

Chapter 1

Introduction

The study of analytic hyperbolic geometry begins in Chap. 2 with the study of gyrogroups. A gyrogroup is a most natural extension of a group into the regime of the nonassociative algebra that we need in order to extend analytic Euclidean geometry into analytic hyperbolic geometry.

Geometry, according to Herodotus, and the Greek derivation of the word, had its origin in Egypt in the mensuration of land, and fixing of boundaries necessitated by the repeated inundations of the Nile. It consisted at first of isolated facts of observation and crude rules for calculation until it came under the influence of Greek thought. Following the introduction of geometry from Egypt to Greece by Thales of Miletus, 640–546 B.C., geometric objects were abstracted, thus paving the way for attempts to give geometry a connected and logical presentation. The most famous of these attempts is that of Euclid, about 300 B.C. [Sommerville (1914), p. 1].

According to the Euclid parallel postulate, given a line L and a point P not on L there is one and only one line L' which contains P and is parallel to L . Euclid's parallel postulate does not seem as intuitive as his other axioms. Hence, it was felt for many centuries that it ought to be possible to find a way of proving it from more intuitive axioms. The history of the study of parallels is full of reproaches against the lack of self-evidence of the Euclid parallel postulate. According to Sommerville [Sommerville (1914), p. 3], Sir Henry Savile referred to it as one of the great blemishes in the beautiful body of geometry [Praelectiones, Oxford, 1621, p. 140]. Following Bolyai and Lobachevsky, however, the parallel postulate became the property that distinguishes Euclidean geometry from non-Euclidean ones.

The Hungarian Geometer János Bolyai (1802–1860) and the Russian Mathematician Nikolai Ivanovich Lobachevsky (1793–1856) independently

worked out a geometry that seemed consistent and yet negated the Euclidean parallel postulate, published in 1832 and 1829. Carl Friedrich Gauss (1777–1855), who was the dominant figure in the mathematical world at the time, was probably the first to understand clearly the possibility of a logically and sound geometry different from Euclid's. According to Harold E. Wolfe [Wolfe (1945), p. 45], it was Gauss who coined the term *non-Euclidean geometry*. The contributions of Gauss to the birth of hyperbolic geometry are described by Sonia Ursini in [Ursini (2001)]. According to Duncan M. Y. Sommerville [Sommerville (1914), p. 24], the ideas inaugurated by Bolyai and Lobachevsky did not attain any wide recognition for many years, and it was only after Baltzer had called attention to them in 1867 that non-Euclidean geometry began to be seriously accepted and studied. In 1871 Felix Klein suggested calling the non-Euclidean geometry of Bolyai and Lobachevsky *hyperbolic geometry* [Sommerville (1914), p. 25]. The discovery of hyperbolic geometry and its development is one of the great stories in the history of mathematics; see, for instance, the accounts of [Rosenfeld (1988)] and [Gray (1989)] for details.

For several centuries (Euclidean) geometry and (associative) algebra developed slowly as distinct mathematical disciplines. In 1637 the French mathematician and philosopher René Descartes published his *La Géométrie* which introduced a theory that unifies (associative) algebra and (Euclidean) geometry, where points are modeled by n -tuples of numbers, n being the dimension of the geometry. The unifying theory is now called analytic (Euclidean) geometry. In full analogy, the aim of this book is to introduce *analytic hyperbolic geometry*, and its applications, as a theory that unifies nonassociative algebra and the hyperbolic geometry of Bolyai and Lobachevsky, where points are modeled by n -tuples of numbers as explained in Figs. 8.1–8.2, p. 262.

1.1 A Vector Space Approach to Euclidean Geometry and A Gyrovector Space Approach to Hyperbolic Geometry

Commonly, three methods are used to study Euclidean geometry:

- (1) *The Synthetic Method*: This method deals directly with geometric objects (figures). It derives some of their properties from other properties by logical reasoning.
- (2) *The Analytic Method*: This method uses a coordinate system, expressing properties of geometric objects by numbers (coordinates).

It derives properties from other properties by numerical expressions and equations, numerical results being interpreted in terms of geometric objects [Boyer (2004)].

