mixed systems: effort-flow modeling, modeling and simulation of distributed parameters systems, open and closed loop control
- numerical solutions for inverse problems using regularization and singular value decomposition methods
- dynamic calibration of sensors

- transient response of under-actuated and under-sensed systems
- active vibration control in flexible structures
- acoustic fields monitoring and control
- thermo-dynamic fields in thermal process control
- magnetic fields in magnetic levitation.

Figure 1.1 shows the schematic diagram of a distributed parameters mechatronic system. In Fig. 1.1 system variables are measured by transducers, signal conditioned and converted from analog to digital form and transmitted to a computer. The computer performs real time monitoring and control as well as signal analysis and has two types of outputs, one for actuator commands and the other for system monitoring display.

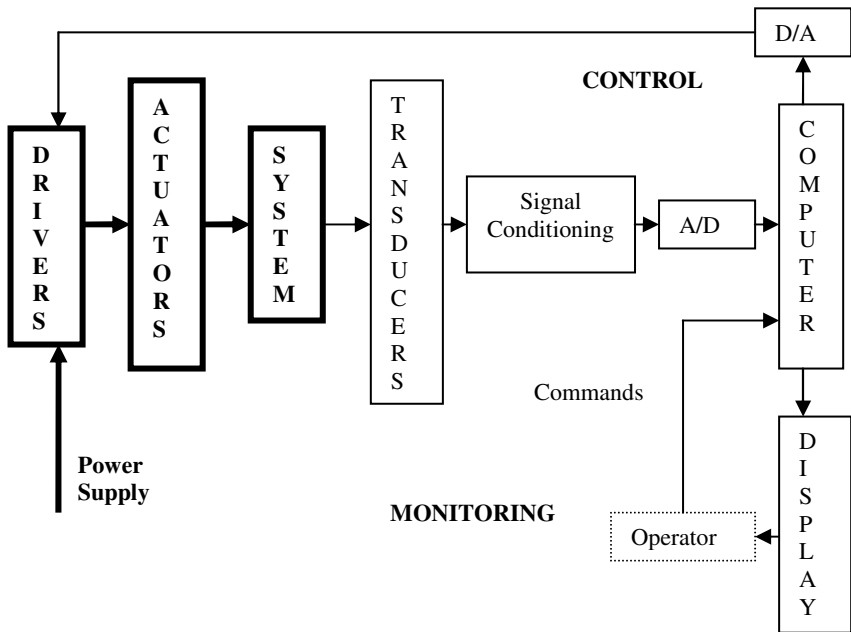


Fig. 1.1 Schematic diagram of a distributed parameters mechatronic system

The commands are either operator commands, or computed commands that are shown applied in a closed loop control configuration.

Computer output for control, after conversion from digital to analog form, sends commands to actuators.

Control commands are signals sent to drivers that modulate the power from an external power supply for the actuators.

An advanced mechatronics approach has to take into account that physical systems are inherently distributed parameters systems and that only some of these systems can be represented by a lumped parameters model. Lumped parameters mechatronic systems were already investigated extensively in several mechatronics books [1-9]. Figure 1.1 refers to a distributed parameters mixed system that can be represented by partial differential equations [25, 44, 110]. Numerous distributed parameters systems are mixed systems. Examples analyzed in this text are: acoustic, thermal, fluid, magnetic systems and flexible structures.

1.2 Signals versus Power Transmission. Lumped Parameters Modeling of Mechatronic Systems

Integration of systems is achieved transmission of signals and power between subsystems.

Distributed parameters systems modeling require modeling of propagation delays, boundaries effects, 3D interactions etc, which are not present in a lumped parameters model or in its block diagram counterpart. Lumped parameters systems, described by Linear Time Invariant (LTI) Ordinary Differential Equations (ODE), are reviewed in this section, in order to identify specific needs for modeling distributed parameters systems.

Block diagrams contain variables associated to the unidirectional links between blocks. These variables can be seen as signals containing the information transmitted from the output of one block to the input of another block. In control engineering signal flow graphs are sometimes used as an equivalent alternative form to block diagrams.

What is important in communication systems is only the information contained in the signals, not the power transmitted by the carrier of this information. In this case, blocks represent transformations applied to the

signal transmitted, for example delays, attenuation or filtering. On a communication link, signals can be transmitted bi directionally. Block diagram models represent only unidirectional transmission, from the designated output of one block to the designated input of another block and consequently contain only a direct model, from “cause” to “effect”. The model associated with a block corresponds only to the transfer from the input to the output. This might be acceptable for signal transmission, but for power transmission, which is normally bidirectional, effort-flow models are required.

Inverse model, from desired output to the required input, is obtained by matrix inversion for square LTI systems. Inverse model for non-square LTI systems require pseudo-inverse. For non-linear systems, no closed form solution might be available for model inversion.

In other engineering systems, the power transmitted by the carrier becomes important, and the equations describing their dynamics are written for variables like force and velocity in mechanical systems and voltage and current in electrical circuits. Equations using these variables can also be used for block diagram modeling. Again, while power often flows bi-directionally on a transmission line, a block diagram model can represent only a single direction of the transmission. In fact, state space models, transfer function and block diagram representations require the assignment of the direction of the signal from one component of the model to another.

Example 1.1 Consider first a simple mechanical system example, shown in Fig. 1.2, composed of a mass m , a spring k and a damper b and subject to a force input F . The velocity v is assumed the output.

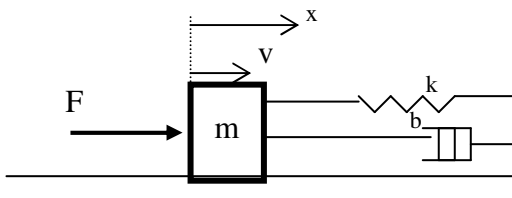


Fig. 1.2 A mass-spring-damper example

Newton's second law gives

$$F = m \frac{d^2}{dt^2} x(t) + b \frac{d}{dt} x(t) + k \cdot x(t)$$

The above differential equation can be written using v as variable

$$F = m \cdot \frac{d}{dt} v(t) + b \cdot v(t) + k \cdot \int_0^t v(\tau) d\tau$$

for

$$v(t) = \frac{d}{dt} x(t)$$

Laplace transform for zero initial conditions gives

$$v(s) = \frac{1}{ms + b + k/s} F(s)$$

Due to the input and output assignments, the same system is modeled differently when the variables F and v change designation. In this case, a simple inversion of the transfer function gives the inverse model

$$F(s) = (m \cdot s + b + k/s) \cdot v(s)$$

In general, however, model inversion does not have a closed form solution, typically for distributed parameters systems. This restricts modularity and interchangeability to modules with identical input and output assignment.

