# Chapter 5

# Plane Anisotropic Elasticity Problems

The structural analysis of composite materials is always based on the analysis of anisotropic materials. The solution of anisotropic elasticity problems is more complicated than isotropic problems. Based on symplectic system of isotropic elasticity, the Hamiltonian system is further extended to plane anisotropic elasticity problems in this chapter and thus a complete symplectic solution methodology for these problems is established. A rational analytical solution for Saint–Venant problems is derived by the expansion of eigenvector subspace of zero eigenvalue.

## 5.1. The Fundamental Equations of Plane Anisotropic Elasticity Problems

Consider in the scope of isotropic elasticity a thin plate with constant thickness and with body forces and forces on side boundary surfaces parallel to the plane of plate. These forces are constant through the thickness of plate. Such problems can be simplified to plane stress problems. On the other hand, for long cylindrical solids with body forces and forces on side boundary surfaces parallel to the cross section of cylinder and all forces are constant in the longitudinal direction, they can be simplified to plane strain problems. For an anisotropic body, the specific geometries and loading conditions above are not the sufficient conditions for simplification to plane elasticity problems. It is necessary for the anisotropic materials to have an elastic symmetric plane which is consistent with the plane of load in order for the problems to be approximately treated as plane elasticity problems.

Consider a thin plate with constant thickness. At any point in the domain, there exists an elastic symmetric plane of material which is parallel to the plane of plate. Let $Oxz$-plane be the mid-plane of plate, i.e. the dimension along the $y$-axis is far smaller than the other two dimensions. Assume

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that the body forces and forces on side boundary surfaces are parallel to the plane and all forces are constant through thickness. Further assume the geometric constraints are also constant through thickness. Such problems can be approximately simplified to plane stress problems. Similar to the plane isotropic elasticity problems, it is required to solve the eight physical quantities $u$, $w$, $\varepsilon_x$, $\varepsilon_z$, $\gamma_{xz}$ and $\sigma_x$, $\sigma_z$, $\tau_{xz}$. As compared to isotropic problems, the only difference here is the stress-strain relation

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{13} & s_{15} \\ s_{13} & s_{33} & s_{35} \\ s_{15} & s_{35} & s_{55} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix} \tag{5.1.1}$$

or in another form

$$\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} b_{11} & b_{13} & b_{15} \\ b_{13} & b_{33} & b_{35} \\ b_{15} & b_{35} & b_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{Bmatrix} \tag{5.1.2}$$

where $b_{ij}$ are the **reduced stiffness coefficients** whose determinant is denoted as $d$ in brief. For an elastic body the deformation energy density is positive definite.

For such problems, the strain-displacement relation and equation of equilibrium remain as Eqs. (4.1.5) and (4.1.6), respectively, and the boundary conditions on $\Gamma_\sigma$ for specified forces and on $\Gamma_u$ for specified displacements remain as Eqs. (4.1.7) and (4.1.8), respectively.

Consider a long cylindrical solid with an elastic symmetric plane of material $Oxz$ at every point and the elastic principal axis $y$ is consistent with the axis of the cylinder. In addition, the body forces and forces on the side boundary surfaces are parallel to the cross section. These forces as well as the geometric constraints are constant along the $y$-axis. Then, such problems can be simplified to plane strain problems. The stress-strain relation of plane strain problems is

$$\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{13} & c_{15} \\ c_{13} & c_{33} & c_{35} \\ c_{15} & c_{35} & c_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{Bmatrix} \tag{5.1.3}$$

while the other fundamental equations remain as Eqs. (4.1.5) to (4.1.8). The fundamental equations and boundary conditions of plane strain problems are the same as those of plane stress problems. The only difference lies on the interpretation of elastic constants. Likewise, the two kinds of problems are treated as similar cases and they are generally referred as the plane anisotropic elasticity problems. The fundamental equations and boundary conditions are Eqs. (5.1.1) or (5.1.2) and Eqs. (4.1.5) to (4.1.8), respectively.

## 5.2. Symplectic Solution Methodology for Anisotropic Elasticity Problems

Consider a rectangular domain as illustrated in Fig. 5.1

$$V\colon \quad 0 \le z \le l, \quad -h \le x \le h \tag{5.2.1}$$

where $l$ is relatively larger. The forces on side boundary surfaces ($x = \pm h$) are

$$\sigma_x = \overline{F}_{x1}(z), \quad \tau_{xz} = \overline{F}_{z1}(z) \quad \text{at } x = -h \tag{5.2.2a}$$

$$\sigma_x = \overline{F}_{x2}(z), \quad \tau_{xz} = \overline{F}_{z2}(z) \quad \text{at } x = h \tag{5.2.2b}$$

And there are body forces $F_x$ and $F_z$ acting along the $x$- and $z$-directions, respectively, in the domain.

There are the corresponding boundary conditions at the two ends $z = 0, l$. For instance, the boundary conditions for specified displacements are

$$w = \overline{w}, \quad u = \overline{u} \quad \text{at } z = 0 \quad \text{or} \quad l \tag{5.2.3a}$$

while the boundary conditions for specified forces are

$$\sigma_z = \overline{\sigma}_z, \quad \tau_{xz} = \overline{\tau}_{xz} \quad \text{at } z = 0 \quad \text{or} \quad l \tag{5.2.3b}$$

The fundamental equations and boundary conditions of anisotropic elasticity problems mentioned above can be derived from the Hellinger–Reissner

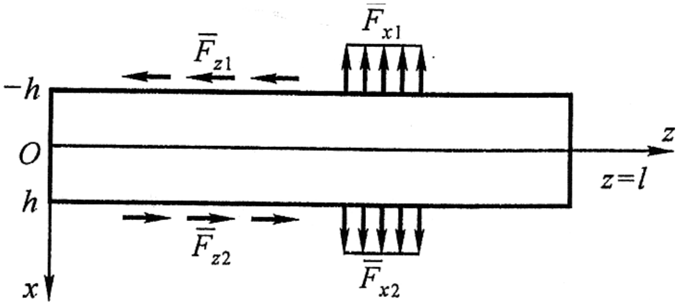

Fig. 5.1. Plane problem in rectangular domain.

variational principle

$$\delta\left\{\int_0^l \int_{-h}^h \left[\sigma_x \frac{\partial u}{\partial x} + \sigma_z \frac{\partial w}{\partial z} + \tau_{xz}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right.\right.$$
$$\left.\left. - v_{\mathrm{c}} - F_x u - F_z w\right]\mathrm{d}x\mathrm{d}z + v_{\mathrm{e}}^1 + v_{\mathrm{e}}^2\right\} = 0 \tag{5.2.4}$$

where

$$v_{\mathrm{c}} = \frac{1}{2}\left(s_{11}\sigma_x^2 + s_{33}\sigma_z^2 + s_{55}\tau_{xz}^2\right) + s_{13}\sigma_x\sigma_z + s_{15}\sigma_x\tau_{xz} + s_{35}\sigma_z\tau_{xz} \tag{5.2.5}$$

$$v_{\mathrm{e}}^1 = -\int_0^l \left\{\left[\overline{F}_{x2}u + \overline{F}_{z2}w\right]_{x=h} - \left[\overline{F}_{x1}u + \overline{F}_{z1}w\right]_{x=-h}\right\}\mathrm{d}z \tag{5.2.6}$$

$$v_{\mathrm{e}}^2 = \begin{cases} -\displaystyle\int_{-h}^h \left[\sigma_z\left(w - \overline{w}\right) + \tau_{xz}\left(u - \overline{u}\right)\right]_{z=0}^l \mathrm{d}x & \text{on } \Gamma_u \\ -\displaystyle\int_{-h}^h \left[\overline{\sigma}_z w + \overline{\tau}_{xz}u\right]_{z=0}^l \mathrm{d}x & \text{on } \Gamma_\sigma \end{cases} \tag{5.2.7}$$

