HOMOGENEOUS BI-LAGRANGIAN MANIFOLDS AND INVARIANT MONGE-AMPÈRE EQUATIONS

DMITRI V. ALEKSEEVSKY
School of Mathematics and Maxwell Insitute for Mathematical Sciences,
University of Edinburgh,
Edinburgh, EH9 3JZ, UNITED KINGDOM
E-mail: D.Aleksee@ed.ac.uk

ANDREA SPIRO
Dip. Matematica e Informatica, Università di Camerino,
I-62032 Camerino (Macerata), ITALY
E-mail: andrea.spiro@unicam.it

In this note, we give a description of the invariant bi-Lagrangian structures on a homogeneous symplectic manifold $(M = G/K, \omega)$ of a semisimple Lie group G. We then use this description in order to obtain invariant generalized Monge-Ampére equations, in the sense of V. Lychagin, on certain homogeneous contact manifolds, associated with homogeneous bi-Lagrangian manifolds.

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1. Bi-Lagrangian manifolds

A *bi-Lagrangian structure* on a symplectic manifold (M, ω) is a decomposition $TM = L^+ + L^-$ of the tangent bundle into a direct sum of integrable Lagrangian distributions (see Ref. 3). Notice that, given a bi-Lagrangian structure $TM = L^+ + L^-$ on (M, ω), the field of endomorphisms $I_x \in \Gamma(\mathrm{End}(T_xM))$, defined by $I|_{L^\pm_x} = \pm Id$ for $x \in M$, is involutive ($I^2 = \mathrm{Id}$), skew-symmetric w.r.t. to the pseudo-Riemannian metric $g = \omega(I\cdot, \cdot)$ and parallel w.r.t. the Levi-Civita connection ∇ of g.

Conversely, a pair (g, I) formed by a metric g and a parallel field I of involutive skew-symmetric endomorphisms (called *para-Kähler structure*) defines a symplectic structure $\omega = g(\cdot, I\cdot)$ together with a bi-Lagrangian structure $TM = L^+ + L^-$, where $L^\pm$ are the (± 1)-eigendistributions of I.

We list now some properties of a bi-Lagrangian manifold $(M, \omega, TM =$

$L^+ + L-$).

I) For any pair of vector fields $X^+ \in \Gamma(L^+)$ and $Y^- \in \Gamma(L^-)$,

$$\nabla_{X^+} Y^- = (\nabla_{X^+} Y^-)_{L^-} = (\nabla_{Y^-} X^+ + [X^+, Y^-])_{L^-} = [X^+, Y^-]_{L^-} \quad (1)$$

where $(\cdot)_{L^-}$ denotes the projection onto L^-.

We denote by $N^\pm = N^\pm(x)$ the maximal integral leaves of the distributions $T^\pm M$ passing through a point $x \in M$. Then (1) implies that the connection induced by ∇ on $L^-|_{N^+} = T^*N^-$ is flat. The identity

$$g(\nabla_{X^+} Y^+, Z^-) = -g(Y^+, \nabla_{X^+} Z^-) \ , \quad \text{for } X^+, Y^+ \in \Gamma(L^+), Z^- \in \Gamma(L^-) \ ,$$

shows that also the induced connections on $TN^+ = L^+|_{N^+}$ and on

$$TM|_{N^+} = L^+|_{N^+} + L^-|_{N^+} = TN^+ + T^*N^+$$

are flat.

II) Assume that each maximal integral leaf $N^\pm(x)$, $x \in M$, is simply connected and that the flat linear connection $\nabla|_{TN^\pm(x)}$ is complete. Then, by (I), each leaf has the structure of an affine space and the exponential map $\exp : L^\pm|_x \longrightarrow N^\pm(x)$ is an isomorphism of affine spaces. Moreover, if we fix a maximal integral leaf $N^+ = N^+(x)$ and we denote by $\exp : L^-|_{N^+} \longrightarrow M$ the restriction of the exponential map of g at the bundle $L^-|_{N^+}$, then the map $\exp : L^-|_{N^+} \longrightarrow M$ is a local diffeomorphism.