1.2.1 Effort Flow Variables and Two Port Models

Two port models were introduced for representing components of electric networks using two terminals for each port. Alternative names for two port components of a network are: four terminal network or quadripole. The two pole port models have associated a current I variable and a voltage V variable that permit the calculation of the power $P = VI$, transferred through the port [8,9].

Example 1.2 For an inductance-resistance L-R circuit supplied by an ideal voltage E source (i.e. with zero internal impedance), the circuit is shown in Fig. 1.3. Obtain the tree cuts diagram.

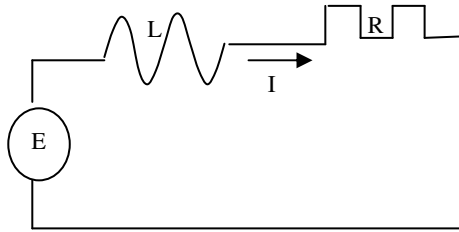


Fig. 1.3 A resistance-inductance R-L circuit

Resistance R and capacitance L components can be represented as separate elements as a result of three cuts (Fig. 1.4).

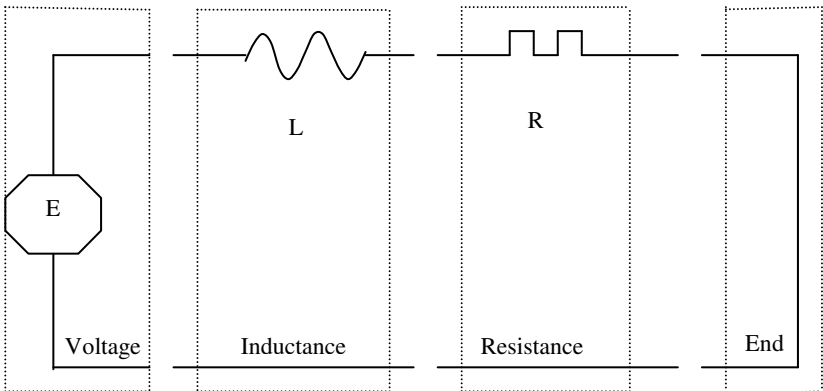


Fig. 1.4 Three cuts in R-L circuit

Example 1.3 A resistance - inductance-capacitance (R – L – C) series circuit subject to voltage V is shown in Fig. 1.5. Obtain $Z(s) = V(s) / i(s)$

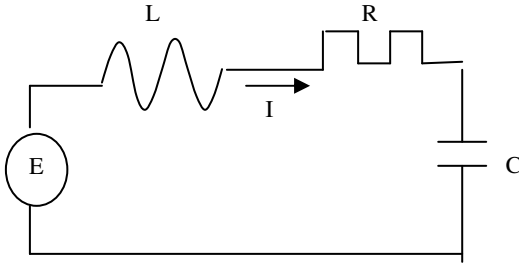


Fig. 1.5 R-L-C circuit

Solution The following voltage drop equation can be written

$$V(t) = Ri(t) + L \frac{d}{dt} i(t) + C \int_0^t i(\tau) d\tau$$

Laplace transform of the above equation for zero initial conditions gives

$$V(s) = (R + Ls + (C/s)) \cdot i(s)$$

The impedance $Z(s)$ of the resistance – inductance – capacitance series circuit is given by

$$Z(s) = V(s) / i(s)$$

or

$$Z(s) = R + L \cdot s + C/s$$

In the case of solid body mechanics, free body diagrams represent components of a multi body system obtained by cutting each “body” from the system and representing boundary effects by local force f and velocity v whose product gives the power $P = f \cdot v$.

Example 1.4 Assume that a flexible horizontal rod is linked to an undefined arbitrary mechanical systems by spherical joints. The rod, cut from these systems, give the free body diagram shown in Fig. 1.6. Obtain the model.

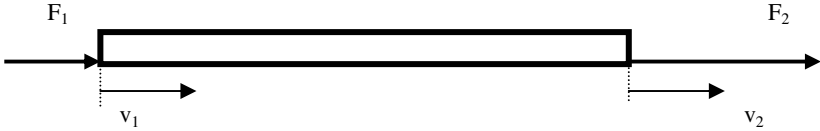


Fig. 1.6 Free body diagram of a rod

In Fig. 1.6 for each cut the internal force F and the absolute velocity v are identified. Assuming the flexible rod represented by a lumped parameters model, shown in Fig. 1.7, the following equations can be written

$$F_1(t) = (x_1(t) - x_2(t)) \cdot k + (v_1(t) - v_2(t)) \cdot b$$

$$F_2(t) = - [(x_1(t) - x_2(t)) \cdot k + (v_1(t) - v_2(t)) \cdot b] = -F_1(t)$$

Even if there is a spring and a damper between the two forces, the equality $F_2(t) = -F_1(t)$ reflects the fact that, in this model, the time-varying force change $F_1(t)$ applied to end 1 appears transmitted instantaneously at end 2, given that lumped parameters models do not account for propagation delay.

For

$$v_1(t) = \frac{d}{dt} x_1(t)$$

$$v_2(t) = \frac{d}{dt} x_2(t)$$

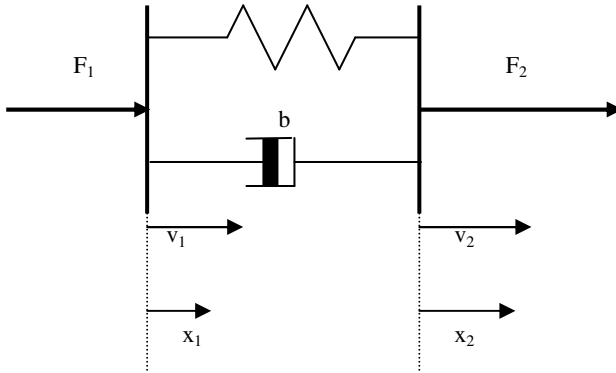


Fig. 1.7 Free body diagram for the rod

the Laplace transform gives

$$x_1(s) = v_1(s)/s$$

$$x_2(s) = v_2(s)/s$$

The two force equations give

$$v_1(s) = - [1/(k/s + b)] \cdot F_2(s) + v_2(s)$$

$$F_1(s) = - F_2(s)$$

These last two equations give the cut variables at end 1, F_1 and v_1 function only of cut variables at end 2, F_2 and v_2 , and parameters b and k , i.e. independent of the dynamics of the systems to which cut 1 and 2 were applied. Lumped parameters mechanical systems can be sectioned by cuts into subsystems interfaced only by force and velocities defined with respect to the cuts. For a flexible torsional shaft, with cut parameters torque T and angular velocity ω , the model is structurally similar:

$$\omega_1(s) = - [1/(k/s + b)] \cdot T_2(s) + \omega_2(s)$$

$$T_1(s) = - T_2(s)$$

A generalization to a variety of engineering systems can be based on the two port components that have associated a flow or through variable “ f ” and an across or effort variable “ e ” giving the power as the product (flow) \cdot (effort) [8]. This is the power passing through the junction of two components associated to a particular port [9].

