Chapter 1

Notation of Knots and Links

1.1 Basic graph theory

This basic introduction to graph theory is written according to the books by R.A. Wilson (2002), N.D. Gilbert and T. Porter (1994), with some changes in definitions, and certain additions.

We will start with the definition of a *graph*:

Definition 1.1. A *graph* G consists of a set of *vertices* $V(G)$ and a set of edges $E(G)$, such that each edge is *incident* with two (not necessarily distinct) vertices.

A graph G can be denoted by $G = G(V, E)$. Two vertices are *adjacent* if there exists an edge join them, and they are the *endpoints* of the edge. Two edges are *adjacent* if they have a common endpoint. An edge which joins a vertex to itself is called a *loop*, k edges which join the same pair of vertices are called k*-multiple* edges, and the corresponding graph is called a *multigraph*. If a multigraph contains only single and 2-multiple (or double) edges, it is called a *2-graph*. A graph is *simple* if it contains no loops and multiple edges. A graph without loops is called *proper*, or *reduced* graph.

If we distinguish the order of the endpoints of edges, treating them as *ordered pairs* of vertices, we obtain *oriented graphs* (or *digraphs*).

As usual, we will draw graphs with enlarged (labelled) dots for the vertices, and straight or curved lines for edges, in such a way that a vertex and an edge are incident *iff* they meet in the diagram. The placement of points in the diagram, and whether the lines representing edges are straight, curved or have to cross one another in any point other then vertex, is irrelevant.

Definition 1.2. The *valence* (or *degree*) of a vertex is the number of edges which are incident to it (in a graph with loops, we usually count a loop-edge twice). The valence of a vertex v will be denoted $d(v)$.

A vertex of a graph G is *single* or *isolated* if its valence is 0. Usually, a non-oriented graph without isolated vertices will be given by a list of *unordered pairs* representing edges. In the case of digraphs, ordered pairs will represent oriented edges. A graph can also be given by its *adjacency list* whose entries are lists, each starting with a vertex followed by vertices adjacent to it, where the order of adjacent vertices is irrelevant.

Definition 1.3. A graph in which all vertices are k-valent is called a *k-valent graph* (or *k-regular graph*).

Since knots are 1-component links ($c = 1$), we will use the term "link" or KL (knot or link) for both knots and links, unless we need to talk about properties specific to knots only.

A graph is $(3,4)$-valent if it contains vertices of valences only 3 or 4. Four-valent graphs will be extremely important for study of KLs as they represent KL shadows.

Among the graphs corresponding to five Platonic regular polyhedra, the tetrahedron, cube and dodecahedron graphs are 3-valent, the octahedron graph is 4-valent, and the icosahedron graph is 5-valent (Fig. 1.1).

Definition 1.4. A graph is *complete* if every pair of vertices is adjacent. A graph is *bipartite* if the vertices can be partitioned into two disjoint sets X and Y such that all the edges join a vertex in X to a vertex in Y. A graph is *complete bipartite* if it contains all possible edges from a vertex in X to a vertex in Y.

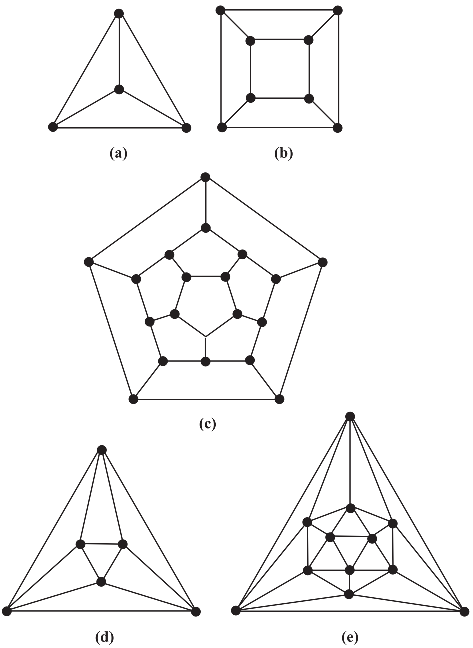
The complete graph on n vertices is usually denoted K_n, while the complete bipartite graph on two sets of m and n vertices is denoted $K_{m,n}$. For example,

$$K_5 = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\},$$

$$K_{3,3} = \{\{1,4\},\{1,5\},\{1,6\},\{2,4\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\}\}.$$

Definition 1.5. A *walk* of length n is a sequence $v_1e_1v_2e_2\ldots v_ne_nv_{n+1}$ of vertices v_i $(1 \leq i \leq n+1)$ and edges e_j $(1 \leq j \leq n)$ such that each is incident to the next.