We are now ready to derive the Hamiltonian system. First we treat the longitudinal $z$-coordinate as the time coordinate and indicate differentiation with respect to $z$ using a dot, i.e. $\dot{(\,)} = \partial/\partial z$. Next, the transverse stress $\sigma_x$ should be eliminated. The variation of Eq. (5.2.4) with respect to $\sigma_x$ is

$$\sigma_x = \frac{1}{s_{11}}\left(\frac{\partial u}{\partial x} - s_{13}\sigma - s_{15}\tau\right) \tag{5.2.8}$$

where $\sigma_z, \tau_{xz}$ are briefly denoted as $\sigma, \tau$. Substituting Eq. (5.2.8) into Eq. (5.2.4) and eliminating $\sigma_x$, yield the mixed energy variational principle of Hamiltonian system

$$\delta\left\{\int_0^l \int_{-h}^h [\sigma\dot{w} + \tau\dot{u} - \mathscr{H}(w, u, \sigma, \tau)]\mathrm{d}x\mathrm{d}z + v_{\mathrm{e}}^1 + v_{\mathrm{e}}^2\right\} = 0 \tag{5.2.9}$$

where the Hamiltonian density function is

$$\mathscr{H} = \frac{1}{2s_{11}d}(b_{55}\sigma^2 + b_{33}\tau^2 - 2b_{35}\sigma\tau) - \frac{1}{2s_{11}}\left(\frac{\partial u}{\partial x}\right)^2$$
$$- \tau\frac{\partial w}{\partial x} + \frac{1}{s_{11}}\frac{\partial u}{\partial x}(s_{13}\sigma + s_{15}\tau) + F_x u + F_z w \tag{5.2.10}$$

Obviously, the dual variables of displacements $w, u$ are stresses $\sigma, \tau$.

The variation of Eq. (5.2.9) yields the Hamiltonian dual equations as

$$\dot{\boldsymbol{v}} = \boldsymbol{H}\boldsymbol{v} + \boldsymbol{Q} \tag{5.2.11}$$

where $\boldsymbol{v}$ is the full state vector

$$\boldsymbol{v} = \{w, u, \sigma, \tau\}^{\mathrm{T}} \tag{5.2.12}$$

$\boldsymbol{H}$ is an operator matrix

$$\boldsymbol{H} = \begin{bmatrix} 0 & \dfrac{s_{13}}{s_{11}}\dfrac{\partial}{\partial x} & \dfrac{b_{55}}{s_{11}d} & -\dfrac{b_{35}}{s_{11}d} \\ -\dfrac{\partial}{\partial x} & \dfrac{s_{15}}{s_{11}}\dfrac{\partial}{\partial x} & -\dfrac{b_{35}}{s_{11}d} & \dfrac{b_{33}}{s_{11}d} \\ 0 & 0 & 0 & -\dfrac{\partial}{\partial x} \\ 0 & -\dfrac{1}{s_{11}}\dfrac{\partial^2}{\partial x^2} & \dfrac{s_{13}}{s_{11}}\dfrac{\partial}{\partial x} & \dfrac{s_{15}}{s_{11}}\dfrac{\partial}{\partial x} \end{bmatrix} \tag{5.2.13}$$

and $\boldsymbol{Q}$ is an inhomogeneous term related to body forces.

$$\boldsymbol{Q} = \{0, \quad 0, \quad -F_z, \quad -F_x\}^{\mathrm{T}} \tag{5.2.14}$$

The boundary conditions (5.2.2) on the two side boundary surfaces can be expressed as

$$\frac{1}{s_{11}}\left(\frac{\partial u}{\partial x} - s_{13}\sigma - s_{15}\tau\right) = \overline{F}_{x1}; \quad \tau = \overline{F}_{z1}, \quad \text{for } x = -h \tag{5.2.15a}$$

$$\frac{1}{s_{11}}\left(\frac{\partial u}{\partial x} - s_{13}\sigma - s_{15}\tau\right) = \overline{F}_{x2}; \quad \tau = \overline{F}_{z2}, \quad \text{for } x = h \tag{5.2.15b}$$

To discuss the property of operator matrix $\boldsymbol{H}$, we denote

$$\begin{aligned} \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle &\overset{\text{def}}{=} \int_{-h}^{h} \boldsymbol{v}_1^{\mathrm{T}} \boldsymbol{J} \boldsymbol{v}_2 \mathrm{d}x \\ &= \int_{-h}^{h} (w_1\sigma_2 + u_1\tau_2 - \sigma_1 w_2 - \tau_1 u_2)\mathrm{d}x \end{aligned} \tag{5.2.16}$$

where $\boldsymbol{J}$ is a unit symplectic matrix

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_2 \\ -\boldsymbol{I}_2 & \boldsymbol{0} \end{bmatrix} \tag{5.2.17}$$

Obviously, Eq. (5.2.16) satisfies the four conditions (1.3.2) of symplectic inner product. Hence, the full state vector $\boldsymbol{v}$ forms a symplectic space in accordance with the definition (5.2.16) of symplectic inner product.

First, we discuss the corresponding homogeneous linear differential equations of Eq. (5.2.11)

$$\dot{\boldsymbol{v}} = \boldsymbol{H}\boldsymbol{v} \tag{5.2.18}$$

and the homogeneous boundary conditions of Eq. (5.2.15) on the side boundary surfaces ($x = \pm h$)

$$\frac{1}{s_{11}}\left(\frac{\partial u}{\partial x} - s_{13}\sigma\right) = 0; \quad \tau = 0 \quad \text{at } x = \pm h \tag{5.2.19}$$

It is obvious via integration by parts that if $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are continuously differentiable full state vectors satisfying the boundary conditions (5.2.19), we then have the identity

$$\langle \boldsymbol{v}_1, \boldsymbol{H}\boldsymbol{v}_2\rangle = \langle \boldsymbol{v}_2, \boldsymbol{H}\boldsymbol{v}_1\rangle \tag{5.2.20}$$

It states that the operator matrix $\boldsymbol{H}$ is the Hamiltonian operator matrix in the symplectic space.

Thus we have transformed the plane anisotropic elasticity problems into Hamiltonian system which can then be solved using the conventional procedure of Hamiltonian system.

Similar to the plane isotropic problems, the method of separation of variables can be applied to solve the system of Eqs. (5.2.18), i.e. assume that

$$\boldsymbol{v}(z, x) = \xi(z)\boldsymbol{\psi}(x) = \mathrm{e}^{\mu z}\boldsymbol{\psi}(x) \tag{5.2.21}$$

where $\mu$ is an eigenvalue and $\boldsymbol{\psi}(x)$ is the eigenvector function. The eigenvalue equation is

$$\boldsymbol{H}\boldsymbol{\psi}(x) = \mu\boldsymbol{\psi}(x) \tag{5.2.22}$$

which must satisfy the homogeneous boundary conditions (5.2.19) on the side boundary surfaces ($x = \pm h$).

As $\boldsymbol{H}$ is a Hamiltonian operator matrix, the eigenvector functions are adjoint symplectic orthogonal. Once the eigenvalue and eigenvector functions are determined, the expansion theorem can be applied to solve this problem.

## 5.3. Eigen-Solutions of Zero Eigenvalue

For problems with free homogeneous boundary conditions (5.2.19) on the side boundary surfaces ($x = \pm h$), there exist the basic eigen-solutions[1] and the Jordan form eigen-solutions of zero eigenvalue. Such solutions in elasticity have significant physical interpretation. To determine the eigen-solutions of zero eigenvalue, we solve the following differential equation

$$\boldsymbol{H}\boldsymbol{\psi}(x) = \mathbf{0} \tag{5.3.1}$$

which can be expanded into

$$\left.\begin{array}{lllll}
0 & +\dfrac{s_{13}}{s_{11}}\dfrac{\partial u}{\partial x} & +\dfrac{b_{55}}{s_{11}d}\sigma & -\dfrac{b_{35}}{s_{11}d}\tau = 0 \\
-\dfrac{\partial w}{\partial x} & +\dfrac{s_{15}}{s_{11}}\dfrac{\partial u}{\partial x} & -\dfrac{b_{35}}{s_{11}d}\sigma & +\dfrac{b_{33}}{s_{11}d}\tau = 0 \\
0 & +0 & +0 & -\dfrac{\partial \tau}{\partial x} = 0 \\
0 & -\dfrac{1}{s_{11}}\dfrac{\partial^2 u}{\partial x^2} & +\dfrac{s_{13}}{s_{11}}\dfrac{\partial \sigma}{\partial x} & +\dfrac{s_{15}}{s_{11}}\dfrac{\partial \tau}{\partial x} = 0
\end{array}\right\} \tag{5.3.2}$$

The eigen-solutions should also satisfy the homogeneous boundary conditions (5.2.19) on the side boundary surfaces.