In the case of distributed parameters systems, the interactions are too complex to be reducible to equivalent simple two-port models.

The direction of the power flow in the junction is bidirectional as opposed to the block diagram description in which signals have unidirectional flow. The same description, using effort-flow two pole ports, is suitable for mixed systems. The theoretical background of this description can be found in Hamiltonian dynamics for obtaining power transfer equations [8]. While effort-flow cuts permit to define power transfer between mixed subsystems, Hamiltonian and Lagrange dynamics permit simultaneous modeling of mixed systems, for example of electromechanical systems [9].

1.2.2 Newton-Euler and Kirchhoff Equations for a Mixed Electro-Mechanical System

Effort-flow representation of mixed systems permits easy application of Newton-Euler equations of motion and Kirchhoff equations for electric circuits. Power transfer conservation law at the conversions of electrical and mechanical energies permits to integrate the two models in an joint electro-mechanical model. The simplified diagram of Permanent Magnet-Direct Current (PM-DC) motor is shown in Fig. 1.8. The stator consists of a pair of magnetic poles N-S. The rotor consists of coils of conducting wires connected through the segments of a collector to a DC power supply.

Figure 1.8 shows the cut from a mechanical load (with cut variables torque T and angular velocity $\omega = d\theta/dt$) as well as the cut from a DC power supply (with cut variables voltage u and current i). The rotor is modeled mechanically as a rigid body with a moment of inertia “ J ” and a viscous friction coefficient “ b ” accounting for the air drag and viscous friction in the lubricated bearings. The electric model of the rotor is

given by the lumped parameters R and L , the rotor winding circuit from the electrical cut (u, i) towards the mechanical cut (T, ω).

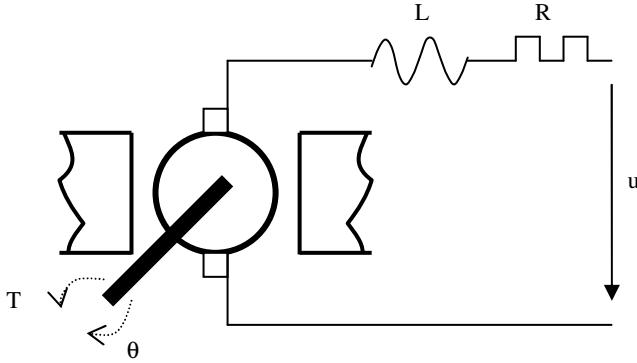


Fig. 1.8 The diagram of a PM- DC motor

The conversion of the electrical energy from the DC power source into the mechanical energy supplied to the load takes place in the DC motor, in particular in the electromagnetic field of the air gap between the stator and the rotor. Forces applied on rotor coils are generated as a result of the current i flowing through the rotor winding surrounded by the magnetic field produced by the PM of the stator. At the same time, the so called back electromotive force (back e.m.f.) are induced voltages in the moving rotor winding moving in the magnetic field. These two effects in a PM-DC motor can be modeled by separating the mechanical subsystem and the electrical subsystem, each being modeled by two port elements, as shown in Figs. 1.9 and 1.10, respectively.

In the left hand side of Fig. 1.9, torque components are represented around a cross section of the shaft. T_r denotes the torque generated in the electromagnetic field of the and acting on the rotor, while U_r represents the back electromagnetic force (back e.m.f.) induced by the magnetic field in the rotor winding in opposite to the supply voltage u . The torque T , and angular velocity ω are the cut variables toward the mechanical load, while the voltage u and the current i are the cut variables toward the DC power supply.

Example 1.5 Obtain the model equations.

The free body diagram and the two port circuit facilitate the derivation of the model equations.

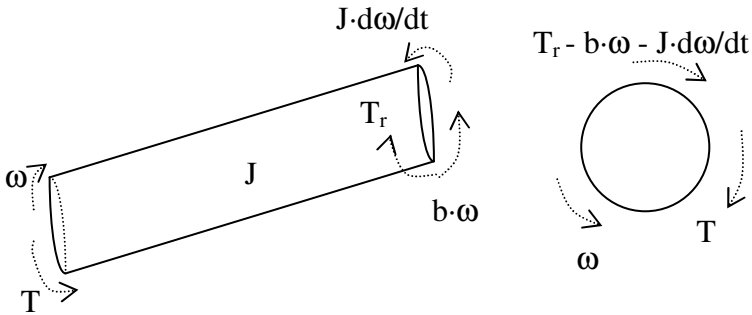


Fig. 1.9 Free body diagram for the mechanical part of the DC motor

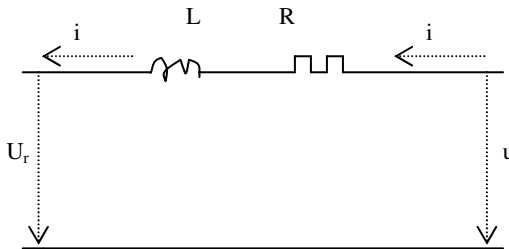


Fig. 1.10 Two port (U_r, i) and (u, i) circuit of the electrical part of the DC motor

Two algebraic equations result from the lumped-parameters model of the electro-mechanic conversion phenomena

$$T_r = k_m \cdot i$$

$$U_r = k_e \cdot \omega$$

where k_m [Nm/A] is the torque constant and k_e [Vs/rad] is the electrical constant.

In case of ideal conversion efficiency, $\eta = 1$, of the electrical power $U_r \cdot i$ into mechanical power $T_r \cdot \omega$,

$$\eta = (U_r \cdot i) / (T_r \cdot \omega) = 1$$

which gives

$$T_r \cdot \omega = U_r \cdot i$$

Using the above two algebraic equations, the following relationship is obtained

$$k_m \cdot i \cdot \omega = k_e \cdot \omega \cdot i$$

such that, in appropriate metric units, k_m in [Nm / A] and k_e in [Vs / rad], are of equal value

$$k_m = k_e$$

Power losses occur due to winding resistance, magnetic losses, friction etc. In the case of negligible losses, ideal power conversion can be assumed ($\eta = 1$).

For the mechanical part, shown in the free body diagram of Fig. 1.11, Newton second law gives:

$$J \frac{d\omega}{dt} = T_r - b \cdot \omega - T$$

For the electrical part shown in Fig. 1.12, the voltage drop equation gives:

$$u = L \frac{di}{dt} + R \cdot i + U_r$$

The last two differential equations and the two algebraic equations regarding the electro-mechanic conversion form a system of four

differential-algebraic equations containing six variables T , ω , T_r , U_r , i and u . This system of four differential-algebraic equations represents the analytical model of the PM-DC motor.