- (3) *The Vector Method*: The vector method occupies a middle position between the synthetic and the analytic method. It deals with geometric objects directly and derives properties from other properties by computation with vector expressions and equations [Hausner (1998)].

Euclid treated his Euclidean geometry synthetically. Similarly, also Bolyai and Lobachevsky treated their hyperbolic geometry synthetically. Because progress in geometry needs computational facilities, the invention of analytic geometry by Descartes (1596–1650) made simple approaches to more geometric problems possible. Later, further simplicity for geometric calculations became possible by the introduction of vectors and their addition by the parallelogram law. The parallelogram law for vector addition is so intuitive that its origin is unknown. It may have appeared in a now lost work of Aristotle (384–322). It was also the first corollary in Isaac Newton’s (1642–1727) “*Principia Mathematica*” (1687), where Newton dealt extensively with what are now considered vectorial entities, like velocity and force, but never with the concept of a vector. The systematic study and use of vectors were a 19th and early 20th century phenomenon. Vectors were born in the first two decades of the 19th century with the geometric representations of complex numbers. The development of the algebra of vectors and of vector analysis as we know it today was first revealed in sets of notes made by J. Willard Gibbs (1839–1903) for his students at Yale University.

The synthetic and analytic methods for the study of Euclidean geometry are accessible to the study of hyperbolic geometry as well. Hitherto, however, the vector method had been deemed inaccessible to that study.

In the years 1908 – 1914, the period which experienced a dramatic flowering of creativity in the special theory of relativity, the Croatian physicist and mathematician Vladimir Varičak (1865 – 1942), professor and rector of Zagreb University, showed that this theory has a natural interpretation in hyperbolic geometry [Varičak (1910a)]. However, much to his chagrin, he had to admit in 1924 [Varičak (1924), p. 80] that the adaption of vector algebra for use in hyperbolic geometry was just not feasible, as Scott Walter notes in [Walter (1999b), p. 121]. Vladimir Varičak’s hyperbolic geometry program, cited by Pauli [Pauli (1958), p. 74], is described by Walter in

[Walter (1999b), pp. 112–115].

Following Varičák’s 1924 realization that, unlike Euclidean geometry, the hyperbolic geometry of Bolyai and Lobachevsky does not admit vectors, there are in the literature no attempts to treat hyperbolic geometry vectorially. There are, however, few attempts to treat hyperbolic geometry analytically [Jackson and Greenspan (1955); Patrick (1986)], dating back to Sommerville’s 1914 book [Sommerville (1914); Sommerville (1919)]. Accordingly, following Bolyai and Lobachevsky, most books on hyperbolic geometry treat the geometry synthetically, some treat it analytically, but no book treats it vectorially.

Fortunately, some 80 years since Varičák’s 1924 realization, the adaption of vectors for use in hyperbolic geometry, where they are called *gyrovectors*, has been accomplished in [Ungar (2000a); Ungar (2001b)], allowing Euclidean and hyperbolic geometry to be united [Ungar (2004c)]. Following the adaption of vector algebra for use in hyperbolic geometry, the hyperbolic geometry of Bolyai and Lobachevsky is now effectively regulated by gyrovector spaces just as Euclidean geometry is regulated by vector spaces. Accordingly, we develop in this book a gyrovector space approach to hyperbolic geometry that is fully analogous to the common vector space approach to Euclidean geometry [Hausner (1998)]. In particular, we find in this book that gyrovectors are equivalence classes of directed gyrosegments, Def. 5.4, p. 133, that add according to the gyroparallelogram law, Figs. 8.25–8.26, p. 322, just like vectors, which are equivalence classes of directed segments that add according to the common parallelogram law.

It should be remarked here that in applications to Einstein’s special theory of relativity, Chaps. 10–11 and 13, Einsteinian velocity gyrovector spaces are 3-dimensional gyrovector spaces fully analogous to Newtonian velocity vectors. Hence, in particular, relativistic gyrovectors are different from the common 4-vectors of relativity physics. In fact, the passage from n -gyrovectors to $(n + 1)$ -vectors is illustrated in Remark 4.21, p. 122, and employed in the study of the special relativistic Lorentz transformation group in Chap. 11. The 4-vectors are important in special relativity and in its extension to general relativity. Early attempts to employ 4-vectors in gravitation, 1905–1910, are described in [Walter (2005)].