Solving Eq. (5.3.2) and substituting into the boundary conditions (5.2.19) yield the basic eigen-solutions of zero eigenvalue as

$$\boldsymbol{\psi}_{0f}^{(0)} = \{1, \quad 0, \quad 0, \quad 0\}^{\mathrm{T}} \tag{5.3.3}$$

$$\boldsymbol{\psi}_{0s}^{(0)} = \{0, \quad 1, \quad 0, \quad 0\}^{\mathrm{T}} \tag{5.3.4}$$

It shows that there are two chains which are denoted by subscripts $f$ and $s$, respectively. These two eigenvectors are the solutions of the original equation (5.2.18) and boundary conditions (5.2.19)

$$\boldsymbol{v}_{0f}^{(0)} = \boldsymbol{\psi}_{0f}^{(0)}, \quad \boldsymbol{v}_{0s}^{(0)} = \boldsymbol{\psi}_{0s}^{(0)} \tag{5.3.5}$$

which are physically interpreted as the rigid body translations along the $z$- and $x$-directions, respectively.

Besides the basic eigenvectors, the remaining eigenvectors are all Jordan form eigenvectors which form two chains. To obtain the first-order Jordan form eigen-solution on chain one, we solve the following equation with homogeneous boundary conditions (5.2.19)

$$\boldsymbol{H}\boldsymbol{\psi}_{0f}^{(1)} = \boldsymbol{\psi}_{0f}^{(0)} \tag{5.3.6}$$

The solution is

$$\begin{aligned}\boldsymbol{\psi}_{0f}^{(1)} &= \left\{w_{0f}^{(1)} \quad u_{0f}^{(1)} \quad \sigma_{0f}^{(1)} \quad \tau_{0f}^{(1)}\right\}^{\mathrm{T}} \\ &= \left\{\frac{s_{35}}{s_{33}}x, \quad \frac{s_{13}}{s_{33}}x, \quad \frac{1}{s_{33}}, \quad 0\right\}^{\mathrm{T}}\end{aligned} \tag{5.3.7}$$

The first-order Jordan form eigenvector $\boldsymbol{\psi}_{0f}^{(1)}$ is not the solution of the original equation which, however, can be used to construct solution to the original equation by

$$\boldsymbol{v}_{0f}^{(1)} = \boldsymbol{\psi}_{0f}^{(1)} + z\boldsymbol{\psi}_{0f}^{(0)} \tag{5.3.8}$$

The components of displacement and stress are

$$w = z + \frac{s_{35}}{s_{33}}x, \quad u = \frac{s_{13}}{s_{33}}x, \quad \sigma = \frac{1}{s_{33}}, \quad \tau = 0 \tag{5.3.9}$$

This solution is physically interpreted as the simple axial tension. As the material is anisotropic and in general $s_{35} \neq 0$, the deformed cross section remains a plane but is not parallel to the original cross section.

It is easy to verify that $\boldsymbol{\psi}_{0f}^{(1)}$ and $\boldsymbol{\psi}_{0s}^{(0)}$ are symplectic orthogonal while $\boldsymbol{\psi}_{0f}^{(1)}$ and $\boldsymbol{\psi}_{0f}^{(0)}$ are mutually symplectic adjoint

$$k_1 = \left\langle \boldsymbol{\psi}_{0f}^{(0)}, \ \boldsymbol{\psi}_{0f}^{(1)} \right\rangle = \frac{2h}{s_{33}} \neq 0 \tag{5.3.10}$$

Hence there is no second-order Jordan form eigen-solution on this Jordan chain and the chain is terminated. It should be noted here that $k_1$ has a special physical meaning, i.e. the extensional rigidity of cross section.

Similarly, the first-order Jordan form eigen-solution on chain two can be obtained by solving the following equation with homogeneous boundary conditions (5.2.19)

$$\boldsymbol{H}\boldsymbol{\psi}_{0s}^{(1)} = \boldsymbol{\psi}_{0s}^{(0)} \tag{5.3.11}$$

The solution is

$$\boldsymbol{\psi}_{0s}^{(1)} = \{-x, \quad 0, \quad 0, \quad 0\}^{\mathrm{T}} \tag{5.3.12}$$

Likewise, the first-order Jordan form eigenvector $\boldsymbol{\psi}_{0s}^{(1)}$ is not the solution of the original equation which, however, can be use to construct solution to the original equation by

$$\boldsymbol{v}_{0s}^{(1)} = \boldsymbol{\psi}_{0s}^{(1)} + z\boldsymbol{\psi}_{0s}^{(0)} \tag{5.3.13}$$

The components of displacement and stress are

$$w = -x, \quad u = z, \quad \sigma = 0, \quad \tau = 0 \tag{5.3.14}$$

The solution is physically interpreted as the in-plane rigid body rotation.

Through direct examination, we know that $\boldsymbol{\psi}_{0s}^{(1)}$ is symplectic orthogonal to both $\boldsymbol{\psi}_{0f}^{(0)}$ and $\boldsymbol{\psi}_{0s}^{(0)}$. Hence, there exists the second-order Jordan form eigen-solution which can be obtained by solving the following equation with homogeneous boundary conditions (5.2.19)

$$\boldsymbol{H}\boldsymbol{\psi}_{0s}^{(2)} = \boldsymbol{\psi}_{0s}^{(1)} \tag{5.3.15}$$

The solution is

$$\begin{aligned}\boldsymbol{\psi}_{0s}^{(2)} &= \{w_{0s}^{(2)}, \quad u_{0s}^{(2)}, \quad \sigma_{0s}^{(2)}, \quad \tau_{0s}^{(2)}\}^{\mathrm{T}} \\ &= \left\{-\frac{s_{35}}{2s_{33}}x^2 + g_1, \quad -\frac{s_{13}}{2s_{33}}x^2 + g_2, \quad -\frac{1}{s_{33}}x, \quad 0\right\}^{\mathrm{T}}\end{aligned} \tag{5.3.16}$$

Hence, the solution of the original equations can be constructed as

$$\boldsymbol{v}_{0s}^{(2)} = \boldsymbol{\psi}_{0s}^{(2)} + z\boldsymbol{\psi}_{0s}^{(1)} + \frac{1}{2}z^2\boldsymbol{\psi}_{0s}^{(0)} \tag{5.3.17}$$

The components of displacement and stress are

$$\left.\begin{aligned} w &= -\frac{s_{35}}{2s_{33}}x^2 + g_1 - xz \\ u &= -\frac{s_{13}}{2s_{33}}x^2 + g_2 + \frac{1}{2}z^2 \\ \sigma &= -\frac{1}{s_{33}}x \\ \tau &= 0 \end{aligned}\right\} \tag{5.3.18}$$

From the cross-sectional stress distribution, we know that there are only normal stresses but not shear stresses on the cross section. The normal stresses result in a constant force couple. Hence, Eq. (5.3.18) represents the solution of pure bending.