The elimination of internal variables T_r and U_r results in a model reduced to two differential equations with four variables of the two cuts (T , ω) and (u , i):

$$k_m \cdot i = J \frac{d\omega}{dt} + b \cdot \omega + T$$

$$u = L \frac{di}{dt} + R \cdot i + k_e \cdot \omega$$

Most DC motors have negligible L , such that the model, for $L = 0$, is reduced to:

$$J \frac{d\omega}{dt} = k_m \cdot i - b \cdot \omega - T$$

$$u = R \cdot i + k_e \omega$$

These equations, obtained using effort-flow cuts, permit the determination of the electrical power $u \cdot i$ and mechanical power $T \cdot \omega$ transferred between these subsystems.

1.2.3 Lagrange Equations for a Mixed Electro-Mechanical System

Lagrange equations are given by [11]:

$$\frac{d}{dt} \frac{\partial}{\partial \dot{q}_r} [K] - \frac{\partial}{\partial q_r} [K] + \frac{\partial}{\partial q_r} [U] = Q_r \quad \text{for } r = 1, 2, \dots, N$$

where

K is kinetic energy

U is potential energy

q_r is the generalized coordinate k

Q_r is the generalized force corresponding to the work done by the generalized coordinate q_r (or voltage in the case of the electrical generalized coordinate)

N is the total number of generalized coordinates needed to completely describe in time the components of the system.

For an electromechanical system with one generalized coordinate x for the mechanical part and one generalized coordinate Q for the electrical part, Lagrange equations for the mechanical and electrical parts of the system are given by [9, 11]:

$$\frac{d}{dt} \frac{\partial}{\partial \dot{x}} [K_m + K_e] - \frac{\partial}{\partial x} [K_m + K_e] + \frac{\partial}{\partial x} [U_m + U_e] = F$$

$$\frac{d}{dt} \frac{\partial}{\partial \dot{Q}} [K_m + K_e] - \frac{\partial}{\partial Q} [K_m + K_e] + \frac{\partial}{\partial Q} [U_m + U_e] = V$$

where

$K_m + K_e$ are the electric and mechanical kinetic energies

$U_m + U_e$ are the electric and mechanical potential energies

x is the generalized displacement variable (angular displacement)

$\dot{x} = v$ is the generalized velocity (angular velocity)

Q is the charge in capacitive components

$\dot{Q} = i$ is the current

F is the generalized force (dissipative and applied forces or torques)

V is the voltage (dissipative voltage drop and applied voltage)

Example 1.6 Obtain the model for the DC motor using Lagrange equations.

Lagrange equations for a PM- DC motor

For the DC motor shown in Fig. 1.8, Lagrange equations are

$$\frac{d}{dt} \frac{\partial}{\partial \dot{\theta}} [K_m + K_e] - \frac{\partial}{\partial \theta} [K_m + K_e] + \frac{\partial}{\partial \theta} [U_m + U_e] = F$$

$$\frac{d}{dt} \frac{\partial}{\partial \dot{Q}} [K_m + K_e] - \frac{\partial}{\partial Q} [K_m + K_e] + \frac{\partial}{\partial Q} [U_m + U_e] = V$$

or, taking into account that

$$\dot{\theta} = \omega$$

and

$$\dot{Q} = i$$

$$\frac{d}{dt} \frac{\partial}{\partial \dot{\omega}} [K_m + K_e] - \frac{\partial}{\partial \theta} [K_m + K_e] + \frac{\partial}{\partial \theta} [U_m + U_e] = F$$

$$\frac{d}{dt} \frac{\partial}{\partial \dot{i}} [K_m + K_e] - \frac{\partial}{\partial Q} [K_m + K_e] + \frac{\partial}{\partial Q} [U_m + U_e] = V$$

where

$$K_m(\omega) = J \cdot \omega^2/2$$

$$U_m = 0$$

$$F(\omega, i) = -b \cdot \omega + k_m \cdot i - T$$

$$K_e(i) = L \cdot i^2/2$$

$$U_e = 0$$

$$V(I, \omega) = u - R \cdot i - k_e \cdot \omega$$

Partial derivatives are

$$\frac{\partial}{\partial \omega} [K_m + K_e] = J\omega$$

$$\frac{\partial}{\partial \theta} [K_m + K_e] = 0$$

$$\frac{\partial}{\partial \theta} [U_m + U_e] = 0$$

$$\frac{\partial}{\partial i} [K_m + K_e] = Li$$

$$\frac{\partial}{\partial Q} [K_m + K_e] = 0$$

$$\frac{\partial}{\partial Q} [U_m + U_e] = 0$$

such that, for $k_m = k_e$, Lagrange equations result as follows

$$\frac{d}{dt} (J \cdot \omega) = k_m \cdot i - b \cdot \omega - T$$

$$\frac{d}{dt} (L \cdot i) = u - R \cdot i - k_e \cdot \omega$$

These are the same as the equations derived for the same DC motor using Effort-Flow representation of mixed systems and Newton-Euler equations of motion and Kirchhoff equations for electric circuits. Dissipative components are the dissipative voltage drop Ri and the dissipative generalized force, in this case the dissipative reaction torque $b \cdot \omega$.

Indeed, Lagrangian dynamics approach does not require effort-flow cuts neither for the mechanical subsystem nor for the electrical subsystem, and no internal variables were defined for such cases. For the interface between electrical and mechanical subsystems, applied torques T (external load torque) and $k_e \cdot i$ (motor torque) and applied voltages u (external voltage) and $k_e \cdot \omega$ (induced voltage) had to be however identified and this requires in fact the definition of the effort-flow cut at this interface.

Example 1.7 Figure 1.11 shows a plunger solenoid consisting of a solenoid of inductance $L(x)$, dependent of the displacement x of the plunger from the non-energized position $x = 0$. The motion of the plunger along x is due to the plunger induced force, caused by the solenoid current i . The current flows in the electric circuit R - $L(x)$ subject to the applied external voltage $u(t)$. On the mechanical side, the plunger of mass M consists of a flexible rod with stiffness coefficient k supported by a lubricated bearing with viscous friction coefficient b . Obtain the model using Lagrange equations.