LINKNOT - Knot Theory by Computer



(a) (b) (c) (d) (e)

Fig. 1.1 (a) Tetrahedron, (b) cube, (c) dodecahedron, (d) octahedron, (e) icosahedron graph.

A walk is *closed* if $v_1 = v_{n+1}$, and *open* otherwise. A *trail* is a walk in which all edges are distinct, a *circuit* is a closed trail with at least one edge. A *path* is a trail in which all vertices are distinct (except v_1 and v_{n+1} if a trail is closed). A *cycle* is a circuit with all distinct vertices (except v_1 and v_{n+1}). An *Euler's circuit* is a walk that uses each edge of a graph exactly once.

Definition 1.6. Two vertices are *connected* if there is a walk from one to the other.

The relation of *vertex connectivity* is an *equivalence relation* (reflexive, symmetric and transitive) which partitions a set of vertices $V(G)$ into equivalence classes called (connected) *components* of G.

Definition 1.7. A graph is *connected* if every pair of vertices is connected (i.e., if all vertices belong to one component).

A *tree* is a connected graph with no cycles.

The *vertex connectivity* $k(G)$ of a graph G is the minimum number of vertices that need to be removed together with their incident edges in order to obtain disconnected or 1-vertex graph (in the case when G can not be disconnected by removing vertices).

Definition 1.8. A graph is called *k-vertex connected* (or just k-connected) if at least k vertices need to be removed in order to disconnect the graph, or to obtain a 1-vertex graph.

Definition 1.9. A connected graph G is *k-edge connected*, if at least k edges need to be removed in order to disconnect the graph. The *edge connectivity* of a graph G is the minimum number of edges which need to be deleted in order to disconnect the graph.

A *subgraph* $G' = G'(V', E')$ of a graph $G = G(V, E)$ is a graph such that $V' \subset V$, $E' \subset E$, and both endpoints of each edge from E' belong to V'.

Definition 1.10. Two graphs G and G' are *isomorphic* ($G \simeq G'$) if there is a one-to-one correspondence between their vertices and one-to-one correspondence between their edges, which preserves incidence.

If the edge e in G corresponds to the edge e' in G', then the endpoints of e correspond to the endpoints of e'.

Definition 1.11. A graph G is *plane* if it is drawn in plane (or on the sphere) with no two edges crossing each other, and it is *planar* if it is isomorphic to a plane graph.

A simple planar graph can be embedded in $\Re^2$ so that each edge is a straight line (for the proof see, e.g., Cromwell, 2004).

Stereographic projection caries plane embeddings to embeddings on a sphere and *vice versa*.

Definition 1.12. An *embedding* of a graph G is a drawing of G on a certain surface in which the edges do not intersect.

A non-planar graph can be always embedded on some surface, other then the plane (or sphere). For example, all graphs of polyhedra (Fig. 1.1) are planar, and the graphs K_5 and $K_{3,3}$ are non-planar.

Definition 1.13. An *automorphism* of a graph G is any isomorphism of G to itself.

All automorphisms of a graph G make its *automorphism group* denoted as $Aut(G)$.

In every graph drawing, edges are drawn by broken and/or smooth lines, where the break points or nugatory edge crossings in graphs which are not plane should not be perceived as vertices. Usually, graph vertices are labelled and/or denoted by dots.

Definition 1.14. An embedding of a graph induces a *map* M: the division of the unbounded surface on which the graph is embedded into disjoint simply-connected *regions* called *faces*. A face with two vertices and two edges will be called a *bigonal face* (or just *bigon*).

The *dual* $D(M)$ of a given map M can be constructed in the following way: in the map M you draw a vertex of $D(M)$ in the interior of each region of M (including the exterior region), and you join them by edges, one edge of $D(M)$ crossing each edge of M. The graph $D(M)$ is called the *dual* of M. In the case of polyhedra and their corresponding graphs, you can join up the points in the interior of adjacent faces of a polyhedron P to obtain the dual polyhedron $D(P)$. Doing this a second time gets you back to a polyhedron $D(D(P))$ isomorphic to P. Different planar embeddings of a planar graph G may give different dual graphs $D(G)$, so there exist isomorphic graphs with non-isomorphic duals (Fig. 1.2).

A component of graph G is its maximal connected subgraph, and a component of map M is its maximal connected submap.