By examining, we know that $\boldsymbol{\psi}_{0s}^{(2)}$ is symplectic orthogonal to both $\boldsymbol{\psi}_{0f}^{(0)}$ and $\boldsymbol{\psi}_{0s}^{(0)}$ and, therefore, there exists the third-order Jordan form eigensolution. Solving the following equation with homogeneous boundary conditions (5.2.19)

$$\boldsymbol{H}\boldsymbol{\psi}_{0s}^{(3)} = \boldsymbol{\psi}_{0s}^{(2)} \tag{5.3.19}$$

yields the third-order Jordan form solution as

$$\begin{aligned}\boldsymbol{\psi}_{0s}^{(3)} &= \left\{ w_{0s}^{(3)}, \quad u_{0s}^{(3)}, \quad \sigma_{0s}^{(3)}, \quad \tau_{0s}^{(3)} \right\}^{\mathrm{T}} \\ &= \left\{ \begin{array}{l} \dfrac{(s_{33}s_{55} + s_{13}s_{33} - 2s_{35}^2)x^3 + (2s_{35}^2 - 3s_{33}s_{55})h^2 x}{6s_{33}^2} - g_2 x \\ \dfrac{(s_{15}s_{33} - 2s_{13}s_{35})x^3 + (2s_{13}s_{35} - 3s_{15}s_{33})h^2 x}{6s_{33}^2} \\ \dfrac{s_{35}}{3s_{33}^2}(h^2 - 3x^2) \\ \dfrac{1}{2s_{33}}(x^2 - h^2) \end{array} \right\}\end{aligned} \tag{5.3.20}$$

Hence, the solution of the original equations can be constructed as

$$\boldsymbol{v}_{0s}^{(3)} = \boldsymbol{\psi}_{0s}^{(3)} + z\boldsymbol{\psi}_{0s}^{(2)} + \frac{1}{2}z^2\boldsymbol{\psi}_{0s}^{(1)} + \frac{1}{6}z^3\boldsymbol{\psi}_{0s}^{(0)} \tag{5.3.21}$$

The components of displacement and stress are

$$\left.\begin{array}{l} w = w_{0s}^{(3)} + zw_{0s}^{(2)} - \dfrac{1}{2}xz^2 \\ u = u_{0s}^{(3)} + zu_{0s}^{(2)} + \dfrac{1}{6}z^3 \\ \sigma = \sigma_{0s}^{(3)} + z\sigma_{0s}^{(2)} \\ \tau = \tau_{0s}^{(3)} \end{array}\right\} \tag{5.3.22}$$

From the cross-sectional stress distribution, we know that the shear forces are constant. Hence, Eq. (5.3.22) represents the solution of constant shear force bending.

As $\boldsymbol{\psi}_{0s}^{(3)}$ and $\boldsymbol{\psi}_{0f}^{(0)}$ are symplectic orthogonal while $\boldsymbol{\psi}_{0s}^{(3)}$ and $\boldsymbol{\psi}_{0s}^{(0)}$ are mutually symplectic adjoint

$$k_2 = \langle \boldsymbol{\psi}_{0s}^{(3)}, \ \boldsymbol{\psi}_{0s}^{(0)} \rangle = -\langle \boldsymbol{\psi}_{0s}^{(2)}, \ \boldsymbol{\psi}_{0s}^{(1)} \rangle = \frac{2h^3}{3s_{33}} \neq 0 \tag{5.3.23}$$

Hence this Jordan chain is terminated. It should noted here that $k_2$ has a special physical meaning, i.e. the flexural rigidity of cross section.

We have obtained all six eigen-solutions of zero eigenvalue. By choosing appropriate constants $g_1$ and $g_2$

$$g_1 = -\frac{s_{35}h^2}{6s_{33}}; \quad g_2 = \frac{(3s_{13}s_{33} + 4s_{35}^2 - 6s_{33}s_{55})h^2}{30s_{33}^2} \tag{5.3.24}$$

we can ensure the symplectic orthogonality of $\boldsymbol{\psi}_{0f}^{(1)}$ and $\boldsymbol{\psi}_{0s}^{(2)}$, as well as $\boldsymbol{\psi}_{0s}^{(2)}$ and $\boldsymbol{\psi}_{0s}^{(3)}$. The other eigenvectors satisfy the adjoint symplectic orthonormal relation. Table 5.1 shows the adjoint symplectic orthonormal relation between the six eigen-solutions where 0 denotes the two quantities are naturally symplectic orthogonal; $g_1$ or $g_2$ denotes that symplectic orthogonality can be fulfilled by an appropriate choice of $g_1$ or $g_2$; and $*$ denotes symplectic adjoint relation.

Here we again observe the adjoint solutions of axial translation, transverse translation and rigid body rotation are, respectively, the solutions of simple tension, constant shear force bending and pure bending. These six eigen-solutions of zero eigenvalue constitute a complete symplectic subspace and they are the basic solutions of the Saint–Venant problem.

Table 5.1. The adjoint symplectic orthonormal relation between eigen-solutions of zero eigenvalue for plane anisotropic elasticity problems.

| | $\boldsymbol{\psi}_{0f}^{(0)}$ | $\boldsymbol{\psi}_{0f}^{(1)}$ | $\boldsymbol{\psi}_{0s}^{(0)}$ | $\boldsymbol{\psi}_{0s}^{(1)}$ | $\boldsymbol{\psi}_{0s}^{(2)}$ | $\boldsymbol{\psi}_{0s}^{(3)}$ |
|---|---|---|---|---|---|---|
| $\boldsymbol{\psi}_{0f}^{(0)}$ | 0 | $*$ | 0 | 0 | 0 | 0 |
| $\boldsymbol{\psi}_{0f}^{(1)}$ | | 0 | 0 | 0 | $g_1$ | 0 |
| $\boldsymbol{\psi}_{0s}^{(0)}$ | | | 0 | 0 | 0 | $*$ |
| $\boldsymbol{\psi}_{0s}^{(1)}$ | | | | 0 | $*$ | 0 |
| $\boldsymbol{\psi}_{0s}^{(2)}$ | | | | | 0 | $g_2$ |
| $\boldsymbol{\psi}_{0s}^{(3)}$ | | | | | | 0 |

## 5.4. Analytical Solutions of Saint–Venant Problems

Consider a plane strip domain ($h \ll l$) with transverse and longitudinal loads. The Saint–Venant principle is always applied to such problems where the effect of self-equilibrium system of forces at two ends is localized in the vicinity of the two ends. It implies that the eigen-solutions of nonzero eigenvalues can be neglected. Only eigen-solutions of zero eigenvalue are adopted in the expansion theorem, as

$$\boldsymbol{v} = a_1\boldsymbol{\psi}_{0f}^{(0)} + a_2\boldsymbol{\psi}_{0f}^{(1)} + a_3\boldsymbol{\psi}_{0s}^{(0)} + a_4\boldsymbol{\psi}_{0s}^{(1)} + a_5\boldsymbol{\psi}_{0s}^{(2)} + a_6\boldsymbol{\psi}_{0s}^{(3)} \tag{5.4.1}$$

or

$$\left.\begin{aligned} w &= a_1 + a_2 w_{0f}^{(1)} - a_4 x + a_5 w_{0s}^{(2)} + a_6 w_{0s}^{(3)} \\ u &= a_2 u_{0f}^{(1)} + a_3 + a_5 u_{0s}^{(2)} + a_6 u_{0s}^{(3)} \\ \sigma &= a_2 \sigma_{0f}^{(1)} + a_5 \sigma_{0s}^{(2)} + a_6 \sigma_{0s}^{(3)} \\ \tau &= a_6 \tau_{0s}^{(3)} \end{aligned}\right\} \tag{5.4.1$'$}$$

where $a_i (i = 1, 2, \ldots, 6)$ are undetermined functions. Substituting Eq. (5.4.1) into the mixed energy variational principle of Hamiltonian system (5.2.9), after simplification and rearrangement, yields

$$\begin{aligned} \delta\Bigg\{ \int_0^l \bigg( & k_1 a_2 \dot{a}_1 + k_2 a_5 \dot{a}_4 - k_2 a_6 \dot{a}_3 - \frac{1}{2} k_1 a_2^2 - \frac{1}{2} k_2 a_5^2 \\ & + k_2 a_4 a_6 - F_{\mathrm{N}} a_1 - W a_2 - F_{\mathrm{S}} a_3 - M a_4 - \theta a_5 - U a_6 \bigg) \mathrm{d}z \\ & + (k_3 a_2 a_5 + k_4 a_5 a_6)\big|_{z=0}^{l} + \tilde{\boldsymbol{v}}_{\mathrm{e}}^2 \Bigg\} = 0 \end{aligned} \tag{5.4.2}$$