This last property follows from the fact that a vector field $J_t = J|_{\gamma_t}$ along the geodesic $\gamma_t = \exp(tv^-)$, for some $v^- \in L^-|_{N^+}$, is a Jacobi field if and only if the components $J^\pm \in L^\pm$ of $J = J^+ + J^-$ satisfy the equations

$$\nabla_{\dot\gamma} \nabla_{\dot\gamma} J^+ = 0 \ , \qquad \nabla_{\dot\gamma} \nabla_{\dot\gamma} J^- + R(\dot\gamma, J^+) \cdot \dot\gamma = 0 \ . \quad (2)$$

From (2) it follows that if $J_0 = 0$ and $J \not\equiv 0$, then either J_t^+ or J_t^- never vanishes on $\mathbb{R} \setminus \{0\}$. Hence $\exp_*|_v$ has trivial kernel for any $v \in L^-|_{N^+}$.

In general, $\exp : L^-|_{N^+} \longrightarrow M$ is a local diffeomorphism and we use the map $\exp^{-1}$ to identify a sufficiently small neighborhood $\mathcal{U} \subset M$ of N^+ with a tubular neighborhood $\mathcal{V} \subset L^-|_{N^+} = T^*N^+$ of the zero section. Consider a system of coordinates (q^i) on N^+ and the associated system of coordinates (q^i, p_i) on T^*N^+. Then the integral leaves $N^\pm$ of the distributions $L^\pm$ are identified with the submanifolds

$$N^+ = \{\, p_1 = \cdots = p_n = 0 \,\} \ , \qquad N^- = \{\, q^1 = \cdots = q^n = 0 \,\} \quad (3)$$

and the symplectic form ω is of the form $\omega = \widetilde{\omega}^i_j dp_i \wedge dq^j$. From $d\omega = 0$, we get that $\omega = \frac{\partial^2 f}{\partial p_i \partial q^j} dp_i \wedge dq^j$ for some smooth function $f(q,p)$ and the coordinates $(q^i, p'_j \overset{\text{def}}{=} \frac{\partial f}{\partial q^j})$ are so that $\omega = dp'_j \wedge dq^i$, i.e. they represent a system of symplectic coordinates. From (3), in such system of symplectic coordinates, the Lagrangian distribution L^- is of the form $L^- = \text{span}\{\frac{\partial}{\partial p'_i}\}$.

III) Let $M = G/K$ be a $2n$-dimensional homogeneous space of a Lie group G and $(\omega, TM = L^+ + L^-)$ a G-invariant bi-Lagrangian structure. Assume, for simplicity, that the maximal integral leaves $N^{\pm}(x)$, $x \in M$, are simply connected. As in (II) we can choose a diffeomeorphism $\varphi : \mathcal{U} \subset M \to \hat{\mathcal{U}} \subset T^*\mathbb{R}^n$ between an open neighborhood of $\mathcal{U}$ of a point $x \in M$ and an open set $\hat{\mathcal{U}}$ of the cotangent bundle $T^*\mathbb{R}^n$ of $\mathbb{R}^n$ such that:

- i) $\varphi_*(\omega) = \omega_{\text{can}} = dp_i \wedge dq^i$;
- ii) $\varphi_*(L^-)$ is the vertical distribution on $\hat{\mathcal{U}} \subset T^*\mathbb{R}^n$;
- iii) any transformation $g \in G$ induces (via the diffeomorphism φ) a local fiber preserving symplectic transformations $\hat{g}$ of $\hat{\mathcal{U}} \subset T^*\mathbb{R}^n$.

2. Invariant Bi-Lagrangian structures on adjoint orbits of semisimple groups and gradations of semi-simple Lie algebras

In the following, for any semisimple Lie algebra $\mathfrak{g}$, we will denote by B the Killing form of $\mathfrak{g}$.

2.1. *Homogeneous symplectic manifolds of a semisimple Lie group*

In this section, we describe the structure of homogeneous bi-Lagrangian manifolds of a semisimple Lie group.