In the same way that vector spaces are commutative groups of vectors that admit scalar multiplication, gyrovector spaces are gyrocommutative gyrogroups of gyrovectors that admit scalar multiplication. Accordingly, the nonassociative algebra of gyrovector spaces is our framework for ana-

lytic hyperbolic geometry just as the associative algebra of vector spaces is the framework for analytic Euclidean geometry. Moreover, gyrovector spaces include vector spaces as a special, degenerate case corresponding to trivial gyroautomorphisms. Hence, our gyrovector space approach forms the theoretical framework for uniting Euclidean and hyperbolic geometry.

1.2 Gyrolanguage

All the analogies that hyperbolic geometry and classical mechanics share in this book, respectively, with Euclidean geometry and relativistic mechanics arise naturally through a single common mechanism represented by the prefix “gyro”. Indeed, in order to elaborate a precise language for dealing with analytic hyperbolic geometry, which emphasizes analogies with classical notions, we extensively use the prefix “gyro”, giving rise to *gyrolanguage*, the language that we use in this book. The resulting gyrolanguage rests on the unification of Euclidean and hyperbolic geometry in terms of analogies they share [Ungar (2004c)]. The prefix “gyro” stems from *Thomas gyration*. The latter, in turn, is a mathematical abstraction of the peculiar relativistic effect known as *Thomas precession* into an operator, called a *gyrator* and denoted “gyr”. The gyrator generates special automorphisms called *gyroautomorphisms*. The effects of the gyroautomorphisms are called gyrations in the same way that the effects of rotation automorphisms are called rotations.

The natural emergence of gyrolanguage is well described by a 1991 letter that the author received from Helmuth Urbantke of the Institute for Theoretical Physics, University of Vienna, sharing with him instructive experience [Ungar (1991b), ft. 36]:

“While giving a seminar about your work, the word *gyromorphism* instead of [Thomas] precession came over my lips. Since it ties in with the many morphisms the mathematicians love, it might appeal to you.”

Helmuth K. Urbantke, 1991

Indeed, we will find in this book that the study of hyperbolic geometry by the gyrolanguage of gyrovector spaces is useful, and that the pursuit of this study entails no pain for unlimited profit.

Analytic Euclidean geometry in n dimensions models points by n -tuples of numbers that form an n -dimensional vector space with an inner prod-

uct and the Euclidean distance function. Accordingly, vector spaces algebraically regulate analytic Euclidean geometry, allowing the principles of (associative) algebra to manipulate Euclidean geometric objects. Contrastingly, synthetic Euclidean geometry is the kind of geometry for which Euclid is famous and that the reader learned in high school.

Analytic hyperbolic geometry in n dimensions is the subject of this book. It models points by n -tuples of numbers that form an n -dimensional gyrovector space with an inner product and a hyperbolic distance function. Accordingly, gyrovector spaces algebraically regulate analytic hyperbolic geometry, allowing the principles of (nonassociative) algebra to manipulate hyperbolic geometric objects. Contrastingly, synthetic hyperbolic geometry is the kind of geometry for which Bolyai and Lobachevsky are famous and that one learns from the literature on classical hyperbolic geometry.

With one exception, proofs are obtained in this book analytically. The exceptional case is the proof of the *gyrotriangle defect identity* which is the identity shown at the bottom of Fig. 8.13, p. 285. Instructively, this identity is verified both analytically, Theorem 8.45, p. 301, and synthetically, Theorem 8.48, p. 304. It is the gyrotriangle defect identity at the bottom of Fig. 8.13 that gives rise to the elegant values of the squared hyperbolic length (gyrolength) of the sides of a hyperbolic triangle (gyrotriangle) in terms of its hyperbolic angles (gyroangles), also shown in Fig. 8.13 as well as in Theorem 8.49 on p. 307.