where

$$k_3 = -\frac{2 s_{35} h^3}{3 s_{33}^2}; \quad k_4 = \frac{2 s_{55} h^5}{15 s_{33}^2} \tag{5.4.3}$$

$$F_{\mathrm{N}} = \int_{-h}^{h} F_z \mathrm{d}x + \overline{F}_{z2} - \overline{F}_{z1} \quad \text{(axial force)} \tag{5.4.4a}$$

$$\begin{aligned} W = & \int_{-h}^{h} [w_{0f}^{(1)}(x) F_z + u_{0f}^{(1)}(x) F_x] \mathrm{d}x \\ & + w_{0f}^{(1)}(h)(\overline{F}_{z2} + \overline{F}_{z1}) + u_{0f}^{(1)}(h)(\overline{F}_{x2} + \overline{F}_{x1}) \end{aligned} \tag{5.4.4b}$$

$$F_{\mathrm{S}} = \int_{-h}^{h} F_x \mathrm{d}x + \overline{F}_{x2} - \overline{F}_{x1} \quad \text{(shear force)} \tag{5.4.4c}$$

$$M = \int_{-h}^{h} (-x) F_z \mathrm{d}x - (\overline{F}_{z2} + \overline{F}_{z1}) h \quad \text{(bending moment)} \tag{5.4.4d}$$

$$\begin{aligned}\theta = & \int_{-h}^{h} [w_{0s}^{(2)}(x) F_z + u_{0s}^{(2)}(x) F_x] \mathrm{d}x \\ & + w_{0s}^{(2)}(h)(\overline{F}_{z2} - \overline{F}_{z1}) + u_{0s}^{(2)}(h)(\overline{F}_{x2} - \overline{F}_{x1})\end{aligned} \tag{5.4.4e}$$

$$\begin{aligned}U = & \int_{-h}^{h} [w_{0s}^{(3)}(x) F_z + u_{0s}^{(3)}(x) F_x] \mathrm{d}x \\ & + w_{0s}^{(3)}(h)(\overline{F}_{z2} + \overline{F}_{z1}) + u_{0s}^{(3)}(h)(\overline{F}_{x2} + \overline{F}_{x1})\end{aligned} \tag{5.4.4f}$$

The boundary term $\boldsymbol{v}_{\mathrm{e}}^2$ is simplified to $\tilde{\boldsymbol{v}}_{\mathrm{e}}^2$. As the boundary conditions for specified forces, $a_2, a_5, a_6$ are not arbitrary variational quantities, the related boundary terms can be neglected and we have

$$\tilde{\boldsymbol{v}}_{\mathrm{e}}^2 = -[\overline{F}_{\mathrm{N}} a_1 + \overline{F}_{\mathrm{S}} a_3 + \overline{M} a_4]_{z=0}^{l} \tag{5.4.5}$$

where $\overline{F}_{\mathrm{N}}$, $\overline{F}_{\mathrm{S}}$ and $\overline{M}$ are respectively the known axial force, shear force and bending moment at both ends ($z = 0$ or $l$).

For specified displacements, the boundary conditions are

$$\begin{aligned}\tilde{\boldsymbol{v}}_{\mathrm{e}}^2 = -[ & k_1 a_1 a_2 + k_2 a_4 a_5 + 2k_3 a_2 a_5 + 2k_4 a_5 a_6 \\ & - k_2 a_3 a_6 - \overline{W} a_2 - \overline{\theta} a_5 - \overline{U} a_6]_{z=0}^{l}\end{aligned} \tag{5.4.6}$$

where

$$\overline{W} = \int_{-h}^{h} \overline{w} \sigma_{0f}^{(1)} \mathrm{d}x \tag{5.4.7a}$$

$$\overline{\theta} = \int_{-h}^{h} \overline{w} \sigma_{0s}^{(2)} \mathrm{d}x \tag{5.4.7b}$$

$$\overline{U} = \int_{-h}^{h} (\overline{w} \sigma_{0s}^{(3)} + \overline{u} \tau_{0s}^{(3)}) \mathrm{d}x \tag{5.4.7c}$$

The variation of Eq. (5.4.2) yields the equations as

$$\left.\begin{aligned} k_1 \dot{a}_2 + F_{\mathrm{N}} = 0 \\ k_1 \dot{a}_1 - k_1 a_2 - W = 0 \end{aligned}\right\} \tag{5.4.8}$$

$$\left.\begin{aligned} k_2\dot{a}_6 - F_{\mathrm{S}} &= 0 \\ k_2\dot{a}_5 - k_2a_6 + M &= 0 \\ k_2\dot{a}_4 - k_2a_5 - \theta &= 0 \\ k_2\dot{a}_3 - k_2a_4 + U &= 0 \end{aligned}\right\} \tag{5.4.9}$$

The six equations above are differential equations in the domain for Saint–Venant problems considering body forces and boundary conditions. The general solutions can be obtained by ordinary integration. In general, equilibrium equations in the domain and boundary conditions cannot be strictly satisfied. The satisfaction only implies the equilibrium of cross section. By using a particular solution to treat in advance the inhomogeneous terms in the domain and at both side bounding surfaces ($x = \pm h$), the problem can be transformed into the corresponding homogeneous equations. In this way, the solution strictly satisfies all differential equations in the domain and boundary conditions at both side bounding surfaces ($x = \pm h$). Hence, the exact elasticity solution can be established.

The integration constants resulted from integrating Eqs. (5.4.8) and (5.4.9) can be determined by the boundary conditions at both ends ($z = 0$ or $l$). The boundary conditions for cases with specified forces at the ends ($z = 0$ or $l$) are

$$\left.\begin{aligned} k_1a_2 &= \overline{F}_{\mathrm{N}} \\ k_2a_6 &= -\overline{F}_{\mathrm{S}} \\ k_2a_5 &= \overline{M} \end{aligned}\right\} \quad \text{at } z = 0 \quad \text{or} \quad l \tag{5.4.10}$$

Obviously, these conditions can be physically interpreted as the equality of axial force, shear force and bending moment to the specified values, respectively.

The boundary conditions for cases with specified displacements at two ends ($z = 0$ or $l$) are

$$\left.\begin{aligned} k_1a_1 + k_3a_5 &= \overline{W} \\ k_2a_4 + k_3a_2 + k_4a_6 &= \overline{\theta} \\ k_2a_3 - k_4a_5 &= -\overline{U} \end{aligned}\right\} \quad \text{at } z = 0 \quad \text{or} \quad l \tag{5.4.11}$$

Obviously, these conditions can be physically interpreted as the equality of equivalent displacement and angle of rotation to the specified values, respectively.

For clamped boundary conditions, $\overline{W} = \overline{\theta} = \overline{U} = 0$ in the expression above. The usual fixed boundary conditions of Saint–Venant problems are

$$w = u = \frac{\partial u}{\partial z} = 0 \quad \text{at } x = 0, \quad z = 0 \quad \text{or} \quad l \tag{5.4.12}$$

or

$$w = u = \frac{\partial w}{\partial x} = 0 \quad \text{at } x = 0, \quad z = 0 \quad \text{or} \quad l \tag{5.4.13}$$

Substitute the general solution of Eqs. (5.4.8) and (5.4.9) into the corresponding boundary conditions yields the analytical solution of Saint–Venant problems. The solutions of two classical problems are presented as follows.

