

# Preface

The seeds of this book were planted in 1988 [Ungar (1988a)], when the author discovered that the seemingly structureless Einstein addition of relativistically admissible velocities possesses a rich grouplike structure that became known as a *gyrocommutative gyrogroup*. It turned out that gyrocommutative gyrogroups and their resulting gyrovector spaces form the setting for hyperbolic geometry just as vector spaces form the setting for Euclidean geometry. Accordingly, following this dramatic breakthrough, this book presents the foundations of analytic hyperbolic geometry from the viewpoint of hyperbolic vectors, called *gyrovectors*, putting it to good use for the extension of Einstein's unfinished symphony of his special theory of relativity. The underlying mathematical tools, gyrogroups and gyrovector spaces, are developed along analogies they share with groups and vector spaces. As a result, a gyrovector space approach to hyperbolic geometry, fully analogous to the standard vector space approach to Euclidean geometry, emerges along with applications in relativistic mechanics that are fully analogous to corresponding applications in classical mechanics.

Owing to its strangeness, some regard themselves as excluded from the profound insights of hyperbolic geometry so that this enormous portion of human achievement is a closed door to them. But this book opens the door on its mission to make the hyperbolic geometry of Bolyai and Lobachevsky widely accessible by introducing a gyrovector space approach to hyperbolic geometry guided by analogies that it shares with the common vector space approach to Euclidean geometry.

Writing this second book on analytic hyperbolic geometry and Einstein's special theory of relativity became possible following the successful adaption of vector algebra for use in hyperbolic geometry in the author's 2001 book "*Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces*" (Kluwer Acad., now Springer). A most convincing way to describe the success of the author's adaption of vector algebra for use in hyperbolic geometry is found in Scott

Walter's review of the author's 2001 book, which is the first forerunner of this book. Therefore, part of Scott Walter's review is quoted below.

Over the years, there have been a handful of attempts to promote the non-Euclidean style for use in problem solving in relativity and electrodynamics, the failure of which to attract any substantial following, compounded by the absence of any positive results must give pause to anyone considering a similar undertaking. Until recently, no one was in a position to offer an improvement on the tools available since 1912. In his [2001] book, Ungar furnishes the crucial missing element from the panoply of the non-Euclidean style: an elegant nonassociative algebraic formalism that fully exploits the structure of Einstein's law of velocity composition. The formalism relies on what the author calls the "missing link" between Einstein's velocity addition formula and ordinary vector addition: Thomas precession . . .

Ungar lays out for the reader a sort of vector algebra in hyperbolic space, based on the notion of a gyrovector. A gyrovector space differs in general from a vector space in virtue of inclusion of Thomas precession, and exclusion of the vector distributive law. As a result, when expressed in terms of gyrovectors, Einstein (noncommutative) velocity addition law becomes "gyrocommutative" . . . . One advantage of this approach is that hyperbolic geometry segues into Euclidean geometry, with notions such as group, vector, and line passing over to their hyperbolic gyro-counterparts (gyrogroup, etc.) . . .

One might suppose that there is a price to pay in mathematical regularity when replacing ordinary vector addition with Einstein's addition, but Ungar shows that the latter supports gyrocommutative and gyroassociative binary operations, in full analogy to the former. Likewise, some gyrocommutative and gyroassociative binary operations support scalar multiplication, giving rise to gyrovector spaces, which provide the setting for various models of hyperbolic geometry, just as vector spaces form the setting for the common model of Euclidean geometry. In particu-

lar, Einstein gyrovector spaces provide the setting for the Beltrami ball model of hyperbolic geometry, while Möbius gyrovector spaces provide the setting for the Poincaré ball model of hyperbolic geometry.

Scott Walter

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Subsequently, a second convincing way to describe the success of the author's adaption of vector algebra for use in hyperbolic geometry is found in Victor Pambuccian's review of the author's 2005 book, which is the second forerunner of this book. Part of this review of the author's 2005 book is therefore quoted below.

This book represents an exposition . . . of an algebraic language in which both hyperbolic geometry and special relativity find an aesthetically pleasing formulation, very much like Euclidean geometry and Newtonian mechanics find them in the language of vector spaces. The aim is thus to provide not just an analytic geometry to enable algebraic computations to answer questions formulated in hyperbolic geometry, . . . but rather to provide a counterpart to the inner product vector space model of Euclidean geometry. The resulting counterparts allow for formulas that look very much like their Euclidean counterparts, with the difference that the vectors are not elements of a vector space, but rather of a gyrovector space, that . . . do not satisfy the familiar commutative, associative rules, but rather “gyro”-variants of these, creating an elaborate “gyrolanguage”, in which all terms familiar from the Euclidean setting get their gyro-counterpart. . . . The author greatly emphasizes the fact that, in this algebraic setting, the original Einstein velocity addition finds a natural home, in which it becomes gyrocommutative [and gyroassociative], and would have perhaps not been abandoned by the advent of Minkowski's space-time reformulation of special relativity, had the gyrolanguage existed at the time.

Victor V. Pambuccian  
Zentralblatt Math.

Analytic hyperbolic geometry, as presented in this book, is now performing better than ever, emphasizing the interdisciplinary collaborations required to further develop this extraordinary mathematical innovation and its applications. But, there have been some challenges during the initial phase of its development, challenging preconceived notions like the dogma of Einsteinian relativity vs. Minkowskian relativity, which was not struck down until the emergence of analytic hyperbolic geometry.

As a mathematical prerequisite for a fruitful reading of this book it is assumed familiarity with Euclidean geometry from the point of view of vectors and, occasionally, with differential calculus and functions of a complex variable. The book is aimed at a large audience. It includes both elementary and advanced topics, and is structured so that it can be enjoyed equally by undergraduates, graduate students, researchers and academics in geometry, algebra, mathematical physics, theoretical physics, and astronomy.

*Abraham A. Ungar*  
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