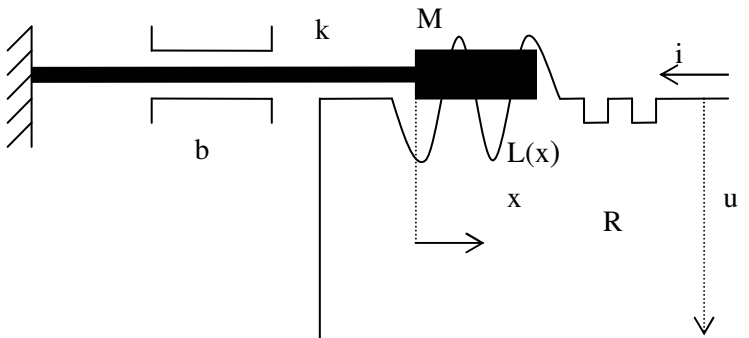


Fig. 1.11 The diagram of a plunger solenoid

Lagrange equations in this case are

$$\frac{d}{dt} \frac{\partial}{\partial \dot{x}} [K_m + K_e] - \frac{\partial}{\partial x} [K_m + K_e] + \frac{\partial}{\partial x} [U_m + U_e] = F$$

$$\frac{d}{dt} \frac{\partial}{\partial \dot{Q}} [K_m + K_e] - \frac{\partial}{\partial Q} [K_m + K_e] + \frac{\partial}{\partial Q} [U_m + U_e] = V$$

where

$$\dot{x} = v$$

and

$$\dot{Q} = i$$

such that

$$\frac{d}{dt} \frac{\partial}{\partial v} [K_m + K_e] - \frac{\partial}{\partial x} [K_m + K_e] + \frac{\partial}{\partial x} [U_m + U_e] = F$$

$$\frac{d}{dt} \frac{\partial}{\partial i} [K_m + K_e] - \frac{\partial}{\partial Q} [K_m + K_e] + \frac{\partial}{\partial Q} [U_m + U_e] = V$$

and

$$K_m(\omega) = M \cdot v^2/2$$

$$U_m = k \cdot x^2/2$$

$$F = -b \cdot v$$

$$K_e = L(x) \cdot i^2/2$$

$$U_e = 0$$

$$V = u(t) - R \cdot i$$

Partial derivatives are

$$\frac{\partial}{\partial v} [K_m + K_e] = M \cdot v$$

$$\frac{\partial}{\partial x} [K_m + K_e] = \frac{i^2}{2} \frac{d}{dx} L(x)$$

$$\frac{\partial}{\partial x} [U_m + U_e] = k \cdot x$$

$$\frac{\partial}{\partial i} [K_m + K_e] = L(x) \cdot i$$

$$\frac{\partial}{\partial Q} [K_m + K_e] = 0$$

$$\frac{\partial}{\partial Q} [U_m + U_e] = 0$$

such that Lagrange equations for the mechanical generalized coordinate x and for the electrical generalized coordinate Q result as follows

$$\frac{d}{dt} M \cdot v - \frac{i^2}{2} \frac{dL(x)}{dx} + k \cdot x = -b \cdot v$$

$$\frac{d}{dt} L(x) \cdot i = u(t) - R \cdot i$$

The term $(i^2 / 2) \cdot dL(x) / dx$ corresponds to the position dependent force applied by the solenoid on the plunger, while the term $d / dt L(x) \cdot i$ corresponds to the position dependent voltage drop on the solenoid inductance.

In reference [11] can be found other examples of Lagrangian dynamics for an electromechanical system in which there is a position

dependent capacitance and for an angular position dependent mutual inductance.

In this section, the same equations of motion of an electro-mechanical system were obtained using two approaches effort-flow cuts with Newton-Kirchhoff dynamics and Lagrangian dynamics. The latter approach is particularly interesting due to the link to Hamiltonian dynamics and Lyapunov stability analysis for mixed systems [9].

Example 1.8 Figure 1.12 shows an electromechanical system composed of a spring, with spring coefficient k , and a coil of radius ρ , with moment of inertia J and with N turns in which flows a current $I = dQ/dt$ [11]. The angular position of the coil with regard to the horizontal plane is θ and varies from 0° to 180° . The coil is subject to a magnetic field produced by a solenoid with n turns in which flows a current $i = dq/dt$. The angular displacement of the coil is due to the induced torque resulting the solenoid current I and coil current i . Resistances of the coil and of the solenoid are R and r , respectively. The coil is subject to a voltage $U(t)$ while the solenoid is subject voltage $u(t)$. Self-inductances L of the coil and l of the solenoid are constant, i.e. independent of the angular position θ of the coil. The mutual inductance $M(\theta)$, between the static solenoid and the rotating coil, is dependent of the angular position θ of the coil

$$M(\theta) = k_{nN} \cdot \pi \cdot \rho^2 \cdot n \cdot N \cdot \sin \theta$$

where k_{nN} is a characteristic constant of the coil.

The coil is supported by a lubricated bearing with viscous friction coefficient B . Obtain the model using Lagrange equations.

Lagrange equations in this case are

$$\frac{d}{dt} \frac{\partial}{\partial \dot{\theta}} [K_m + K_e] - \frac{\partial}{\partial \theta} [K_m + K_e] + \frac{\partial}{\partial \theta} [U_m + U_e] = F$$

$$\frac{d}{dt} \frac{\partial}{\partial \dot{Q}} [K_m + K_e] - \frac{\partial}{\partial Q} [K_m + K_e] + \frac{\partial}{\partial Q} [U_m + U_e] = V$$

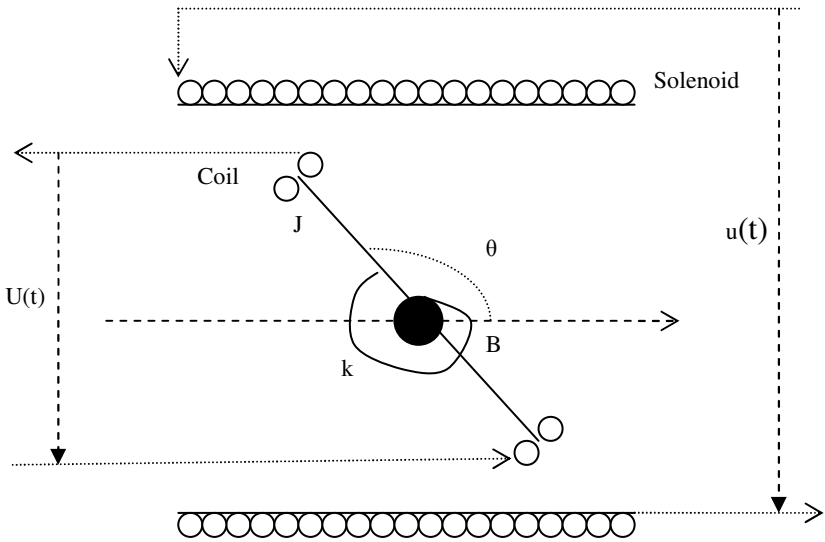


Fig. 1.12 The diagram of rotating spring coil and solenoid system

$$\frac{d}{dt} \frac{\partial}{\partial \dot{q}} [K_m + K_e] - \frac{\partial}{\partial q} [K_m + K_e] + \frac{\partial}{\partial q} [U_m + U_e] = v$$

where

$$\frac{d}{dt} \theta = \omega$$

$$\frac{d}{dt} Q = I$$

$$\frac{d}{dt} q = i$$

such that

$$\frac{d}{dt} \frac{\partial}{\partial \dot{\omega}} [K_m + K_e] - \frac{\partial}{\partial \theta} [K_m + K_e] + \frac{\partial}{\partial \theta} [U_m + U_e] = F$$