### 1. Solution of eccentric tension

Let a cantilever be fixed at $z = 0$ with an eccentric axial force $F$ acting at the other end $z = l$ with eccentricity $e$. Solving Eqs. (5.4.8) and (5.4.9) and substituting into the corresponding boundary conditions (5.4.10) yield

$$\left.\begin{array}{l} a_1 = \dfrac{Fz}{k_1} + f_1; \quad a_2 = \dfrac{F}{k_1}; \quad a_3 = -\dfrac{Fez^2}{2k_2} + f_2 z + f_3 \\ a_4 = -\dfrac{Fez}{k_2} + f_2; \quad a_5 = -\dfrac{Fe}{k_2}; \quad a_6 = 0 \end{array}\right\} \tag{5.4.14}$$

where constants $f_1, f_2, f_3$ represent translation of a rigid body. The constants can be determined by the clamped boundary conditions (5.4.11), (5.4.12) or (5.4.13). The stress field corresponding to the solution (5.4.14) is

$$\sigma_z = \frac{F}{2h} + \frac{3Fex}{2h^3}; \quad \sigma_x = \tau_{xz} = 0 \tag{5.4.15}$$

The stress distribution is the same as that of an isotropic rod. The deflection of the axis of beam is

$$u\,(0, z) = -\frac{3Fes_{33}}{4h^3} z^2 + \begin{cases} \dfrac{F[10zs_{33}s_{35} - e(4s_{35}^2 + 3s_{13}s_{33})]}{20hs_{33}} & \text{for boundary condition (5.4.11)} \\ 0 & \text{for boundary condition (5.4.12)} \\ \dfrac{Fzs_{35}}{2h} & \text{for boundary condition (5.4.13)} \end{cases} \tag{5.4.16}$$

The first part of the expression above is the solution of classical mechanics of material and it is the same as that of isotropic beam. The second part (rigid body translation) is resulted from the different assumptions of fixed end, which are high-order small quantities as compared with the first part. It shows that the different assumptions of fixed end have no effect on the stress distribution and deformation within the domain. They only cause small differences in rigid body translation.

### 2. Solution of cantilever with uniformly distributed load

Let an anisotropic cantilever with uniformly distributed load $q$ acting on one side boundary surface $x = -h$. Assume that $z = 0$ is clamped while $z = l$ is free. First of all we solve the Jordan form $\boldsymbol{H}\boldsymbol{\psi} = \boldsymbol{\psi}_{0s}^{(3)}$ with inhomogeneous boundary conditions to give a particular solution of the original problem, as

$$\left.\begin{aligned}
\tilde{\sigma}_z &= \frac{qx^3}{h^3}\left(\frac{s_{35}^2}{s_{33}^2} - \frac{2s_{13}+s_{55}}{4s_{33}}\right) + \frac{qx}{h}\left[\frac{9\left(2s_{13}+s_{55}\right)}{20s_{33}} - \frac{4s_{35}^2}{5s_{33}^2}\right] \\
&\quad - \frac{qs_{13}}{2s_{33}} - \frac{qs_{35}}{2h^3 s_{33}} z\left(h^2 - 3x^2\right) + \frac{3q}{4h^3} xz^2 \\
\tilde{\sigma}_x &= \frac{q}{4}\left(2 - 3\frac{x}{h} + \frac{x^3}{h^3}\right) \\
\tilde{\tau}_{xz} &= \frac{qs_{35}}{2h^3 s_{33}} x\left(h^2 - x^2\right) + \frac{3q}{4h^3} z\left(h^2 - x^2\right)
\end{aligned}\right\} \tag{5.4.17}$$

We further transform the original equation into a homogeneous one, and then solve the corresponding homogeneous equation. Finally the stress field in the domain is

$$\left.\begin{aligned}
\sigma_z &= \frac{3q}{4h^3} x(l-z)^2 + \frac{qs_{35}}{2h^3 s_{33}}(l-z)(h^2 - 3x^2) \\
&\quad - \left[q\left(\frac{s_{35}^2}{s_{33}^2} - \frac{2s_{13}+s_{55}}{4s_{33}}\right)\left(\frac{3x}{5h} - \frac{x^3}{h^3}\right)\right] \\
\sigma_x &= \left[\frac{q}{4}\left(2 - 3\frac{x}{h} + \frac{x^3}{h^3}\right)\right] \\
\tau_{xz} &= \frac{3q}{4h^3}(z-l)(h^2 - x^2) + \left[\frac{qs_{35}}{2h^3 s_{33}} x(h^2 - x^2)\right]
\end{aligned}\right\} \tag{5.4.18}$$

If the distribution of the external forces at the free end (form a equilibrium system of forces) is the same as Eq. (5.4.18), then Eq. (5.4.18) is the exact elasticity solution of the problem.

If we do not treat the inhomogeneous terms by applying Eq. (5.4.17) in advance but rather solve Eqs. (5.4.8) and (5.4.9) directly, we obtain an approximate solution, i.e. omit all terms in square brackets in Eq. (5.4.18). Obviously, for problems in plane strip domain ($h \ll l$), the terms in square brackets are higher-order small quantities. It clearly shows that the rational method by expanding eigenvectors of zero eigenvalue for solving the Saint–Venant problems is very effective and practical.

As only the eigen-solutions of zero eigenvalue are applied, the boundary conditions at both ends ($z = 0$ or $l$) cannot be satisfied strictly in general. We have to introduce the relaxed boundary conditions (5.4.10) or (5.4.11) where the effect is localized in the vicinity in accordance with the Saint–Venant principle. To strictly satisfy the boundary conditions we have to include the eigen-solutions of nonzero eigenvalues. Moreover, for complex problems such as a general rectangular domain or a short beam, the Saint–Venant principle is no longer applicable because the transverse dimension $h$ is not a higher-order small quantity comparing to the longitudinal dimension $l$. Hence, we need to apply the eigen-solutions of nonzero eigenvalues in the expansion theorem in order to solve the problems.

## 5.5. Eigen-Solutions of Nonzero Eigenvalues

The eigenvalue equation for eigen-solutions of nonzero eigenvalues is Eq. (5.2.22). First, we should solve the eigenvalues $\tilde{\lambda}$, which satisfies the following equation

$$\det\begin{bmatrix} -\mu & \dfrac{s_{13}}{s_{11}}\tilde{\lambda} & \dfrac{b_{55}}{s_{11}d} & -\dfrac{b_{35}}{s_{11}d} \\ -\tilde{\lambda} & \dfrac{s_{15}}{s_{11}}\tilde{\lambda}-\mu & -\dfrac{b_{35}}{s_{11}d} & \dfrac{b_{33}}{s_{11}d} \\ 0 & 0 & -\mu & -\tilde{\lambda} \\ 0 & -\dfrac{1}{s_{11}}\tilde{\lambda}^2 & \dfrac{s_{13}}{s_{11}}\tilde{\lambda} & \dfrac{s_{15}}{s_{11}}\tilde{\lambda}-\mu \end{bmatrix} = 0 \tag{5.5.1}$$

Expanding the determinant yields the eigenvalue equation

$$s_{33}\tilde{\lambda}^4 - 2s_{35}\tilde{\lambda}^3\mu + (s_{55}+2s_{13})\tilde{\lambda}^2\mu^2 - 2s_{15}\tilde{\lambda}\mu^3 + s_{11}\mu^4 = 0 \tag{5.5.2}$$

which has four roots

$$\tilde{\lambda}_i = \lambda_i\mu \quad (i = 1, 2, 3, 4) \tag{5.5.3}$$

For an ideal elastic body, we can prove that for $\lambda_i$ there are only complex roots or pure imaginary roots but no real roots$^2$. There are no repeated roots in general, i.e. there are two different pairs of complex conjugate roots. We only discuss the general case in this section while the other cases can be discussed in a similar way. If there are four different roots $\lambda_i$, the general solution of Eq. (5.2.22) can be expressed as

$$\left.\begin{aligned} w &= \sum_{i=1}^{4} A_i \exp(\lambda_i \mu x) \\ u &= \sum_{i=1}^{4} B_i \exp(\lambda_i \mu x) \\ \sigma &= \sum_{i=1}^{4} C_i \exp(\lambda_i \mu x) \\ \tau &= \sum_{i=1}^{4} D_i \exp(\lambda_i \mu x) \end{aligned}\right\} \tag{5.5.4}$$

where constants $A_i, B_i, C_i, D_i$ are not independent. Substituting Eq. (5.5.4) into Eq. (5.2.22) and choosing $D_i\,(i = 1, 2, 3, 4)$ as independent constants yield

$$\left.\begin{aligned} A_i &= \frac{s_{35}\lambda_i - s_{33}\lambda_i^2 - s_{13}}{\lambda_i \mu} D_i \\ B_i &= \frac{s_{15}\lambda_i - s_{13}\lambda_i^2 - s_{11}}{\lambda_i^2 \mu} D_i \\ C_i &= -\lambda_i D_i \end{aligned}\right\} \quad (i = 1, 2, 3, 4) \tag{5.5.5}$$