While Euclidean geometry has a single standard model, hyperbolic geometry is studied in the literature by several standard models. In this book, analytic hyperbolic geometry appears in three mutually isomorphic models, each of which has its advantages and blindness for selected aspects. These are:

- (I) The Poincaré ball (or disc, in two dimensions) model.
- (II) The Beltrami-Klein ball (or disc, in two dimensions) model.
- (III) The Proper Velocity (PV, in short) space (or plane, in two dimensions) model.

The PV space model of hyperbolic geometry is also known as Ungar space model [Ungar (2001b)]. The terms “Ungar gyrogroups” and “Ungar gyrovector spaces” were coined by Jing-Ling Chen in [Chen and Ungar (2001)] following the emergence of gyrolanguage in [Ungar (1991b)]. Ungar gyrogroups and gyrovector spaces may be used to describe algebraic structures of relativistic proper velocities. Hence, in this book these are called *PV gyrogroups* and *PV gyrovector spaces*.

Before the emergence of gyrolanguage the author coined the term “K-loop” in [Ungar (1989b)] to honor related pioneering work of Karzel in the 1960s, and to emphasize relations with loops that have later been studied in [Krammer (1998); Sabinin, Sabinina and Sbitneva (1998); Issa (1999); Issa (2001)]. With the emergence of gyrolanguage, however, since 1991 the author’s K-loops became “gyrocommutative gyrogroups” following the need to accommodate “non-gyrocommutative gyrogroups” and to emphasize analogies with groups. The ultimate fate of mathematical terms depends on their users. Thus, for instance, some like the term “K-loop” that the author coined in 1989 (as recorded in [Kiechle (2002), pp. 169–170] and, in more detail, in [Sexl and Urbantke (2001), pp. 141–142]), and some prefer using the alternative term “Bruck-loop” (as evidenced, for instance, from MR:2000j:20129 in *Math. Rev.*).

A new term, “dyadic symset”, which has recently emerged from an interesting work of Lawson and Lim in [Lawson and Lim (2004)], turns out to be identical to a two-divisible, torsion-free, gyrocommutative gyrogroup according to [Lawson and Lim (2004), Theorem 8.8]. It thus seems that, as Michael Kinyon notes in his MR:2003d:20109 review in *Math. Rev.* of Hubert Kiechle’s nice introductory book on the “Theory of K-loops” [Kiechle (2002)], “It is unlikely that there will be any convergence of terminology in the near future.”

Since the models of hyperbolic geometry are regulated algebraically by gyrovector spaces just as the standard model of Euclidean geometry is regulated algebraically by vector spaces, the theory of gyrogroups and gyrovector spaces develops in this book an internal ecology. It includes the special gyrolanguage, key examples, definitions and theorems, central themes, and a few gems, like those illustrated in various figures in this book, to amaze both the uninitiated and the practicing expert on hyperbolic geometry and special relativity.

1.3 Analytic Hyperbolic Geometry

One of the tasks of the geometer who is interested in analytic hyperbolic geometry is to construct mathematical models and a theory that correspond to elements of the relativistic and quantum physical world. The criteria for judging the success of our analytic hyperbolic geometry are generality, simplicity, and beauty. These are illustrated in various figures of the book, starting with Figs. 6.1–6.4 on pp. 210–211 of Chap. 6.

In gyrolanguage we prefix a gyro to any term that describes a concept in Euclidean geometry and in associative algebra to mean the analogous concept in hyperbolic geometry and nonassociative algebra. The prefix gyro stems from Thomas gyration. Thomas gyration, in turn, is a special automorphism abstracted from the relativistic effect known as Thomas precession. The destiny of Thomas precession in the foundations of hyperbolic geometry thus began to unfold following its extension by abstraction in [Ungar (1988a); Ungar (1988b); Ungar (1989b); Ungar (1989a)] since 1988.

Following the extension of groups and vector spaces of associative algebra and Euclidean geometry into nonassociative counterparts, gyrolanguage gives rise to gyroterms like gyrogroups and gyrovector spaces, gyrolines and gyroangles, of nonassociative algebra and hyperbolic geometry. Similarly, commutativity and associativity in associative algebra and Euclidean geometry are extended in gyrolanguage to gyrocommutativity and gyroassociativity in nonassociative algebra and hyperbolic geometry.