$$\frac{d}{dt} \frac{\partial}{\partial I} [K_m + K_e] - \frac{\partial}{\partial Q} [K_m + K_e] + \frac{\partial}{\partial Q} [U_m + U_e] = V$$

$$\frac{d}{dt} \frac{\partial}{\partial i} [K_m + K_e] - \frac{\partial}{\partial q} [K_m + K_e] + \frac{\partial}{\partial q} [U_m + U_e] = v$$

and

$$K_m(\omega) = J \frac{\omega^2}{2}$$

$$U_m = k \frac{\theta^2}{2}$$

$$F = -B \cdot \omega$$

$$K_e = L \frac{I^2}{2} + l \frac{i^2}{2} + M(\theta) \cdot I \cdot i = L \frac{I^2}{2} + l \frac{i^2}{2} + k_{nN} \cdot \pi \cdot \rho^2 \cdot n \cdot N \cdot \sin(\theta) \cdot I \cdot i$$

or

$$K_e = L \frac{I^2}{2} + l \frac{i^2}{2} + \alpha \cdot I \cdot i \cdot \sin\theta$$

$$U_e = 0$$

$$V = U(t) - R I$$

$$v = u(t) - r i$$

where the constant α is

$$\alpha = k_{nN} \cdot \pi \cdot \rho^2 \cdot n \cdot N$$

Partial derivatives are

$$\frac{\partial}{\partial \theta} [K_m + K_e] = \frac{\partial}{\partial \theta} \alpha \cdot I \cdot i \cdot \sin \theta = \alpha \cdot I \cdot i \cdot \cos \theta$$

$$\frac{\partial}{\partial Q} [K_m + K_e] = 0$$

$$\frac{\partial}{\partial q} [K_m + K_e] = 0$$

$$\frac{\partial}{\partial \theta} [U_m + U_e] = k \cdot \theta$$

$$\frac{\partial}{\partial I} [K_m + K_e] = L \cdot I + \alpha \cdot i \cdot \sin \theta$$

$$\frac{\partial}{\partial i} [K_m + K_e] = l \cdot i + \alpha \cdot I \cdot \sin \theta$$

$$\frac{\partial}{\partial Q} [K_m + K_e] = 0$$

$$\frac{\partial}{\partial q} [K_m + K_e] = 0$$

$$\frac{\partial}{\partial Q} [U_m + U_e] = 0$$

$$\frac{d}{dt} \frac{\partial}{\partial \omega} [K_m + K_e] = I \cdot \dot{\omega}$$

$$\frac{d}{dt} \frac{\partial}{\partial I} [K_m + K_e] = L \cdot \dot{I} + \alpha \cdot \dot{i} \cdot \sin \theta + \alpha \cdot i \cdot \omega \cdot \cos \theta$$

$$\frac{d}{dt} \frac{\partial}{\partial i} [K_m + K_e] = l \cdot \dot{i} + \alpha \cdot \dot{I} \cdot \sin \theta + \alpha \cdot I \cdot \omega \cdot \cos \theta$$

such that Lagrange equations for the mechanical generalized coordinate θ and for the electrical generalized coordinates Q and q result as follows

$$J \cdot \ddot{\theta} - \alpha \cdot I \cdot i \cdot \cos\theta + k \cdot \theta = -B \cdot \omega$$

$$L \cdot \dot{I} + \alpha \cdot \dot{i} \cdot \sin\theta + \alpha \cdot i \cdot \omega \cdot \cos\theta = U(t) - R \cdot I$$

$$L \cdot \dot{i} + \alpha \cdot \dot{I} \cdot \sin\theta + \alpha \cdot I \cdot \omega \cdot \cos\theta = u(t) - r \cdot i$$

or

$$J \cdot \ddot{\theta} + B \cdot \dot{\theta} + k \cdot \theta = \alpha \cdot I \cdot i \cdot \cos\theta$$

$$L \cdot \dot{I} + R \cdot I = U(t) - \alpha \cdot \dot{i} \cdot \sin\theta - \alpha \cdot i \cdot \omega \cdot \cos\theta$$

$$L \cdot \dot{i} + r \cdot i = u(t) - \alpha \cdot \dot{I} \cdot \sin\theta - \alpha \cdot I \cdot \omega \cdot \cos\theta$$

These three nonlinear differential equations with variables $\theta(t)$, $I(t)$ and $i(t)$ represent the model of the system from Fig. 1.12, given the inputs $U(t)$ and $u(t)$, *i.e.* the direct problem. In practical applications, one of the inputs, $U(t)$ or $u(t)$, can be held constant. For either I or I vanishing, the first equation gives the equilibrium position $\theta = 0$.

In the first equation, for the rotational mechanical subsystem, the term $T = \alpha \cdot I \cdot i \cdot \cos\theta$ represents the torque produced by the magnetic fields interaction of the solenoid with the coil, which is zero when the coil and the solenoid are perpendicular, *i.e.* when $\theta = 90^\circ$, or when the two magnetic fields are parallel. As a result, the angle θ should be limited to the domain

$$-90 + \varepsilon < \theta < 90 - \varepsilon$$

where ε can be obtained from the condition that maximum admissible currents I_{\max} and i_{\max} produce a minimum required torque T_{\min} to be able to rotate the J-B-K mechanical system, *i.e.*

$$T_{\min} = \alpha I_{\max} i_{\max} \cos|\theta - \varepsilon|$$

In second equation, for the moving coil, the terms $\alpha \cdot \dot{i} \cdot \sin\theta + \alpha \cdot i \cdot \omega \cdot \cos\theta$ represent the induced voltages in the coil due to the time varying current and due to the coil angular velocity. Similarly, the terms $\alpha \cdot \dot{I} \cdot \sin\theta + \alpha \cdot I \cdot \omega \cdot \cos\theta$ represent induced voltages in the solenoid due to the time varying current and due to the coil angular velocity.

Example 1.9 Figure 1.13 shows a capacitance with a moving top electrode of mass m and with a gap $X - x$, where X is the gap. The equilibrium position of the top electrode is $x = 0$, when no voltage is applied to the capacitance and the spring is stretched by $m \cdot g / k$ to counterbalance top electrode weight $m \cdot g$. The bottom electrode is sitting on a fixed electric insulator. The top electrode can move vertically with the displacement x , as a result of the time varying voltage applied to the electrodes from a voltage source with $U(t)$ connected through wires with resistance R and inductance L [11]. The top electrode is connected to the moving bottom end of a spring with spring coefficient k . The spring has the top end connected to a fixed insulator. Assume that the structural damping coefficient is b .

The capacity of the time varying gap capacitance is given by

$$C(x) = \frac{k \cdot A}{X - x}$$

where k is the dielectric constant and A is the cross-sectional area of the capacitance. Obtain the model using Lagrange equations.