Every embedding can be described by an *embedding adjacency matrix.* For each entry, the first vertex is followed by a sequence of its adjacent vertices given in the same (left or right) cyclic order. For the end points, only the cyclic permutation corresponding to them is important, and not their particular position in the permutation. For example, a planar embedding of the octahedron graph (Fig. 1.3)

$$O = \{\{1,2\},\{1,3\},\{1,5\},\{1,6\},\{2,3\},\{2,4\},\{2,6\},\{3,4\},\{3,5\},\{4,5\},\{4,6\},\{5,6\}\}$$

is

$$\{\{1,2,6,5,3\},\{2,3,4,6,1\},\{3,1,5,4,2\},\{4,3,5,6,2\},\{5,3,1,6,4\},\{6,2,4,5,1\}\}.$$

After drawing the first vertex 1, we draw its incident edges in the right cyclic order: $\{1,2\}$, $\{1,6\}$, $\{1,5\}$, $\{1,3\}$, then we continue with the vertex 2 and its adjacent edges in the same right cyclic order: $\{2,3\}$, $\{2,4\}$, $\{2,6\}$,

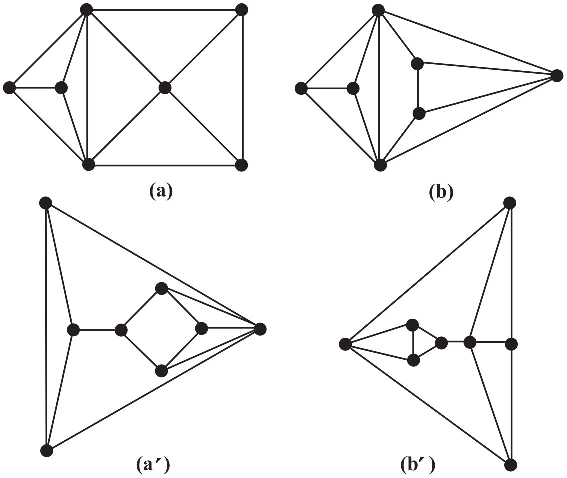


Fig. 1.2 Isomorphic graphs (a), (b), and their non-isomorphic duals (a′), (b′).

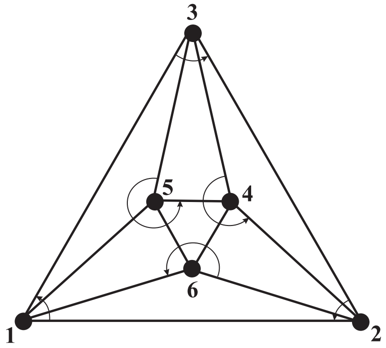


Fig. 1.3 The planar embedding of octahedron graph.

$\{2,1\}$, having in mind that $\{2,1\}$ is already drawn as $\{1,2\}$, *etc.*, until using all edges of the graph.

LinKnot function **fPlanarEmbGraph** gives the planar embedding of a 3-connected planar graph given by a list of unordered pairs. The output is a list that consists of the input graph, its planar embedding, and the faces of the planar embedded graph. The basis of this program is the external program *planarity.exe* written by J.M. Boyer (Boyer and Myrvold, 2005). The *LinKnot* function **DrawPlanarEmbGraph** draws a planar embedding of a graph given by a list of unordered pairs, and the function **DrawPlanarEmbKL** draws a planar embedding of a KL without multiple edges. The basis of these functions is the program *3-Dimensional Convex Drawings of 3-Connected Planar Graphs* by M. Ochiai, N. Imafuji and N. Morimura.

In every plane graph drawn in the plane $\Re^2$ we visually distinguish an external face and internal faces placed inside it. In the octahedron plane graph (Fig. 1.3) the external face is $\{1, 2, 3\}$, and other (internal) faces are placed inside it. Sometimes, especially for the plane graphs obtained from symmetric polyhedra, it is useful to imagine them on a sphere S^3.

Theorem 1.1. *(Euler's formula) Every planar map M with v vertices, e edges, f faces and c components satisfies Euler's formula:*

$$v - e + f = c + 1.$$

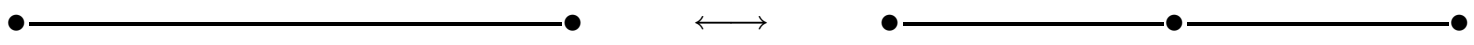
The term $\chi = v - e + f$ obtained from a map on any surface is called the *Euler characteristic* of the surface.

Euler's formula for polyhedra ($c = 1$) was discovered around 1750 by L. Euler and first proven by A.M. Legendre in 1794. The interested reader can find 19 different proofs of Euler's formula in D. Eppstein's *The Geometry Junkyard* (http://www.ics.uci.edu/~eppstein/junkyard/euler/). A few of them are based on *Jordan Curve Theorem*, proved by O. Veblen in 1905:

Theorem 1.2. *(Jordan Curve Theorem) If c is a simple closed curve in $\Re^2$, then $\Re^2 \setminus c$ has two components (an "inside" and "outside"), with the c boundary of each (Jordan, 1887; Veblen, 1905; Hatcher, 2002; Grabowski, 2005).*

The complete proof of Jordan Curve Theorem is given in *Algebraic Topology* by A. Hatcher (2002), and the computer proof in the proof checker *Mizar* required 200 000 lines (Grabowski, 2005).