Theorem 2.1 (Kirillov-Kostant-Souriou). *Let $(M = G/K, \omega)$ be a symplectic homogeneous space of a semisimple Lie group G. Then, up to a covering, M is the adjoint orbit $M = \mathrm{Ad}_G\, h = G/K \subset \mathfrak{g}$, with $K = Z_G(h)$, of some element $h \in \mathfrak{g} = Lie(G)$. Moreover, for any $Z \in M \subset \mathfrak{g}$, the symplectic form ω is given by*

$$\omega_Z(X, Y) = B(Z, [X, Y]) \ , \qquad \textit{for any } X, Y \in T_Z M \subset \mathfrak{g} \ .$$

Theorem 2.2 (Hou-Deng-Kaneyuki-Nishiyama[4]).
Let G be a semisimple (real or complex) Lie group and $(M = \mathrm{Ad}_G\, h, \omega)$ an adjoint orbit of an element $h \in \mathfrak{g} = Lie(G)$, equipped with invariant symplectic structure ω. The manifold M admits a bi-Lagrangian structure if and only if h is a semisimple element.

A $\mathbb{Z}$-gradation

$$\mathfrak{g} = \mathfrak{g}^{-k} + \cdots + \mathfrak{g}^{-1} + \mathfrak{g}^{0} + \mathfrak{g}^{1} + \cdots + \mathfrak{g}^{k} , \qquad [\mathfrak{g}^{i}, \mathfrak{g}^{j}] \subset \mathfrak{g}^{i+j} \tag{4}$$

of a semi-simple Lie algebra $\mathfrak{g}$ is called *fundamental* if the subalgebra

$$\mathfrak{g}^{-} = \mathfrak{g}^{-k} + \cdots + \mathfrak{g}^{-1}$$

is generated by $\mathfrak{g}^{-1}$.

The following theorem reduces the classification of homogeneous bi-Lagrangian manifolds of a (complex or real) semisimple Lie group G to the description of fundamental gradations of its corresponding Lie algebra $\mathfrak{g}$.

Theorem 2.3 (Alekseevsky-Medori[2]). *Let $(M = \mathrm{Ad}_G\, h = G/K, \omega)$ be as in the previous theorem and let $\mathfrak{k} = Z_{\mathfrak{g}}(h) = Lie(K)$. There exists a natural one-to-one correspondence between*

i) invariant bi-Lagrangian structures $TM = L^{+} \oplus L^{-}$;
ii) K-invariant decompositions (called bi-isotropic) of the Lie algebra

$$\mathfrak{g} = \mathfrak{n}^{-} + \mathfrak{k} + \mathfrak{n}^{+} , \tag{5}$$

where $\mathfrak{n}^{\pm}$ are subalgebras such that $B|_{\mathfrak{n}^{\pm}} = 0$;
iii) fundamental K-invariant $\mathbb{Z}$-gradations (4) with $\mathfrak{g}^{0} = \mathfrak{k}$.

More precisely, the bi-isotropic decomposition (5), which corresponds to the fundamental gradation (4), is given by $\mathfrak{n}^{\pm} = \sum_{\pm i > 0} \mathfrak{g}^{i}$ and $\mathfrak{k} = \mathfrak{g}^{0}$, while the bi-Lagrangian decomposition $TM = L^{+} + L^{-}$ associated with (5) is the natural invariant extension of the K-invariant decomposition $T_oM = \mathfrak{n}^{+} + \mathfrak{n}^{-}$ of the tangent space of $M = G/K$, $o = eK$ (under the standard identification $T_oM = \mathfrak{g}/\mathfrak{k} = \mathfrak{n}^{+} + \mathfrak{n}^{-}$).

2.2. *Fundamental gradations of a (complex or real) semisimple Lie algebra* $\mathfrak{g}$

Let $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_{\alpha}$ be a root space decomposition of a complex semisimple Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$. We fix a system of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_\ell\} \subset R$, that is a basis of $\mathfrak{h}^*$ such that any root $\alpha \in R$ has integer coefficients with respect to Π of the same sign (≥ 0 or ≤ 0).

