

Contents

<i>Preface</i>	vii
<i>Acknowledgements</i>	xi
1. Introduction	1
1.1 A Vector Space Approach to Euclidean Geometry and A Gyrovector Space Approach to Hyperbolic Geometry . . .	2
1.2 Gyrolanguage	5
1.3 Analytic Hyperbolic Geometry	7
1.4 The Three Models	9
1.5 Applications in Quantum and Special Relativity Theory . . .	12
2. Gyrogroups	15
2.1 Definitions	16
2.2 First Gyrogroup Theorems	19
2.3 The Associative Gyropolygonal Gyroaddition	23
2.4 Two Basic Gyrogroup Equations and Cancellation Laws . . .	25
2.5 Commuting Automorphisms with Gyroautomorphisms	32
2.6 The Gyrosemidirect Product Group	34
2.7 Basic Gyration Properties	39
3. Gyrocommutative Gyrogroups	51
3.1 Gyrocommutative Gyrogroups	51
3.2 Nested Gyroautomorphism Identities	68
3.3 Two-Divisible Two-Torsion Free Gyrocommutative Gyrogroups	72

3.4	From Möbius to Gyrogroups	75
3.5	Higher Dimensional Möbius Gyrogroups	77
3.6	Möbius gyrations	81
3.7	Three-Dimensional Möbius gyrations	85
3.8	Einstein Gyrogroups	86
3.9	Einstein Coaddition	92
3.10	PV Gyrogroups	93
3.11	Points and Vectors in a Real Inner Product Space	97
3.12	Exercises	98
4.	Gyrogroup Extension	101
4.1	Gyrogroup Extension	101
4.2	The Gyroinner Product, the Gyronorm, and the Gyroboost	105
4.3	The Extended Automorphisms	111
4.4	Gyrotransformation Groups	114
4.5	Einstein Gyrotransformation Groups	117
4.6	PV (Proper Velocity) Gyrotransformation Groups	117
4.7	Galilei Transformation Groups	118
4.8	From Gyroboosts to Boosts	119
4.9	The Lorentz Boost	121
4.10	The $(p:q)$ -Gyromidpoint	123
4.11	The $(p_1:p_2:\dots:p_n)$ -Gyromidpoint	127
5.	Gyrovectors and Cogrovectors	131
5.1	Equivalence Classes	131
5.2	Gyrovectors	132
5.3	Gyrovector Translation	133
5.4	Gyrovector Translation Composition	137
5.5	Points and Gyrovectors	140
5.6	The Gyroparallelogram Addition Law	141
5.7	Cogyrovectors	143
5.8	Cogyrovector Translation	144
5.9	Cogyrovector Translation Composition	148
5.10	Points and Cogyrovectors	151
5.11	Exercises	152
6.	Gyrovector Spaces	153
6.1	Definition and First Gyrovector Space Theorems	153

6.2 Solving a System of Two Equations in a Gyrovector Space	160
6.3 Gyrolines and Cogyrolines	163
6.4 Gyrolines	166
6.5 Gyromidpoints	172
6.6 Gyrocovariance	175
6.7 Gyroparallelograms	177
6.8 Gyrogeodesics	183
6.9 Cogyrolines	186
6.10 Carrier Cogyrolines of Cogyrovectors	197
6.11 Cogyromidpoints	198
6.12 Cogyrogeodesics	199
6.13 Various Gyrolines and Cancellation Laws	203
6.14 Möbius Gyrovector Spaces	205
6.15 Möbius Cogyroline Parallelism	212
6.16 Illustrating the Gyroline Gyration Transitive Law	213
6.17 Turning the Möbius Gyrometric into the Poincaré Metric	216
6.18 Einstein Gyrovector Spaces	218
6.19 Turning Einstein Gyrometric into a Metric	222
6.20 PV (Proper Velocity) Gyrovector Spaces	223
6.21 Gyrovector Space Isomorphisms	225
6.22 Gyrotriangle Gyromedians and Gyrocentroids	228
6.22.1 In Einstein Gyrovector Spaces	229
6.22.2 In Möbius Gyrovector Spaces	233
6.22.3 In PV Gyrovector Spaces	236
6.23 Exercises	238
7. Rudiments of Differential Geometry	239
7.1 The Riemannian Line Element of Euclidean Metric	240
7.2 The Gyroline and the Cogyroline Element	241
7.3 The Gyroline Element of Möbius Gyrovector Spaces	245
7.4 The Cogyroline Element of Möbius Gyrovector Spaces	248
7.5 The Gyroline Element of Einstein Gyrovector Spaces	250
7.6 The Cogyroline Element of Einstein Gyrovector Spaces	253
7.7 The Gyroline Element of PV Gyrovector Spaces	255
7.8 The Cogyroline Element of PV Gyrovector Spaces	257
7.9 Table of Riemannian Line Elements	259

