

RECIPROCAL TRANSFORMATIONS AND INTEGRABLE HAMILTONIAN HYDRODYNAMIC TYPE SYSTEMS

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Following our approach in Ref. 2, I present sufficient conditions so that the reciprocal Hamiltonian structure to a DN system is local.

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1. Dubrovin-Novikov systems and zero curvature equations

Equations of hydrodynamic type

$$u_t^i = \sum_{k=1}^n v_k^i(u) u_x^k, \quad u = (u^1, \dots, u^n), \quad u_x^i = \frac{\partial u^i}{\partial x}, \quad u_t^i = \frac{\partial u^i}{\partial t}. \quad (1)$$

naturally arise in applications such as gas dynamics, hydrodynamics, chemical kinetics, the Whitham averaging procedure, differential geometry and topological field theory.^{4,7,8,16,17} Dubrovin and Novikov⁷ showed that Eq. (1) is a Hamiltonian (DN) system with Hamiltonian $H[u] = \int h(u) dx$, if there exists a flat contravariant metric $g^{il}(u)$ in \mathbb{R}^n with Christoffel symbols $\Gamma_{jk}^i(u)$, such that the matrix $v_k^i(u)$ can be represented in the form

$$v_k^i(u) = \sum_{s=1}^n \left(g^{il}(u) \frac{\partial^2 h}{\partial u^l \partial u^k}(u) - \sum_{s=1}^n g^{ik}(u) \Gamma_{sk}^l(u) \frac{\partial h}{\partial u^l}(u) \right). \quad (2)$$

If $n = 2$, Eq. (1) are the Euler equations for ideal compressible fluids, the system can be put in diagonal form and is integrable by the hodograph method. For arbitrary n , Tsarev¹⁶ proved that a DN system as in Eqs. (1), (2) can be integrated by a generalized hodograph method only if it may be transformed to the diagonal form

$$u_t^i = v^i(u) u_x^i, \quad i = 1, \dots, n. \quad (3)$$

In the latter case, moreover the flat metric is diagonal, the Hamiltonian satisfies

$$\frac{\partial^2 h}{\partial u^i \partial u^j} = \Gamma_{ij}^i(u) \frac{\partial h}{\partial u^i} + \Gamma_{ji}^j(u) \frac{\partial h}{\partial u^j}, \quad (4)$$

and each solution $p(u)$ to Eq. (4) generates a conserved quantity for the DN system (1), (2) and all of these symmetries commute.

From a differential geometric point of view, a non-degenerate flat diagonal metric in \mathbb{R}^n is associated to an orthogonal coordinate system $u^i = u^i(x_1, \dots, x_n)$. Upon introducing the Lamé coefficients

$$H_i^2(u) = \sum_k \left(\frac{\partial x^i}{\partial u^k} \right)^2,$$

the metric tensor in the coordinate system u^i is diagonal $ds^2 = \sum_{i=1}^n H_i^2(u) (du^i)^2$, and the zero curvature conditions $R_{il,im}(u) = 0$ ($i \neq l \neq m \neq i$) and $R_{il,il}(u) = 0$ ($i \neq l$) form an overdetermined system:

$$\begin{aligned} \frac{\partial^2 H_i}{\partial u^l \partial u^m} &= \frac{1}{H_l} \frac{\partial H_l}{\partial u^m} \frac{\partial H_i}{\partial u^l} + \frac{1}{H_m} \frac{\partial H_m}{\partial u^l} \frac{\partial H_i}{\partial u^m}, \quad (5) \\ \frac{\partial}{\partial u^l} \frac{\partial H_i}{H_l \partial u^l} + \frac{\partial}{\partial u^i} \frac{\partial H_l}{H_i \partial u^i} + \sum_{m \neq i, l} \frac{1}{H_m^2} \frac{\partial H_i}{\partial u^m} \frac{\partial H_l}{\partial u^m} &= 0. \quad (6) \end{aligned}$$

Bianchi and Cartan showed that a general solution to the zero curvature equations (5), (6) can be parametrized locally by $n(n-1)/2$ arbitrary functions of two variables. If the Lamé coefficients $H_i(u)$ are known, one can find $x^i(u^1, \dots, u^n)$ solving the linear undetermined problem (embedding equations)

$$\frac{\partial^2 x^i}{\partial u^k \partial u^l} = \Gamma_{kl}^k(u) \frac{\partial x^i}{\partial u^k} + \Gamma_{lk}^l(u) \frac{\partial x^i}{\partial u^l}, \quad \frac{\partial^2 x^i}{\partial (u^l)^2} = \sum_k \Gamma_{ll}^k(u) \frac{\partial x^i}{\partial u^k}. \quad (7)$$

Comparison of Eqs. (4) and (7) implies that the flat coordinates for the metric $g_{ii}(u) = (H^i(u))^2$ are the Casimirs of the corresponding Hamiltonian operator. Finally, Zakharov¹⁹ showed that the dressing method may be used to determine the solutions to the zero curvature equations up to Combescure transformations.