Any disjoint decomposition $\Pi = \Pi^0 \cup \Pi^1$ of Π defines a fundamental gradation of $\mathfrak{g}$ as follows. First, define the function $d : R \to \mathbb{Z}$ by

$$d|_{\Pi^0} = 0 , \quad d|_{\Pi^1} = 1 , \quad d(\alpha) = \sum k_i d(\alpha_i) , \quad \text{for any } \alpha = \sum k_i \alpha_i \in R .$$

Then the fundamental gradation is given by

$$\mathfrak{g}^0 = \mathfrak{h} + \sum_{\alpha\in R,\ d(\alpha)=0} \mathfrak{g}_\alpha\ , \qquad \mathfrak{g}^i = \sum_{\alpha\in R,\ d(\alpha)=i} \mathfrak{g}_\alpha\ .$$

Notice that *any fundamental gradation of* $\mathfrak{g}$ *is conjugated to a unique gradation of such a form.*

Any real semisimple Lie algebra $\hat{\mathfrak{g}}$ is a real form of a complex semisimple Lie algebra γ, that is $\hat{\mathfrak{g}}$ is the fixed point set $\hat{\mathfrak{g}} = \mathfrak{g}^\sigma$ of some antilinear involution σ of $\mathfrak{g}$, i.e. an antilinear involutive map $\sigma : \mathfrak{g} \to \mathfrak{g}$, which is an automorphism of $\mathfrak{g}$ as a Lie algebra over $\mathbb{R}$. We can always assume that σ preserves a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and induces an automorphism of the root system R. A root $\alpha \in R$ is called *compact* (or *black*) if $\sigma\alpha = -\alpha$. It is always possible to choose a system of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ such that, for any non-compact root $\alpha_i \in \Pi$, the corresponding root $\sigma\alpha_i$ is a sum of one non-compact root $\alpha_j \in \Pi$ and a linear combination of compact roots from Π. The roots α_i and α_j are called *equivalent*.

Proposition 2.4 (Alekseevsky - Medori[2]). *Let* $\mathfrak{g}$ *be a complex semisimple Lie algebra* $\mathfrak{g}$, $\sigma : \mathfrak{g} \to \mathfrak{g}$ *an antilinear involution and* $\mathfrak{g}^\sigma$ *the corresponding real form. The gradation of* $\mathfrak{g}$, *associated with a decomposition* $\Pi = \Pi^0 \cup \Pi^1$, *defines a gradation* $\mathfrak{g}^\sigma = \sum(\mathfrak{g}^i)^\sigma$ *of* $\mathfrak{g}^\sigma$ *if and only if* Π^1 *consists of non-compact roots and any two equivalent roots are either both in* Π^0 *or both in* Π^1.

2.3. *Examples of fundamental gradations*

2.3.1. *Fundamental gradations of* $\mathfrak{sl}(V)$

Let V be a (complex or real) vector space and $V = V^1 + \cdots + V^k$ a decomposition of V into a direct sum of subspaces. It defines a fundamental gradation $\mathfrak{sl}(V) = \sum_{i=-k}^{k} \mathfrak{g}^i$ of the Lie algebra $\mathfrak{sl}(V)$, where

$$\mathfrak{g}^i = \{A \in \mathfrak{sl}(V),\ AV^j \subset V^{i+j},\ j = 1, \ldots, k\ \}\ .$$

Any fundamental gradation of $\mathfrak{sl}(V)$ is of this form.

2.3.2. *Fundamental gradations of* $\mathfrak{g}_2$

The root system of the complex exceptional Lie algebra $\mathfrak{g}_2$ has the form $R = \{\pm\varepsilon_i, \pm(\varepsilon_i - \varepsilon_j),\ i, j =, 2, 3\}$ where the vectors ε_i satisfy

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0,\ \varepsilon_i^2 = 2/3,\ (\varepsilon_i, \varepsilon_j) = -1/3,\ i \neq j\ .$$

Consider the system of simple roots $\Pi = \{\alpha_1 = -\varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3\}$. The corresponding system of positive roots is

$$R^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$

There are three fundamental gradations for the complex Lie algebra $\mathfrak{g}_2$. For any of such gradations, we give below the subset $\Pi^1 \subset \Pi$ and the level sets $R^i := \{\alpha \in R, d(\alpha) = i\}$ of the grading function $d : R \to \mathbb{Z}$.