8. Gyrotrigonometry	261
8.1 Vectors and Gyrovectors are Equivalence Classes	261
8.2 Gyroangles	263
8.3 Gyrovector Translation of Gyrorays	275
8.4 Gyrorays Parallelism and Perpendicularity	282
8.5 Gyrotrigonometry in Möbius Gyrovector Spaces	284
8.6 Gyrotriangle Gyroangles and Side Gyrolengths	296
8.7 The Gyroangular Defect of Right Gyroangle Gyrotriangles .	300
8.8 Gyroangular Defect of the Gyrotriangle	301
8.9 Gyroangular Defect of the Gyrotriangle – a Synthetic Proof	304
8.10 The Gyrotriangle Side Gyrolengths in Terms of its Gyroangles	307
8.11 The Semi-Gyrocircle Gyrotriangle	314
8.12 Gyrotriangular Gyration and Defect	316
8.13 The Equilateral Gyrotriangle	318
8.14 The Möbius Gyroparallelogram	321
8.15 Gyrotriangle Defect in the Möbius Gyroparallelogram . . .	324
8.16 Gyroparallelograms Inscribed in a Gyroparallelogram	330
8.17 Möbius Gyroparallelogram Addition Law	333
8.18 The Gyrosquare	336
8.19 Equidefect Gyrotriangles	342
8.20 Parallel Transport	344
8.21 Parallel Transport vs. Gyrovector Translation	350
8.22 Gyrocircle Gyrotrigonometry	353
8.23 Cogyroangles	356
8.24 The Cogyroangle in the Three Models	362
8.25 Parallelism in Gyrovector Spaces	363
8.26 Reflection, Gyroreflection, and Cogyroreflection	365
8.27 Tessellation of the Poincaré Disc	367
8.28 Bifurcation Approach to Non-Euclidean Geometry	369
8.29 Exercises	371
9. Bloch Gyrovector of Quantum Information and Computation	375
9.1 The Density Matrix for Mixed State Qubits	375
9.2 Bloch Gyrovector	381
9.3 Trace Distance and Bures Fidelity	390
9.4 The Real Density Matrix for Mixed State Qubits	392
9.5 Extending the Real Density Matrix	395

9.6 Exercises	396
10. Special Theory of Relativity: The Analytic Hyperbolic Geometric Viewpoint	
Part I: Einstein Velocity Addition and its Consequences	397
10.1 Introduction	399
10.2 Einstein Velocity Addition	401
10.3 From Thomas Gyration to Thomas Precession	403
10.4 The Relativistic Gyrovector Space	407
10.5 Gyrogeodesics, Gyromidpoints and Gyrocentroids	409
10.6 The Midpoint and the Gyromidpoint – Newtonian and Einsteinian Mechanical Interpretation	411
10.7 Einstein Gyroparallelograms	418
10.8 The Relativistic Gyroparallelogram Law	424
10.9 The Parallelepiped	427
10.10 The Pre-Gyroparallelepiped	430
10.11 The Gyroparallelepiped	433
10.12 The Relativistic Gyroparallelepiped Addition Law	438
10.13 Exercises	443
11. Special Theory of Relativity: The Analytic Hyperbolic Geometric Viewpoint	
Part II: Lorentz Transformation and its Consequences	445
11.1 The Lorentz Transformation and its Gyro-Algebra	445
11.2 Galilei and Lorentz Transformation Links	452
11.3 $(t_1:t_2)$ -Gyromidpoints as CMM Velocities	454
11.4 The Hyperbolic Theorems of Ceva and Menelaus	460
11.5 Relativistic Two-Particle Systems	465
11.6 The Covariant Relativistic CMM Frame Velocity	471
11.7 The Relativistic Invariant Mass of an Isolated Particle System	477
11.8 Relativistic CMM and the Kinetic Energy Theorem	485
11.9 Additivity of Relativistic Energy and Momentum	488
11.10 Bright (Baryonic) and Dark Matter	491
11.11 Newtonian and Relativistic Kinetic Energy	494
11.11.1 The Newtonian Kinetic Energy	494
11.11.2 The Relativistic Kinetic Energy	495
11.11.3 Consequences of Classical Kinetic Energy Conservation During Elastic Collisions	496

11.11.4	Consequences of Relativistic Kinetic Energy Conservation During Elastic Collisions	498
11.11.5	On the Analogies and a Seeming Disanalogy	501
11.12	Barycentric Coordinates	502
11.13	Einsteinian Gyrobarycentric Coordinates	505
11.14	The Proper Velocity Lorentz Group	508
11.15	Demystifying the Proper Velocity Lorentz Group	513
11.16	The Standard Lorentz Transformation Revisited	516
11.17	The Inhomogeneous Lorentz Transformation	517
11.18	The Relativistic Center of Momentum and Mass	520
11.19	Relativistic Center of Mass: Example 1	527
11.20	Relativistic Center of Mass: Example 2	529
11.21	Dark Matter and Dark Energy	531
11.22	Exercises	532
12.	Relativistic Gyrotrigonometry	537
12.1	The Relativistic Gyrotriangle	537
12.2	Law of Gyrocosines, <i>SSS</i> to <i>AAA</i> Conversion Law	542
12.3	The <i>AAA</i> to <i>SSS</i> Conversion Law	542
12.4	The Law of Gyrosines	544
12.5	The Relativistic Equilateral Gyrotriangle	544
12.6	The Relativistic Gyrosquare	545
12.7	The Einstein Gyrosquare with $\theta = \pi/3$	547
12.8	The <i>ASA</i> to <i>SAS</i> Conversion Law	550
12.9	The Relativistic Gyrotriangle Defect	551
12.10	The Right-Gyroangled Gyrotriangle	552
12.11	The Einsteinian Gyrotrigonometry	554
12.12	The Relativistic Gyrotriangle Gyroarea	558
12.13	The Gyrosquare Gyroarea	560
12.14	The Gyrotriangle Constant Principle	561
12.15	Ceva and Menelaus, Revisited	563
12.16	Saccheri Gyroquadrilaterals	566
12.17	Lambert Gyroquadrilaterals	570
12.18	Exercises	575
13.	Stellar and Particle Aberration	577
13.1	Particle Aberration: The Classical Interpretation	579
13.2	Particle Aberration: The Relativistic Interpretation	583

13.3 Particle Aberration: The Geometric Interpretation	593
13.4 Relativistic Stellar Aberration	596
13.5 Exercises	599
<i>Notation And Special Symbols</i>	601
<i>Bibliography</i>	605
<i>Index</i>	621