The most celebrated result about the planarity of graphs is *Pontryagin-Kuratowski's Theorem.* Two graphs G and G' are *isomorphic modulo vertices of degree 2* if G is isomorphic to a graph G'' obtained from G' by the addition or deletion of vertices with just two incident edges:



Theorem 1.3. *(Pontryagin-Kuratowski's Theorem) Let G be a finite graph. G is planar iff it contains no subgraph isomorphic modulo vertices of degree 2 to K_5 or $K_{3,3}$ (Kuratowski, 1930).*

Short proof of the sufficiency part of this theorem is given by Y. Makarychev (1997), and the complete proof can be found, e.g., in *Graphs on Surfaces* by B. Mohar and C. Thomassen (2001).

The transformations described above are a *subdivision* and a *contraction* of a graph edge. A *subdivision* of a graph G is a graph obtained from G by a finite number of the following operations. Let v, w be the vertices of G which are connected by the edge vw. Introduce a new vertex x and replace the edge vw by two edges vx and xw, i.e., insert a vertex x in the middle of an existing edge vw.

Definition 1.15. Replacing two adjacent vertices by a single vertex of a graph is an operation called *elementary contraction.* The new vertex is joined to every other vertex which was joined to one or both original two vertices. A *contraction* of G is any graph that can be obtained from G by a finite sequence of elementary contractions.

The same operations, subdivision and contraction, can be applied on any line segment AB in $\Re^3$ replacing it by two line segments AC and CB or *vice versa.*

In the language of contraction, Pontryagin-Kuratowski's Theorem can be formulated as:

Theorem 1.4. *A graph G is planar iff it contains no subgraph which has K_5 or $K_{3,3}$ as a contraction.*

In considering KLs, a special kind of contraction where edges forming a bigon (Definition 1.14) are contracted simultaneously will play an important role. We will call such contraction a *bigon collapse.*

As well as a plane or sphere, we may consider other *smooth* surfaces, which can be *orientable*, and like a sphere have an inside and an outside, or can be *non-orientable*, such as the projective plane or Klein bottle. An orientable surface can be thought of as a sphere with g handles ($g = 0, 1, 2, \ldots$), and the number of handles g is the *genus* of the surface. For a torus or "sphere with a handle", $g = 1$, for a double torus $g = 2$, *etc.* The simplest non-orientable surface is the projective plane, which may be regarded as a sphere with antipodes identified, or as a hemisphere with opposite peripheral points identified, or as a sphere with a cross-cap. For a non-orientable surface without boundary, the genus g is given by the formula $g = 2 - \chi$, and for an orientable surface without boundary by the formula $g = \frac{2-\chi}{2}$ (see, e.g., Coxeter and Moser, 1980).

Instead of embedding a graph into a plane or sphere, we may try any any other smooth (orientable or non-orientable) surface. An embedding of a graph G in a surface can be constructed by the method known as the *Edmonds algorithm*, named after J. Edmonds who described it in 1960 (Edmonds, 1960; Gilbert and Porter, 1994).

As an input, the *LinKnot* function **fEdmonds** uses unoriented graph given by the list of unordered pairs (edges) and calculates its embeddings (given by labelled polygons), the Euler characteristic of the surface and its genus. From the output, we can draw the corresponding embedding. For example, for $K_{3,3}$ given by the list of unordered pairs, the function **fEdmonds** gives the result

$$\{\{\{1,4,2,5,3,6\},\{1,4,3,6,2,5\},\{1,5,3,4,2,6\}\},0,1\}.$$

Since the surface has Euler characteristic 0 and genus 1, according to the classification of surfaces, the graph G is embedded on a torus.

Hence, the embedding of the non-planar graph $K_{3,3}$ on a torus (Fig. 1.4) is given by

$$\{\{1,6,4,5\},\{2,6,4,5\},\{3,6,4,5\},\{4,1,2,3\},\{5,1,2,3\},\{6,1,2,3\}\}.$$

In a similar way we obtain the embedding

$$\{\{1,2,4,3\},\{1,2,5,4\},\{1,3,2,5\},\{1,4,3,5\},\{2,3,5,4\}\},$$

of the non-planar graph K_5 on a torus.

We can also consider colored (or weighted) graphs:

Definition 1.16. A *vertex k-coloring* of a graph is a coloring of the vertices by k colors, and an *edge k-coloring* is a coloring of the edges by k colors. A coloring with two colors (usually black and white) will be called a (vertex or edge) *bicoloring*.

If the colors are treated as *weights* assigned to vertices or edges of a graph, such a graph is a *weighted graph*.