It then follows that the classification of flat diagonal metrics $ds^2 = g_{ii}(u)(du^i)^2$ is an important preliminary step in the classification of integrable Hamiltonian systems of hydrodynamic type. An important technical point is that all known examples of integrable Hamiltonian systems of hydrodynamic type possess a pair of compatible flat metrics and have been

obtained in the framework of semisimple Frobenius manifolds (axiomatic theory of integrable Hamiltonian systems).⁴⁻⁶ In the latter case, one of the metrics is Egorov (*i.e.* its rotation coefficients are symmetric).

It is then of a certain interest to find and classify new examples of integrable Hamiltonian systems of hydrodynamic type which do not belong to the above class. Reciprocal transformations act non trivially on Hamiltonian structures; following our approach in Refs. 1,2, I explain below their possible role in the above picture.

2. Reciprocal transformations and integrable DN systems

Reciprocal transformations have been introduced by Rogers and Shadwick¹⁴ and are an important class of nonlocal transformations which act on hydrodynamic-type systems.^{1-3,10-15,18} These transformations originate from gas dynamics - the simplest example being the passage from Eulerian to Lagrangian coordinates in one-dimensional gas dynamics - and they change the independent variables of a system.

Let the integrable DN system in Riemann invariant form

$$u_t^i = v^i(u)u_x^i, \quad i = 1, \dots, n, \quad (8)$$

admits conservation laws

$$B(u)_t = A(u)_x, \quad N(u)_t = M(u)_x \quad (9)$$

with $B(u)M(u) - A(u)N(u) \neq 0$. In the new independent variables \tilde{x} and \tilde{t} defined by

$$d\tilde{x} = B(u)dx + A(u)dt, \quad d\tilde{t} = N(u)dx + M(u)dt, \quad (10)$$

the reciprocal system is still diagonal and takes the form

$$u_{\tilde{t}}^i = \frac{B(u)v^i(u) - A(u)}{M(u) - N(u)v^i(u)} u_{\tilde{x}}^i = \hat{v}^i(u)u_{\tilde{x}}^i. \quad (11)$$

Moreover, the metric of the initial systems $g_{ii}(u)$ transforms to

$$\hat{g}_{ii}(u) = \left(\frac{M(u) - N(u)v^i(u)}{B(u)M(u) - A(u)N(u)} \right)^2 g_{ii}(u) \quad (12)$$

and all conservations laws and commuting flows of the original system (8) may be recalculated in the new independent variables.

If the reciprocal transformation is linear (*i.e.* A, B, N, M are constant functions), then the reciprocal to a flat metric is still flat and locality and compatibility of the associated Hamiltonian structures are preserved (see Refs. 13,17,18).

Under a general reciprocal transformation, the Hamiltonian structure does not behave trivially and a thorough study of reciprocal Hamiltonian structures is still an open problem. Ferapontov¹⁰ takes a reciprocal transformation where the conservation laws in Eq. (11) are a linear combination of the Casimirs, momentum and Hamiltonian densities and gives sufficient conditions so that the reciprocal to the flat metric $g_{ii}(u)$ in Eq. (12) is a constant curvature metric. Finally, Ferapontov and Pavlov¹² construct the Riemann curvature tensor and the nonlocal Hamiltonian operator associated to the reciprocal to Eq. (2).

The classification of the reciprocal Hamiltonian structures is complicated by the fact that a DN system as in Eqs. (1)-(2) also possesses an infinite number of nonlocal Hamiltonian structures. It is then possible that two DN systems are linked by a reciprocal transformation and that the flat metrics of the first system are not reciprocal to the flat metrics of the second. In Ref. 1, we constructed such an example: the genus one modulation (Whitham-CH) equations associated to Camassa-Holm in Riemann invariant form ($n = 3$ in Eq. (8)). We proved that the Whitham-CH equations are a DN-system and possess a pair of compatible flat metrics (none of the metrics is Egorov). We also proved the connection via a reciprocal transformation of the Whitham-CH equations to the modulation equations associated to the first negative flow of the Korteweg de Vries hierarchy (Whitham-KdV₋₁). In Ref. 1, finally we also found the relation between the Poisson structures of the Whitham-KdV₋₁ and the Whitham-CH equations: both systems possess a pair of compatible flat metrics, and the two flat metrics of the first system are respectively reciprocal to the constant curvature and conformally flat metrics of the second (and vice versa).