1) $\mathbf{\Pi^1 = \Pi}$:

$$R^0 = \emptyset, \quad R^1 = \{\alpha_1, \alpha_2\}, \quad R^2 = \{\alpha_1 + \ a_2\},$$

$$R^3 = \{2\alpha_1 + \alpha_2\}, \quad R^4 = \{3\alpha_1 + \alpha_2\}, \quad R^5 = \{3\alpha_1 + 2\alpha_2\};$$

2) $\mathbf{\Pi^1} = \{\alpha_1\}$:

$$R^0 = \{\alpha_2\}, R^1 = \{\alpha_1, \alpha_1+\alpha_2\}, R^2 = \{2\alpha_1+\alpha_2\}, R^3 = \{3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\};$$

3) $\mathbf{\Pi^1} = \{\alpha_2\}$:

$$R^0 = \{\alpha_1\},\ R^1 = \{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\},\ R^2 = \{3\alpha_1 + 2\alpha_2\}.$$

There are just two real forms of the complex Lie algebra $\mathfrak{g}_2$: the compact form, which has no non-trivial gradation, and the normal form $\mathfrak{g}_2^g$, which has a diagonalizable Cartan subalgebra and no compact roots. The above listed gradations of the complex Lie algebra $\mathfrak{g}_2$ define three gradations of the real Lie algebra $\mathfrak{g}_2^g$.

3. Generalized Monge-Ampère equations and effective forms on contact manifolds

Let $J^1 = \mathbb{R}^{2n+1}$ be the space of 1-jets of real functions $u : \mathbb{R}^n \to \mathbb{R}$, with standard coordinates $(q^1, \dots, q^n, u, p_1, \dots, p_n)$ and natural contact form $\vartheta = du - p_i dq^i$. V. Lychagin[5] associates with any differential n-form $\Theta \in \Omega^n(J^1)$ a differential operator

$$\Delta_\Theta : C^\infty(\mathbb{R}^n; \mathbb{R}) \to \Omega^n(\mathbb{R}^n), \qquad \Delta_\Theta(h) \overset{\text{def}}{=} \Theta|_{j^1(h)}, \text{ for any } h \in C^\infty(\mathbb{R}^n; \mathbb{R})$$

called *Monge-Ampère operator determined by* Θ. Denote by $F_\Theta(x, j^2(h)) = F_\Theta\left(x, h, \frac{\partial h}{\partial x^i}, \frac{\partial^2 h}{\partial x^{i2}}\right)$ the unique function such that

$$\Delta_\Theta(h) = F_\Theta(x, j^2(h)) dx^1 \wedge \cdots \wedge dx^n .$$

The corresponding second order equation $F_\Theta(x, j^2(h)) = 0$ is called *generalized Monge-Ampère equation associated with* Θ.

Example 3.1. (i) Let $n = 2$ and $\Theta = dp_1 \wedge dp_2$. The associated equation is the standard Monge-Ampère equation $F_\Theta(x, j^2(h)) = \det\left(\frac{\partial u}{\partial x^i \partial x^j}\right) = 0$.

(ii) Let $\Theta = \sum_i (-1)^{i+1} dp_i \wedge dx^1 \wedge \ldots \widehat{i} \ldots \wedge dx^n$. The associated equation is the Laplace equation

$$F_\Theta(x, j^2(h)) = \Delta h = 0 \ .$$

Two n-forms Θ and Θ' that define the same Monge-Ampère operators are called *equivalent*.

Recall that the restriction $\omega \overset{\text{def}}{=} d\theta|_{\mathcal{D}}$ of $d\theta$ to the contact distribution $\mathcal{D} := \ker \theta$ is non-degenerate. Denote by $\mathcal{T}$ be the Reeb vector field on J^1, i.e. the vector field defined by

$$\vartheta(\mathcal{T}) \equiv 1 \ , \qquad \mathcal{T} \lrcorner\, d\vartheta = 0 \ . \tag{6}$$

A differential n-form Θ_{eff} on J^1 is called *effective* if

$$\mathcal{T} \lrcorner\, \Theta_{\text{eff}} = 0 \ , \qquad (\omega)^{-1} \lrcorner\, (\Theta_{\text{eff}}|_{\mathcal{D}}) = 0 \ . \tag{7}$$

Theorem 3.2 (Lepage - Lychagin[5]). *Any n-form $\Theta \in \Omega^n(J^1)$ is equivalent to a unique effective n-form Θ_{eff}, given by*

$$\Theta_{\text{eff}} = \Theta - \vartheta \wedge (\mathcal{T} \lrcorner\, \Theta) + d\vartheta \wedge \Theta_1$$

where Θ_1 is a uniquely defined $(n-2)$-form. In particular, the generalized Monge-Ampère equations are in natural one-to-one correspondence with the effective n-forms on J^1.