We sometimes abuse gyrolanguage a bit and drop the prefix gyro when it competes with a classical term. Thus, for instance, “points” of a gyrovector space remain points rather than gyropoints, as they should be called in gyrolanguage. But, “vectors” of a gyrovector space are called gyrovectors since they do not exist classically. Furthermore, we use the terms gyrogeodesics and (hyperbolic) geodesics interchangeably since, for instance, the gyrogeodesics (also called *gyrolines*) of Möbius gyrovector spaces are nothing else but the familiar geodesics of the Poincaré model of hyperbolic geometry.

The most impressive examples of the need to abuse gyrolanguage a bit come (i) from the gyro-Euclidean geometry, which is nothing else but the hyperbolic geometry of Bolyai and Lobachevsky and (ii) from the gyro-mass, which is nothing else but the Einstein relativistic mass. We certainly do not recommend to abandon the classical term “hyperbolic geometry” in favor of its gyrolanguage equivalent term “gyro-Euclidean geometry” and, similarly, we do not recommend to abandon the term “relativistic mass” in favor of its gyrolanguage equivalent term “gyro-mass”.

In contrast, we find it useful to adopt the term “gyrotrigonometry”. It is, in fact, hyperbolic trigonometry, but it is more similar, in terms of analogies, to Euclidean trigonometry than to traditional hyperbolic trigonometry, which is expressed in terms of the familiar hyperbolic functions \cosh and \sinh [McCleary (2002), p. 52].

Three other examples come from gyrolines, gyroangles, and gyrotrian-

gles, which coincide with hyperbolic lines, hyperbolic angles, and hyperbolic triangles respectively. When a gyroterm in gyrolanguage competes with a classical term, abuse of gyrolanguage may occur. Some gyroterms that compete with classical terms cannot be abandoned since they come with dual counterparts that, classically, are not recognized as duals since their duality symmetries can only be captured by gyrotheoretic techniques. Thus, for instance, gyrolines, gyroangles, and gyrotriangles are associated with their corresponding dual counterparts, cogyrolines, cogyroangles, and cogyrotriangles.

Gyrolanguage abuse must be done with care, as the example of the gyrocosine function in Fig. 8.13, p. 285, indicates. The definition of the gyrocosine of a gyroangle is presented in Fig. 8.13. We cannot view it as the “hyperbolic” cosine of a hyperbolic angle since the term “hyperbolic cosine” is already in use in a different sense. Abusing notation, we use the same notation for the trigonometric functions and their gyro-counterparts. Thus, for instance, the gyrocosine function in Fig. 8.13 is denoted by \cos . This notation for the elementary gyrotrigonometric functions \cos , \sin , \tan , etc. is justified since the gyrotrigonometric functions are interrelated by the same identities that interrelate the trigonometric functions.

Thus, for instance, the trigonometric identity $\cos^2 \alpha + \sin^2 \alpha = 1$ (along with all other trigonometric identities between elementary trigonometric functions) remains valid in gyrotrigonometry as well. Furthermore, in the conformal model of the Poincaré ball, corresponding gyroangles and angles have the same measure, so that the elementary trigonometric functions are identical with their gyro-counterpart in all the hyperbolic models that are isomorphic (in the sense of gyro-algebra) to the Poincaré ball model, as verified in Theorem 8.3, p. 264.

1.4 The Three Models

There are infinitely many models of hyperbolic geometry. The three models that we study in this book, described below, are particularly interesting.

(I) The Poincaré ball model of hyperbolic geometry is algebraically regulated by Möbius gyrovector spaces where Möbius addition plays a role analogous to the role that vector addition plays in vector spaces. The geodesics of this model (gyrolines) are Euclidean circular arcs (with finite or infinite radius, the latter being diameters of the ball.) that intersect the boundary of the ball orthogonally, shown in Fig. 8.13, p. 285, for the

two-dimensional ball, and in Fig. 8.38, p. 348, for the three-dimensional ball. The model is conformal to the Euclidean model in the sense that the measure of the hyperbolic angle between two intersecting gyrolines is equal to the measure of the Euclidean angle between corresponding intersecting tangent lines, Figs. 8.3–8.5, pp. 265–269.