Lagrange equations in this case are

$$\frac{d}{dt} \frac{\partial}{\partial \dot{x}} [K_m + K_e] - \frac{\partial}{\partial x} [K_m + K_e] + \frac{\partial}{\partial x} [U_m + U_e] = F$$

$$\frac{d}{dt} \frac{\partial}{\partial \dot{Q}} [K_m + K_e] - \frac{\partial}{\partial Q} [K_m + K_e] + \frac{\partial}{\partial Q} [U_m + U_e] = V$$

where

$$\frac{dx}{dt} = v$$

$$\frac{dQ}{dt} = I$$

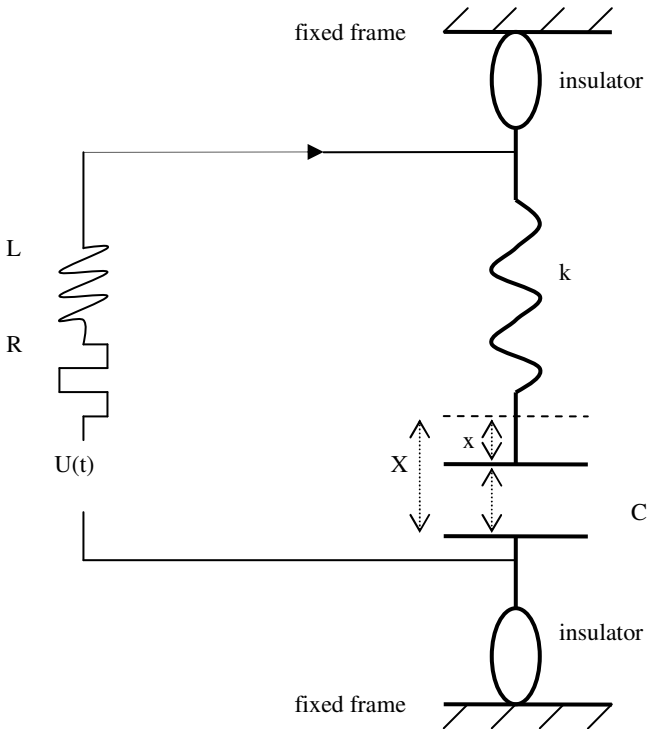


Fig. 1.13 Diagram of a system consisting of a capacitance and a spring

such that

$$\frac{d}{dt} \frac{\partial}{\partial v} [K_m + K_e] - \frac{\partial}{\partial x} [K_m + K_e] + \frac{\partial}{\partial x} [U_m + U_e] = F$$

$$\frac{d}{dt} \frac{\partial}{\partial I} [K_m + K_e] - \frac{\partial}{\partial Q} [K_m + K_e] + \frac{\partial}{\partial Q} [U_m + U_e] = V$$

and

$$K_m(v) = m \frac{v^2}{2}$$

$$U_m(x) = k \frac{x^2}{2}$$

$$F = -b \cdot v$$

$$K_e(I) = L \frac{I^2}{2}$$

$$U_e(x) = Q^2 \frac{X - x}{2 \cdot A \cdot k}$$

$$U_e(x) = Q^2 \cdot (X - x) / (2 \cdot A \cdot c)$$

Partial derivatives are

$$\frac{\partial}{\partial x} [K_m + K_e] = \frac{\partial}{\partial x} \left(m \frac{v^2}{2} + L \frac{I^2}{2} \right) = 0$$

$$\frac{\partial}{\partial Q} [K_m + K_e] = 0$$

$$\frac{\partial}{\partial x} [U_m + U_e] = \frac{\partial}{\partial x} \left[k \frac{x^2}{2} + Q^2 \cdot \frac{(X - x)}{2 \cdot A \cdot k} \right] = k \cdot x - \frac{Q^2}{2 \cdot A \cdot k}$$

$$\frac{\partial}{\partial v} [K_m + K_e] = \frac{\partial}{\partial v} \left(m \frac{v^2}{2} + L \frac{I^2}{2} \right) = m \cdot v$$

$$\frac{\partial}{\partial I} [K_m + K_e] = \frac{\partial}{\partial I} \left(m \frac{v^2}{2} + L \frac{I^2}{2} \right) = L \cdot I$$

$$\frac{\partial}{\partial Q} [K_m + K_e] = 0$$

$$\frac{\partial}{\partial Q} [U_m + U_e] = \frac{\partial}{\partial Q} \left[kx^2/2 + Q^2 \frac{X-x}{2 \cdot A \cdot k} \right] = Q \frac{X-x}{A \cdot k}$$

$$\frac{d}{dt} \frac{\partial}{\partial v} [K_m + K_e] = m \cdot \dot{v}$$

$$\frac{d}{dt} \frac{\partial}{\partial I} [K_m + K_e] = L \cdot \dot{I}$$

Lagrange equations for the mechanical generalized coordinate θ and for the electrical generalized coordinates Q and q result as follows

$$m \cdot \dot{v} + k \cdot x - \frac{Q^2}{2 \cdot A \cdot k} = -b \cdot v$$

$$L \cdot \dot{I} + Q \frac{X-x}{A \cdot k} = U(t) - R \cdot I$$

The following two second order nonlinear differential equations with unknowns $x(t)$ and $Q(t)$ and $q(t)$ represent the model of the system from Fig. 1.13, given the time varying input voltage $U(t)$.

$$m \cdot \ddot{x} + b \cdot \dot{x} + k \cdot x = F(Q)$$

$$L \cdot \ddot{Q} + R \cdot \dot{Q} + \frac{Q}{C(x)} = U(t)$$

where $C(x)$ is the time varying gap dependant capacitance with

$$C(x) = \frac{k \cdot A}{X - x}$$

and

$$F(Q) = \frac{Q^2}{2 \cdot k \cdot A}$$

is the charge dependant force applied by the moving electrode to the bottom end of the spring.

The two nonlinear differential equations with variables x and Q permit to model the effect of time varying external voltage $U(t)$ on the displacement $x(t)$ of the moving top electrode, i.e. a direct problem.

1.3 Local Sensing and Actuation in Spatially Continuous Systems

Spatially continuous systems, can be modeled using either effort-flow cuts or Lagrangian dynamics. These models are needed for the design of systems or for their real-time monitoring and control.

Continuous systems can be modeled with lumped parameters models or with distributed parameters models, depending on the acceptable level of accuracy and modeling difficulties. In both cases, the number of inputs can be lower than the number of degrees of freedom, resulting in under-actuation or lower number of outputs than states, resulting in under-sensing. The issue of local sensing and actuation has to be investigated in both cases. Control of these systems can be either open loop or closed loop. Under-actuation and under-sensing have consequences on the performance of both types of systems, but is a particularly difficult problem to solve for distributed parameters models [18].