Further substituting Eq. (5.5.5) into Eq. (5.5.4) yields

$$\left.\begin{aligned} w &= \sum_{i=1}^{4} \left[ \frac{s_{35}\lambda_i - s_{33}\lambda_i^2 - s_{13}}{\lambda_i \mu} D_i \exp(\lambda_i \mu x) \right] \\ u &= \sum_{i=1}^{4} \left[ \frac{s_{15}\lambda_i - s_{13}\lambda_i^2 - s_{11}}{\lambda_i^2 \mu} D_i \exp(\lambda_i \mu x) \right] \\ \sigma &= \sum_{i=1}^{4} [-\lambda_i D_i \exp(\lambda_i \mu x)] \\ \tau &= \sum_{i=1}^{4} [D_i \exp(\lambda_i \mu x)] \end{aligned}\right\} \tag{5.5.6}$$

From Eq. (5.2.8), we obtain

$$\sigma_x = \sum_{i=1}^{4} \left[ -\frac{1}{\lambda_i} D_i \exp(\lambda_i \mu x) \right] \tag{5.5.7}$$

Substituting Eqs. (5.5.6) and (5.5.7) into the homogeneous boundary conditions (5.2.19) of side bounding surfaces yields

$$\left.\begin{aligned} \sum_{i=1}^{4} [D_i \exp(-\lambda_i \mu h)] = 0 \\ \sum_{i=1}^{4} [-D_i \exp(-\lambda_i \mu h)]/\lambda_i = 0 \\ \sum_{i=1}^{4} [D_i \exp(\lambda_i \mu h)] = 0 \\ \sum_{i=1}^{4} [-D_i \exp(\lambda_i \mu h)]/\lambda_i = 0 \end{aligned}\right\} \tag{5.5.8}$$

For nontrivial solution, the determinant of this coefficient matrix vanishes. Denote $\beta = 2\mu h$, by rearranging and simplifying, we obtain

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \exp(\lambda_1\beta) & \exp(\lambda_2\beta) & \exp(\lambda_3\beta) & \exp(\lambda_4\beta) \\ \lambda_1 \exp(\lambda_1\beta) & \lambda_2 \exp(\lambda_2\beta) & \lambda_3 \exp(\lambda_3\beta) & \lambda_4 \exp(\lambda_4\beta) \end{vmatrix} = 0 \tag{5.5.9}$$

The equation above is the transcendental equation for solving nonzero eigenvalues in which numerical methods are required. Substituting the eigenvalues into Eq. (5.5.8) yields the trivial solution of $D_i$ and, hence, the corresponding eigenvector functions are obtained.

All eigen-solutions of nonzero eigenvalues are covered in the Saint–Venant principle. Together with the eigen-solutions of zero eigenvalue, they constitute a complete adjoint symplectic orthonormal basis and the expansion theorem is then applicable. This is very important for the solution method. The solution method for solving isotropic elasticity problem can be applied to solve general anisotropic elasticity problems. Here we observe that there is no essential difference between the solution method of anisotropic problems and that of isotropic problems except derivation of the former is somewhat more complicated. This approach differs very much from the classical semi-inverse method because it is more rational. Therefore, the Hamiltonian system and symplectic mathematical methods have tremendous potential applications.

## 5.6. Introduction to Hamiltonian System for Generalized Plane Problems

Consider a homogeneous anisotropic infinite cylindrical solid with rectangular cross section. Let $y$-axis be the axial direction of this column and the external load be independent of $y$. This is a plane strain problem if $y$ is the elastic principal axis. For general anisotropic materials, however, the displacement $v$ along the $y$-direction will warp and, therefore, the problem is a **generalized plane strain problem**$^3$.

As the geometric properties, material properties and external loads are all independent of $y$-coordinate, all components of stress, strain and displacement are only functions of $x, z$. Displacement $v$ has a term $\varepsilon_{y0}$ which is linearly dependent on $y$ but independent of $x, z$. The value of $\varepsilon_{y0}$ can be determined from zero axial force on the cross section. For brevity, this term is not taken into consideration here.

For generalized plane strain problems, the strain-displacement relations (2.2.2) can be simplified as

$$\left.\begin{array}{c}\varepsilon_x = \dfrac{\partial u}{\partial x}; \quad \varepsilon_y = 0; \quad \varepsilon_z = \dfrac{\partial w}{\partial z} \\ \gamma_{xy} = \dfrac{\partial v}{\partial x}; \quad \gamma_{xz} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}; \quad \gamma_{yz} = \dfrac{\partial v}{\partial z}\end{array}\right\} \tag{5.6.1}$$

These strain components are functions of $x, z$. The stress-strain relation is Eq. (2.3.1). Hence the deformation energy along the $y$-direction is

$$\begin{aligned} V_\varepsilon = \int_0^l \int_0^h \Big[&\frac{1}{2}c_{11}\varepsilon_x^2 + c_{13}\varepsilon_x\varepsilon_z + c_{14}\varepsilon_x\gamma_{xy} + c_{15}\varepsilon_x\gamma_{xz} + c_{16}\varepsilon_x\gamma_{yz} + \frac{1}{2}c_{33}\varepsilon_z^2 \\ &+ c_{34}\varepsilon_z\gamma_{xy} + c_{35}\varepsilon_z\gamma_{xz} + c_{36}\varepsilon_z\gamma_{yz} + \frac{1}{2}c_{44}\gamma_{xy}^2 + c_{45}\gamma_{xy}\gamma_{xz} + c_{46}\gamma_{xy}\gamma_{yz} \\ &+ \frac{1}{2}c_{55}\gamma_{xz}^2 + c_{56}\gamma_{xz}\gamma_{yz} + \frac{1}{2}c_{66}\gamma_{yz}^2\Big]\mathrm{d}x\mathrm{d}z = \int_0^l \int_0^h \mathscr{L}\,\mathrm{d}x\mathrm{d}z \end{aligned} \tag{5.6.2}$$

where strain components (5.6.1) have been substituted. Now the $z$-coordinate is treated as the time coordinate and an overdot denotes differentiation with respect to $z$, i.e. $(\dot{})=\mathrm{d}/\mathrm{d}x$. First we introduce the dual variables

$$\frac{\partial \mathscr{L}}{\partial \dot{u}} = c_{15}\frac{\partial u}{\partial x} + c_{35}\dot{w} + c_{45}\frac{\partial v}{\partial x} + c_{55}\dot{u} + c_{55}\frac{\partial w}{\partial x} + c_{56}\dot{v} = \tau_{xz} \tag{5.6.3a}$$

$$\frac{\partial \mathscr{L}}{\partial \dot{v}} = c_{16}\frac{\partial u}{\partial x} + c_{36}\dot{w} + c_{46}\frac{\partial v}{\partial x} + c_{56}\dot{u} + c_{56}\frac{\partial w}{\partial x} + c_{66}\dot{v} = \tau_{yz} \tag{5.6.3b}$$

$$\frac{\partial \mathscr{L}}{\partial \dot{w}} = c_{13}\frac{\partial u}{\partial x} + c_{33}\dot{w} + c_{34}\frac{\partial v}{\partial x} + c_{35}\dot{u} + c_{35}\frac{\partial w}{\partial x} + c_{36}\dot{v} = \sigma_{z} \tag{5.6.3c}$$

