Introduction

There are a variety of ways to think about mathematics. On the one hand, for people in the natural sciences, mathematics might be seen as a tool used to describe reality. For example, we know that the orbit of planets around the sun is an ellipse. Physicists can use mathematics to find the force needed to move a mass a particular distance. There are constants from nature that are used as part of the description of the world. In this setting, mathematics seems just an obvious way to proceed. Newton probably believed that with some added developments in mathematics, most problems in physics would be solved. This view of mathematics would suggest that mathematics is barely more than a reflection of reality. But there are other ways to look at mathematics.

One alternative way to view mathematics is that it is a language with particular symbols and a particular syntax. To learn mathematics is to learn this language. This language is used in some disciplines by some people who find it a convenient way to converse about problems in that discipline. There are two observations that should be made about this view. First is that a language is productive in a field of study if it promotes the gaining of new knowledge, knowledge that could not be gained from other forms of discourse. Note that the idea that new knowledge must be gained may be somewhat difficult to assign to the particular language. The underlying ideas are the same in any language. The rules of logic are the same in mathematics as in other modes of thought. It is not immediately clear what new ideas could be generated by the use of mathematics in a field of study. On the other hand, in physics it is pretty clear that there is no direct reasoning process that would lead one to think the curve of least time between two points would be a cycloid. Some kind of reasoning using the tools of mathematics is essential to obtaining this result. But what? And can this extra thing in mathematics carry over to the study of nonphysical systems? A second concern about mathematics is that if the language is difficult, it may restrict those who can participate in the discussion. It may be seen as a barrier to entry in the field. There is surely a good deal of feeling that mathematics has reduced the accessibility of economics and some natural sciences. Not everyone is comfortable in the presence of those who speak mathematics — just as not everyone is comfortable in the presence of those who speak the language of philosophy. Economists and others have a need to be sure that the use of mathematics is not a barrier to entry, and indeed there are a variety of areas of economics where the mathematics is not the driving force in the field of study. So we understand that mathematics is a language and that, in some settings, it can propel the discipline in ways few other devices can. Will this be true in economics? Or does economics suffer from physics envy? We return to this question shortly.

One other view of mathematics is that mathematics is a model. This idea is one that flies in the face of the earlier vision of mathematics as a reflection of reality. In what sense is mathematics a model? After all, once we concede the existence of the natural numbers, addition and multiplication quickly give us the rationals, and the square root gives us the irrationals. What kind of a model is this? In a real sense, it is a model just like any other. It starts with definitions and axioms and generates testable conclusions. The conclusions are examinable against the real world. There will be areas of the world where the model will not apply. For example, the geometry of Euclid, the plane geometry, seems real enough. But does it work on a globe? In fact not, and there is a well-developed geometry for spheres. And there are other non-Euclidian geometries. In short, even the calculus is a model of the world. In the early 1900s there was a fierce debate among mathematicians about the question of whether all of mathematics was essentially derivable from a set of axioms. Would it be possible to set out some axioms that would be consistent and from which all of mathematics could be derived? Further would it be possible, given these axioms, to decide if any proposition was true or false? It turns out that this does not happen. There is no set of axioms that can be used to generate all of mathematics and there are some propositions whose truth we cannot decide. In this sense, mathematics is a model and one that is useful because of the outcomes it provides.

One final observation. Students often think of mathematics as abstract and wonder why anyone would study this subject. Of what use can these ideas be? In fact, mathematics, while a pleasant study on its own, is more valuable because it is useful somewhere. The parts of mathematics that do not find a use, die. They are gone until there is some reason to bring them back. Perhaps the one exception to this rule is the study of number theory, which until the need for ciphers for the safety of commercial transactions over the internet, had little direct value.

The mathematics presented here will not be encyclopedic. But we will try to bring in the ideas we will need to move forward later. Thus we do not try to introduce ideas for their own beauty, but because they will be used in the development of the mathematical fabric useful in economics. Finally, we will use the mathematics to develop some tools for use in economics.

Is it possible that the study of economics, the study of a certain aspect of human behavior, is susceptible to mathematical description? At one level, the question is clearly preposterous. Who knows any individual who acts in a way that could be described by some function or some mathematical relation or whose decision process could be described by a mathematical method? I am not aware of any economist who believes that we can capture the thought process of a human deciding what to buy or what to produce by some mathematical method. However, the economists assume that rational actors make decisions based on the maximization of some function, possibly subject to constraint (leading naturally to the use of calculus). But if economists do not believe that we can describe behavior by mathematics, why do they subscribe to the maximization hypothesis?

One answer, I believe, is this. No economist much cares how a particular individual decides what to do. Watch a consumer standing before the vending machine. The consumer stands looking at the contents of the machine for a while holding a dollar. Finally, the consumer puts the dollar in the machine and looks some more. At long last, the consumer sighs, pushes a combination of buttons, and claims her prize. It is seriously doubtful that anyone, including the consumer, can give a coherent version of what happened. Surely no outside observer can tell what went on in the consumer's mind. How would the most careful scientist model this choice process? No model is likely to be very good at telling what this consumer would do the next time she stood before the vending machine. And more to the point, what would we know even if we could tell how *this* consumer chooses? If we are interested in the determination of prices, knowing how one consumer chooses will help us little. But if we could represent this consumer's choice process in some rather abstract way, a way that might capture the rough elements of how consumers generally choose, then we might have something. Note what has happened. Although we are trying to model how one consumer chooses, we are not worried about a particular consumer. Our model should be applicable to a variety of consumers (but maybe not all consumers). Because consumers are different, they will not choose in exactly the same manner. Thus the model becomes looser to account for and allow differences in individual behavior. Now in some way this is what the maximizing hypothesis does. It allows us to look at a variety of consumers without much hope of getting it right for any one, but broadly speaking we can use this model to say something about how they will choose, and the model will be rich enough to help us understand their choice process. At least, in some rough terms, the model will allow us to isolate the elements of the choice process that we should focus on if we are to understand how the choice is made.

Why maximization and not some other hypothesis? Consumers (and others) are widely seen to do the best they can with what they have. It is a serious leap to go from that statement to the hypothesis that individuals maximize, but at least maximization is hinted at by the previous statement. Again, few economists view consumers or firms as actually carrying out the maximizing behavior as it is represented mathematically. But there is a feeling that it captures what consumers are, in some general way, trying to do. And the hypothesis yields testable conclusions about economic behavior.

To carry out this program, we will need to know some mathematics. We will start now. But where to start? We will start with some elementary concepts. But we will move quickly to the algebra of matrices, the algebra of linear functions, and finally to the calculus. Our applications of mathematics to economics will take up the last part of these notes with the hope that a student can be patient for the results.