1.3.1 *Lumped Parameters Models with Under-Actuation and Under-Sensing*

Lumped parameters models for linear case can be written in the form of linear ordinary differential equations (ODE):

$$d\mathbf{X}(t) / dt = \mathbf{A}(t) \cdot \mathbf{X}(t) + \mathbf{B}(t) \cdot \mathbf{u}(t) + \mathbf{G}(t) \cdot \mathbf{w}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t) \cdot \mathbf{X}(t) + \mathbf{D}(t) \cdot \mathbf{u}(t)$$

where

$\mathbf{X}(t)$ = n-vector of states with given initial conditions $\mathbf{x}(0)$

$\mathbf{u}(t)$ = m-vector of inputs

$\mathbf{w}(t)$ = d-vector of disturbances

$\mathbf{y}(t)$ = p-vector of outputs

$\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{G}(t)$, $\mathbf{C}(t)$, $\mathbf{D}(t)$ = time varying matrices.

Lumped models for nonlinear case can also be written in the form of linear ordinary differential equations (ODE):

$$d\mathbf{X}(t) / dt = \mathbf{F}(\mathbf{X}(t), \mathbf{u}(t), \mathbf{w}(t))$$

$$\mathbf{y}(t) = \mathbf{H}(\mathbf{X}(t), \mathbf{u}(t))$$

where \mathbf{F} and \mathbf{H} are nonlinear functions.

The number of states, n , is finite and, consequently, lumped parameters models which are a simplified representation of continuous systems. Certainly, spatial resolution is in the former case limited. Under-actuation results from fewer inputs m than the number of degrees of freedom N , i.e. $m < N$, and under-sensing from fewer outputs p than the number of states, i.e. $p < n$. A continuous system would have infinite values for n and N , consequently, finite number of actuators and sensors will always result in this case in under-actuation and under-sensing. Given the complexities of distributed parameters models, under-actuation and under-sensing issues are easier to be analyzed using in a first approximation lumped linear models represented by ordinary differential equations (ODE) with time invariant (LTI) parameters.

1.3.2 *Distributed Parameters Models with Under-Actuation and Under-Sensing*

Distributed parameters models can be take a large variety of mathematical forms. A generic form is:

$$\delta \mathbf{X}(x, y, z, t) / \delta t = \mathbf{F}(\mathbf{X}(x, y, z, t), \nabla \mathbf{X}(x, y, z, t), \nabla^2 \mathbf{X}(x, y, z, t), \dots, \mathbf{w}(t))$$

subject to boundary conditions

$$\mathbf{G}(\mathbf{X}(x_b, y_b, z_b, t), \mathbf{u}(t)) = 0$$

and intial conditions

$$\mathbf{I}(\mathbf{X}(x, y, z, 0), \mathbf{u}(0)) = 0$$

While, output equation is

$$\mathbf{H}(\mathbf{y}(x_m, y_m, z_m, t), \mathbf{X}(x, y, z, t), \mathbf{u}(t))$$

where ∇ is the partial differentiation operator, with regard to x, y, z , variables and the function G and the subscript b refer to boundary conditions, while the function I defines initial conditions. It can be observed that control variables $\mathbf{u}(t)$ appear in this case only in the boundary conditions, a typical case in practice where the continuous system is actuated only from specific system boundaries. Similarly, the outputs \mathbf{y} are typically measured in some specific points x_m, y_m, z_m . These limitations regarding local actuation and sensing pose specific challenges to the design and performance of controllers and for the integration of spatially continuous systems.

1.4 Centralized versus Local Control

Local sensing and actuation of systems with large or infinite number of states is linked also to the issue of centralized versus local control. A finite number of actuators can be controlled either at the actuator location or using a centralized control for all actuators.

Local controllers use collocated actuators and sensors, have the advantage of easier design and tuning and tend to produce predictable local system behavior, but are not optimal for the system as a whole. Moreover, dynamic couplings in the system can result in inefficient or unstable system behavior. Centralized control can be designed optimally, but suffers from unavoidable simplifications of the system model on which they are based and requires often a prohibitively large number of signal transmissions [19]. These issues are critical for continuous systems distributed over a large area or for formations.

Problems

1. Consider the system shown in Fig. 1.2 but with added viscous friction between the mass M and the ground, with viscous friction coefficient B . Obtain $v(s)$ given $F(s)$.
2. For the system shown in Fig. 1.5, obtain the four cuts representation.
3. For the free body diagram shown in Fig. 1.7, consider that the mass of the rod is not negligible and that is concentrated equally at the two ends of the diagram as M_1 and M_2 . Obtain the equations for v_1 and F_1 function of v_2 and F_2 .
4. For the DC motor shown in Fig.1.8, assume that the shaft is flexible, such that in the free body diagram from Fig. 1.9 a torsional spring coefficient K is in series with the moment of inertia J .
 - a. Obtain the model with two differential equations for the cut variables (T, ω) and (u, i)
 - b. Verify that the same model is obtained using Lagrange equations.
5. Assume that the plunger solenoid from Fig. 1.11 has the plunger of mass M connected by a spring, with spring coefficient K , to a

- right hand side rigid wall. Obtain the Lagrange equations of motion.
6. For electromechanical system shown in Fig. 1.12, the mutual inductance between the static solenoid and the rotating coil is $M(\theta) = k_{nN} \cdot \pi \cdot \rho^2 \cdot n \cdot N \cdot \sin \theta$. The coil, of moment of inertia J , actuates a flexible shaft supported at one end by a lubricated bearing with viscous friction coefficient B . The shaft, with torsional stiffness coefficient K , has a load with a moment of inertia J , and has itself a negligible moment of inertia, relative to the two end moments of inertia. Obtain Lagrange equations for this system.
 7. Consider the system shown in Fig. 1.13, which consists of a capacitance with a moving top electrode of mass m and with a gap $X - x$, where X is the gap for the equilibrium position $x = 0$, when no voltage is applied to the capacitance, and the spring is stretched by $m \cdot g / k$ to counterbalance top electrode weight $m \cdot g$. The bottom electrode is sitting on a fixed insulator. The top electrode is moving vertically with the displacement x , as a result of the time varying voltage applied to the electrodes from a voltage source with $U(t)$ connected through wires with resistance R and inductance L . The top electrode is connected to the moving bottom end of a spring with spring coefficient k and in parallel with a damper with damping coefficient b . The spring has the top end connected to a fixed insulator. The capacity of the time varying gap capacitance is $C(x) = c \cdot A / (X - x)$, where c is a constant dependent of the insulator between the electrodes. Obtain the model using Lagrange equations.
 8. For a multi-DOF linear lumped parameters mechanical system, the system is considered under-actuated if:
 - a. there are fewer actuators than the number of states
 - b. there are as many actuators as the number of states
 - c. there are as many actuators as the number of degrees of freedom.