Möbius addition is a natural generalization of the Möbius transformation without rotation of the complex open unit disc from the theory of functions of a complex variable, as we will see in Sec. 3.5. Thus, although more than 150 years have passed since August Ferdinand Möbius first studied the transformations that now bear his name [Ahlfors (1984)][Mumford, Series and Wright (2002), p. 71], this book demonstrates that the rich structure he thereby exposed is still far from being exhausted. The story of the road from Möbius to gyrogroups [Ungar (2008)] is found in Secs. 3.4–3.5.

(II) The Beltrami-Klein ball model of hyperbolic geometry is algebraically regulated by Einstein gyrovector spaces where Einstein addition plays a role analogous to the role that vector addition plays in vector spaces. The geodesics of this model (gyrolines) are Euclidean straight lines in the ball, Fig. 6.8, p. 220.

Einstein addition, in turn, is the standard velocity addition of relativistically admissible velocities that Einstein introduced in his 1905 paper [Einstein (1905)] that founded the special theory of relativity. In this book, accordingly, the presentation of Einstein's special theory of relativity is solely based on Einstein velocity addition law, taking the reader to the immensity of the underlying hyperbolic geometry. Thus, more than 100 years after Einstein introduced the relativistic velocity addition law that now bears his name, this book demonstrates that placing Einstein velocity addition centrally in special relativity theory is an old idea whose time has returned [Ungar (2006a)].

Einstein's failure to recognize and advance the gyrovector space structure that underlies his relativistic velocity addition law contributed to the eclipse of his velocity addition of relativistically admissible 3-velocities, creating a void that could be filled only with Minkowskian relativity, Minkowski's reformulation of Einstein's special relativity based on the Lorentz transformation of 4-velocities and on spacetime [Walter (2008)].

The approach to special relativity from Einstein velocity addition fills a noticeable gap in the relativity physics arena. Thus, for instance,

- (1) the seemingly notorious Thomas precession, which is either ignored or studied as an isolated phenomenon in most relativity physics

- books; and
- (2) the seemingly confusing relativistic mass, which does not mesh up with Minkowskian relativity

mesh extraordinarily well with the analytic hyperbolic geometric approach to Einsteinian relativity [Ungar (2005a)]. The term “Minkowskian relativity”, as opposed to Einsteinian relativity, was coined by L. Pyenson in [Pyenson (1982), p. 146]. The historical struggle between Einsteinian relativity and Minkowskian relativity is skillfully described by S. Walter in [Walter (1999b)] where, for the first time, the term “Minkowskian relativity” appears in a title.

Rather than being notorious and confusing, Thomas precession and Einstein’s relativistic mass provide unexpected insights that are not easy to come by, by means other than analytic hyperbolic geometric techniques.

The remarkable fit between geometry and physics that Figs. 10.2, p. 412, and 10.3, p. 413, exhibit is not fortuitous. It demonstrates that the relativistic mass plays in relativistic mechanics and its underlying hyperbolic geometry the same important role that the Newtonian mass plays in classical mechanics and its underlying Euclidean geometry. The relativistic mass is thus an asset rather than a liability. The relativistic center of momentum and gyrobarocentric coordinates associated with the relativistic mass are studied in Chap. 11.

(III) The PV space model of hyperbolic geometry is governed by PV gyrovector spaces where PV addition plays a role analogous to the role that vector addition plays in vector spaces. The geodesics of this model (gyrolines) are Euclidean hyperbolas with asymptotes that intersect at the space origin, Fig. 6.12, p. 223. PV addition turns out to be the “proper velocity” addition of proper velocities in special relativity. As opposed to (i) *coordinate velocities* in special relativity, measured by observer’s time and composed by Einstein addition, (ii) *proper velocities* in special relativity are measured by traveler’s time and composed by PV addition.