Let now M be a $(2n+1)$-dimensional manifold endowed with a contact form θ and denote by $\mathcal{T}$ and $\mathcal{D}$ the Reeb vector and the contact distribution $\mathcal{D} = \ker \theta$ determined by θ. We call *effective form* any n-form $\Theta_{\text{eff}} \in \Omega^n(M)$ that satisfies (7). By Darboux's theorem, there are local coordinates (q^i, u, p_j) on M such that $\theta = du - \sum p_i dq^i$. In other words, we can always locally identify (M, θ) with the space J^1, equipped with the standard contact form. In particular, any such local identification allows to consider generalized Monge-Ampère equations associated with effective n-forms on M.

4. Invariant Monge-Ampère equations on homogeneous bi-Lagrangian manifolds

4.1. *Invariant effective forms on contactifications of homogeneous symplectic manifolds*

Let (M, ω) be a $2n$-dimensional symplectic manifold. A differential n-form Θ on M is called *effective* if $\omega^{-1} \lrcorner\, \Theta = 0$.

Consider a homogeneous symplectic manifold $(M = G/K, \omega)$. It is possible to associate with (M, ω) a homogeneous contact manifold $(\tilde{M} = \tilde{G}/K, \theta)$, called *contactification of* (M, ω). It is defined as a homogeneous principal $\mathbb{R}$-bundle over $M = G/K$, with an invariant connection form $\theta : T\tilde{M} \to \mathbb{R}$, whose curvature form $d\theta$ is equal to $d\theta = \pi^*\omega$ (see e.g. Ref.1). By definition, θ is a $\widetilde{G}$-invariant contact form on $\widetilde{M}$. Such a homogeneous contactification is uniquely determined (up to a covering) and it is described as follows. The Lie algebra $\widetilde{\mathfrak{g}}$ of $\widetilde{G}$ is a central extension $\widetilde{\mathfrak{g}} = \mathbb{R}Z + \mathfrak{g}$ of $\mathfrak{g}$, with Lie brackets $\widetilde{[\cdot,\cdot]}$ defined by

$$\widetilde{[X,Y]} = \omega_{eK}(\hat{X},\hat{Y})Z + [X,Y] \ , \qquad X, Y \in \mathfrak{g} = Lie(G) \ ,$$

where $\hat{X}$, $\hat{Y} \in \mathfrak{X}(M)$ are the infinitesimal transformations associated with $X, Y \in \mathfrak{g}$, while θ is the unique $\widetilde{G}$-invariant contact 1-form such that

$$\theta_{eK}(Z) = 1 \ , \qquad \theta_{eK}|_{\mathfrak{g}} = 0 \ .$$

By construction, the Reeb vector $\mathcal{T}$ of θ is the infinitesimal transformation $\mathcal{T} = \hat{Z}$ associated with Z and the contact distribution $\mathcal{D}$ coincides with the unique $\widetilde{G}$-invariant distribution

$$\mathcal{D}_{eK} = \mathrm{Span}\{ \ \hat{X}_{eK} \ , \ X \in \mathfrak{g} \ \} \ .$$

Proposition 4.1. *Let* $(M = G/K, \omega)$ *be a homogeneous* $2n$*-dimensional symplectic manifold and* $(\tilde{M} = \tilde{G}/K, \theta)$ *the associated contactification. Then, there exists a natural 1-1 correspondence between* $\tilde{G}$*-invariant effective* n*-forms* Θ *on* $\tilde{M}$ *and* G*-invariant effective* n*-forms* Θ^M *on* M.

Furthermore, if the subgroup K *is connected and the homogeneous manifold* $M = G/K$ *is reductive, with a reductive decomposition* $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$*, then the invariant effective* n*-forms on* M *are in natural one to one correspondence with the* $ad_{\mathfrak{k}}$*-invariant exterior* n*-forms* $\Theta_o \in \Lambda^n\mathfrak{m}^*$ *that are effective, i.e. so that*

$$\omega_o^{-1} \lrcorner \, \Theta_o = 0 \ , \tag{8}$$

where we denoted by $\omega_o \in \Lambda^2\mathfrak{m}^*$ *the value of* ω *at the point* $o = eK \in G/K$.