The primal variable $\boldsymbol{q}$ and the dual variable $\boldsymbol{p}$ are

$$\boldsymbol{q} = \{u, \quad v, \quad w\}^{\mathrm{T}}, \quad \boldsymbol{p} = \{\tau_{xz}, \quad \tau_{yz}, \quad \sigma_z\}^{\mathrm{T}} \tag{5.6.4}$$

respectively. Denote

$$\boldsymbol{C}_d = \begin{bmatrix} c_{55} & c_{56} & c_{35} \\ c_{56} & c_{66} & c_{36} \\ c_{35} & c_{36} & c_{33} \end{bmatrix}, \quad \boldsymbol{C}_e = \begin{bmatrix} c_{11} & c_{14} & c_{15} \\ c_{14} & c_{44} & c_{45} \\ c_{15} & c_{45} & c_{55} \end{bmatrix} \tag{5.6.5}$$

and

$$\boldsymbol{C}_t = \begin{bmatrix} c_{15} & c_{45} & c_{55} \\ c_{16} & c_{46} & c_{56} \\ c_{13} & c_{34} & c_{35} \end{bmatrix} \tag{5.6.6}$$

then the Lagrange function $\mathscr{L}$ can be expressed as

$$\mathscr{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{C}_d\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{C}_t\left(\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}x}\right) + \left(\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}x}\right)^{\mathrm{T}}\boldsymbol{C}_e\left(\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}x}\right) \tag{5.6.7}$$

Next, from Eq. (5.6.3), we obtain

$$\dot{\boldsymbol{q}} = -\boldsymbol{C}_d^{-1}\boldsymbol{C}_t\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}x} + \boldsymbol{C}_d^{-1}\boldsymbol{p} \tag{5.6.8}$$

According to Legendre's transformation, the Hamiltonian function is

$$\begin{aligned}\mathscr{H}(\boldsymbol{q}, \boldsymbol{p}) &= \boldsymbol{p}^{\mathrm{T}}\dot{\boldsymbol{q}} - \mathscr{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \\ &= \frac{1}{2}\boldsymbol{p}^{\mathrm{T}}\boldsymbol{C}_d^{-1}\boldsymbol{p} - \boldsymbol{p}^{\mathrm{T}}\boldsymbol{C}_d^{-1}\boldsymbol{C}_t\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}x} - \left(\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}x}\right)^{\mathrm{T}}\boldsymbol{B}_c\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}x}\end{aligned} \tag{5.6.9}$$

where

$$\boldsymbol{B}_c = \boldsymbol{C}_e - \boldsymbol{C}_t^{\mathrm{T}} \boldsymbol{C}_d^{-1} \boldsymbol{C}_t \tag{5.6.10}$$

It is noted that matrices $\boldsymbol{C}_e$ and $\boldsymbol{C}_d$ are diagonal principal submatrices of the matrix of three-dimensional elastic constants. Both of them are symmetric and positive definite matrices and their inverse matrices exist. Furthermore, $\boldsymbol{B}_c$ is a symmetric matrix.

Having derived the Hamiltonian density function, we can express the Hamiltonian variational principle (or mixed energy variational principle) as

$$\begin{aligned} \delta \Bigg\{ \int_0^l \int_0^h [\boldsymbol{p}^{\mathrm{T}} \dot{\boldsymbol{q}} - \mathscr{H}(\boldsymbol{q}, \boldsymbol{p}) - \boldsymbol{X}^{\mathrm{T}} \boldsymbol{q}] \mathrm{d}x \mathrm{d}z \\ - \int_0^l [(\overline{\boldsymbol{X}}_2^{\mathrm{T}} \boldsymbol{q})_{x=h} - (\overline{\boldsymbol{X}}_1^{\mathrm{T}} \boldsymbol{q})_{x=-h}] \mathrm{d}z \Bigg\} = 0 \end{aligned} \tag{5.6.11}$$

where $\boldsymbol{q}, \boldsymbol{p}$ are independent variable vector functions which has mutually independently variationals. The body force is $\boldsymbol{X}$ and the surface tractions are $\overline{\boldsymbol{X}}_1, \overline{\boldsymbol{X}}_2$. These external forces are in self-equilibrium, independent of $y$-coordinate and without components along the $y$-axis.

Performing variation on Eq. (5.6.11) and integrating by parts yield the Hamiltonian dual system of equations as

$$\begin{Bmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{D} \\ \boldsymbol{B} & -\boldsymbol{A}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{Bmatrix} - \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{X} \end{Bmatrix} \tag{5.6.12}$$

where

$$\left. \begin{aligned} &\boldsymbol{A} = -\boldsymbol{C}_d^{-1} \boldsymbol{C}_t \frac{\partial}{\partial x}, \quad &&\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{C}_t^{\mathrm{T}} \boldsymbol{C}_d^{-1} \frac{\partial}{\partial x} \\ &\boldsymbol{B} = -\boldsymbol{B}_c \frac{\partial^2}{\partial x^2}, \quad &&\boldsymbol{D} = \boldsymbol{C}_d^{-1} \end{aligned} \right\} \tag{5.6.13}$$

The boundary conditions are

$$\boldsymbol{C}_t^{\mathrm{T}} \boldsymbol{C}_d^{-1} \boldsymbol{p} + \boldsymbol{B}_c \frac{\partial \boldsymbol{q}}{\partial x} = \overline{\boldsymbol{X}}_2 \quad \text{at } x = h \tag{5.6.14a}$$

$$\boldsymbol{C}_t^{\mathrm{T}} \boldsymbol{C}_d^{-1} \boldsymbol{p} + \boldsymbol{B}_c \frac{\partial \boldsymbol{q}}{\partial x} = \overline{\boldsymbol{X}}_1 \quad \text{at } x = -h \tag{5.6.14b}$$

Introducing the full state vector $\boldsymbol{v}$ and operator matrix $\boldsymbol{H}$, we have

$$\boldsymbol{v} = \begin{Bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{Bmatrix}, \quad \boldsymbol{H} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{D} \\ \boldsymbol{B} & -\boldsymbol{A}^{\mathrm{T}} \end{bmatrix} \tag{5.6.15}$$

The Hamiltonian dual equation (5.6.12) can be abbreviated to

$$\dot{\boldsymbol{v}} = \boldsymbol{H}\boldsymbol{v} + \boldsymbol{h}; \quad \boldsymbol{h}^{\mathrm{T}} = \{\mathbf{0}^{\mathrm{T}}, -\boldsymbol{X}^{\mathrm{T}}\} \tag{5.6.16}$$

The operator matrix $\boldsymbol{H}$ is independent of external load. Hence we investigate the homogeneous linear differential equation

$$\dot{\boldsymbol{v}} = \boldsymbol{H}\boldsymbol{v} \tag{5.6.17}$$

and the homogeneous boundary conditions on the side boundary surfaces

$$\boldsymbol{C}_t^{\mathrm{T}}\boldsymbol{C}_d^{-1}\boldsymbol{p} + \boldsymbol{B}_c\frac{\partial \boldsymbol{q}}{\partial x} = \mathbf{0} \quad \text{at } x = \pm h \tag{5.6.18}$$

Using an approach similar to the preceding few chapters, we can prove that the operator matrix $\boldsymbol{H}$ is a Hamiltonian operator matrix of symplectic geometric space.

At this point, we have derived the Hamiltonian system from the generalized plane problems. The system can be solved in a general way as discussed although the derivation is more complicated.

For example, the method of separation of variables can be applied to solve the homogeneous equation (5.6.17). Let

$$\boldsymbol{v}(z,x) = \xi(z)\boldsymbol{\psi}(x) \tag{5.6.19}$$

and substituting into Eq. (5.6.17) yield

$$\xi(z) = \mathrm{e}^{\mu z} \tag{5.6.20}$$

and the eigenvalue equation

$$\boldsymbol{H}\boldsymbol{\psi}(x) = \mu\boldsymbol{\psi}(x) \tag{5.6.21}$$

where $\mu$ is the undetermined eigenvalue and $\boldsymbol{\psi}(x)$ is the eigenvector which has to satisfy the homogeneous boundary conditions (5.6.18) on the side boundary surfaces ($x = \pm h$).

It has been mentioned repeatedly that the eigenvalue problem of Hamiltonian operator matrix has certain characteristics. The eigenvectors are adjoint symplectic orthonormal. The solution can be obtained via eigenvector expansion theorem.

A zero eigenvalue with Jordan form eigenvectors exists for a problem with free boundary conditions (5.6.18) on the side boundary surfaces ($x = \pm h$). For the present problem, there are eight eigenvectors of zero eigenvalue. These eigenvectors form three chains and they are adjoint symplectic orthonormal. Hence, they form a symplectic subspace. The details are omitted. Interested readers are referred to the related chapters in monograph[4].

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