In view of the above results, in Ref. 2 we have started to classify the reciprocal transformations which transform a DN system to a DN system, under the condition that the flat metric tensor $\tilde{g}(u)$ of the transformed system is reciprocal to a metric tensor $g(u)$ of the initial system, which is either flat or of constant Riemannian curvature or conformally flat.

In particular, in Ref. 2, we classify which conservation laws may preserve the flatness of the metric when the transformation changes only one independent variable and we get the following theorem:

Theorem 2.1. *Assume that the $u_t^i = v^i(u)u_x^i$, $i = 1, \dots, n$, is a DN system with flat diagonal metric $g_{ii}(u)$ and conservation laws $B(u)_t = A(u)_x$, $N(u)_t = M(u)_x$ such that $B(u)M(u) - A(u)N(u) \neq 0$. Let $(\nabla B)^2(u) = \sum_{m=1}^n g^{mm}(u)(\partial_m B)^2$. Let the reciprocal transformation and the*

reciprocal metric tensor \tilde{g} be as in Eqs. (11) and (12). Then

(i) If $N \equiv 0$, $M \equiv 1$, the reciprocal metric tensor \tilde{g} is flat if and only if:

- (a) B and A are constant functions;
- (b) $B(u)$ is a Casimir for the metric $g_{ii}(u)$ and $(\nabla B)^2(u) = 0$;
- (c) $B(u)$ is a density of momentum for the metric $g_{ii}(u)$ and $(\nabla B)^2(u) = 2B(u)$;

(ii) If $B = 1$, $A = 0$, the reciprocal metric tensor \tilde{g} is flat if and only if:

- (a) N and M are constant functions;
- (b) $N(u)$ is a Casimir for the metric $g_{ii}(u)$ and $(\nabla N)^2(u) = 0$;
- (c) $N(u) = \kappa H(u)$, where $H(u)$ is a density of Hamiltonian associated to the metric g^{ii} and $(\nabla N(u))^2 = 2\kappa M(u)$.

To prove the above theorem we express the zero curvature equations (5) and (6) of the reciprocal metric tensor (12) in function of the initial metric $g(u)$ and of the conservation laws in the reciprocal transformation (10).

We use the same technique also to produce sufficient conditions for a flat reciprocal metric \tilde{g} in the case of reciprocal transformations of both independent variables: assuming that $B(u)$ is either a Casimir or a momentum density or a Hamiltonian associated to the initial metric $g_{ii}(u)$, we² prove that $N(u)$ is a linear combination of the Casimirs, momentum and Hamiltonian density for the initial metric $g_{ii}(u)$.

To construct explicit examples, we use Dubrovin⁵ classification of flat metrics on Hurwitz spaces (spaces of meromorphic functions on Riemann surfaces): the genus g -KdV modulation equations correspond to the case in which the Riemann invariants (u_1, \dots, u_{2g+1}) of the integrable DN system (8) are the ramification points of genus g hyperelliptic Riemann surfaces. Indeed, we² find new examples of flat pencils of metrics associated to the reciprocal to genus g -KdV modulation equations ($n = 2g + 1$ in Eq. (8)), plugging the explicit expressions of the Casimirs, the conservation laws and the flat metrics associated to the genus g -KdV modulation equations into the zero curvature conditions for the reciprocal metrics. In particular, we show that the genus g Camassa-Holm modulation equations are a DN system with a pair of compatible flat non-Egorov metrics, generalizing our results in Ref. 1 to any genus $g \geq 1$.

There are of course still many open problems connected to the classification of reciprocal Hamiltonian structures. The results in Refs. 2,10 suggest that Casimirs, momentum and Hamiltonian densities have a privileged role in reciprocal transformations which preserve locality of the Hamiltonian

structure. So a natural question is: do there exist two DN systems connected by a reciprocal transformation not in the above class? What about other types of transformations among hydrodynamic systems?

Finally, several systems of evolutionary PDEs arising in physics may be written as perturbations of hyperbolic systems of PDEs and their classification in case of Hamiltonian perturbations has recently been started by Dubrovin, Liu and Zhang.⁹ It would also be interesting to investigate the role of reciprocal transformations in this perturbation scheme.

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