The power and elegance of the gyrovector space approach to hyperbolic geometry is convincingly illustrated in this book by analogies with the common vector space approach to Euclidean geometry. Aesthetic criteria are fundamental to the development of mathematical ideas [Penrose (2005), p. 22]. The conversion law from gyrotriangle gyroangles α, β, γ to their corresponding gyrotriangle side gyrolengths $\|\mathbf{a}\|, \|\mathbf{b}\|, \|\mathbf{c}\|$ in a gyrotriangle ABC is shown in Fig. 8.13, p. 285, and in the AAA to SSS Conversion Law, Theorem 8.49, p. 307. It presents one of the examples of an extraordinary

unexpected hidden beauty that analytic hyperbolic geometry reveals.

We thus encounter in this book remarkable interrelations between truth and beauty, which are abundant in the areas of analytic hyperbolic geometry, and which share remarkable analogies with corresponding classical results in the areas of analytic Euclidean geometry.

1.5 Applications in Quantum and Special Relativity Theory

The applicability in physics of the gyrovector space approach to hyperbolic geometry is demonstrated in Chaps. 9 and 10-13.

Chapter 9 demonstrates that Bloch vector of quantum computation theory is, in fact, a gyrovector rather than a vector. This discovery of the relationship between “Bloch vector” and the Poincaré model of hyperbolic geometry led Péter Lévay to realize in [Lévay (2004a)] and [Lévay (2004b)] that the so called *Bures metric* in quantum computation is equivalent to the metric that results from the hyperbolic distance function.

Like Möbius addition, Einstein velocity addition is neither commutative nor associative. Hence, the study of special relativity in the literature follows the lines laid down by Minkowski, in which the role of Einstein velocity addition and its interpretation in the hyperbolic geometry of Bolyai and Lobachevsky are ignored [Barrett (1998)]. The breakdown of commutativity and associativity in Einstein velocity addition, thus, poses a significant problem. Einstein’s opinion about significant problems in science is well known:

The significant problems we have cannot be solved at
the same level of thinking with which we created them.

Albert Einstein (attributed)

Indeed, it is the gyrovector space approach to Einstein’s special relativity and to hyperbolic geometry that resolves the significant problem of commutativity and associativity breakdown in Einstein velocity addition. In this novel approach,

- (1) Einstein velocity addition emerges triumphant as a gyrocommutative, gyroassociative binary operation between gyrovectors in hyperbolic geometry; fully analogous to
- (2) Newton velocity addition, which is a commutative, associative binary operation between vectors in Euclidean geometry.

Chapters 10 and 11 are collectively entitled “Special theory of relativity: the analytic hyperbolic geometric viewpoint” (Parts I and II). They demonstrate that the gyrovector space approach, which unifies Euclidean and hyperbolic geometry, unifies some aspects of classical and relativistic mechanics as well.

Chapter 12, entitled “Relativistic Gyrotrigonometry”, presents the gyrotrigonometry of hyperbolic geometry in the Beltrami-Klein ball model or, equivalently, in Einstein gyrovector spaces. The analogies relativistic gyrotrigonometry shares with the standard, Euclidean trigonometry are remarkable. Naturally, relativistic gyrotrigonometry is isomorphic with gyrotrigonometry in the Poincaré ball model of hyperbolic geometry.

Chapter 13, entitled “Stellar and particle aberration”, employs the universe as our laboratory, where the relativistic stellar aberration effect is our experiment. It is shown in this chapter that within the frame of Einstein’s special theory of relativity, velocity composition in the universe obeys the gyrotriangle law of gyrovector addition. This result is fully analogous to the well known result that, within the frame of classical mechanics, Newtonian velocity addition is given by the triangle law of vector addition.

Geometry is at the foundation of physics. In particular, hyperbolic geometry finds natural home at the foundation of Einstein’s special relativity theory just as Euclidean geometry lies at the foundation of Newtonian physics. By listening to the sounds of relativistic velocities and their composition by Einstein velocity addition law, analytic hyperbolic geometry thus significantly extends Einstein’s unfinished symphony.