From this proposition, the problem of describing the invariant effective n-forms on the homogeneous contact manifold $(\tilde{M} = \tilde{G}/K, \theta)$ and associated invariant generalized Monge-Ampére equations reduces to the classification of $ad_{\mathfrak{k}}$-invariant effective n-forms on the vector space $\mathfrak{m}$. In the next section, we describe such invariant effective forms on homogeneous bi-Lagrangian manifolds of the normal real form of the exceptional group G_2.

4.2. *Invariant effective forms on homogeneous bi-Lagrangian manifolds*

Consider the homogeneous bi-Lagrangian manifolds M_1, M_2 and M_3 of the normal real form $G_2^{\mathfrak{g}}$ of the exceptional group G_2, associated with the fundamental gradations (1), (2) and (3) in §2.3.

The manifolds M_2 and M_3 are both 10-dimensional. The space of invariant 5-forms on M_3 is 2-dimensional. In order to give explicit expressions of two generators Θ and Θ' for this space, let us pass to the complexification and consider a basis (E^*_α) of $(\mathfrak{m}^{\mathbb{C}})^*$, which is dual to a basis of root vectors (E_α) of $\mathfrak{m}^{\mathbb{C}} = \sum_{i\neq 0} \mathfrak{g}^i$. Let also

$$f^*_\pm \overset{\text{def}}{=} E^*_{\pm(3\alpha_1+2\alpha_2)} ,$$

$$e^*_{\pm 1} = E^*_{\pm\alpha_2} , \quad e^*_2 = E^*_{\pm(\alpha_1+\alpha_2)} , \quad e^*_3 = E^*_{\pm(2\alpha_1+\alpha_2)} , \quad e^*_4 = E^*_{\pm(3\alpha_1+\alpha_2)} .$$

In this notation, one generator is

$$\Theta = \varepsilon^{ijk\ell} f^*_+ \wedge e^*_{+i} \wedge e^*_{-j} \wedge e^*_{-k} \wedge e^*_{-\ell}$$

while the other 5-form Θ' is obtained from Θ by changing all + signs into − and vice versa. Both forms Θ and Θ' are effective, i.e. satisfy (8) w.r.t. to the unique (up to a scalar multiple) invariant symplectic form ω_o of M^3.

The manifold M_2 also has a 2-dimensional space of invariant 5-forms. A pair of generators is given by the 5-form

$$\hat{\Theta} = E^*_{2\alpha_1+\alpha_2} \wedge E^*_{-\alpha_1} \wedge E^*_{-\alpha_1-\alpha_2} \wedge$$
$$\wedge \left(E^*_{3\alpha_1+\alpha_2} \wedge E^*_{-3\alpha_1-\alpha_2} - E^*_{3\alpha_1+2\alpha_2} \wedge E^*_{-3\alpha_1-2\alpha_2}\right)$$

and the form $\hat{\Theta}'$, obtained from $\hat{\Theta}$ by replacing any element $E^*_{k_1\alpha_1+k_2\alpha_2}$ by $E^*_{-k_1\alpha_1-k_2\alpha_2}$. However, neither $\hat{\Theta}$ nor $\hat{\Theta}'$ is effective w.r.t. to the unique invariant symplectic form of M_2.

Finally, consider the 12-dimensional manifold M_1. This manifold admits several invariant 6-forms and a 2-dimensional space of invariant symplectic forms. The following is an example of an invariant 6-form, which is effective w.r.t. any of the invariant symplectic forms:

$$\widetilde{\Theta} = E^*_{3\alpha_1+2\alpha_2} \wedge E^*_{2\alpha_1+\alpha_2} \wedge E^*_{-\alpha_1} \wedge E^*_{-\alpha_2} \wedge E^*_{-\alpha_1-\alpha_2} \wedge E^*_{-3\alpha_1-\alpha_2} .$$

The complete classification of invariant symplectic forms and related effective forms can be obtained by direct computations.

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