Chapter 1

Spin Waves and Equations of Ferromagnetic Spin Chain

1.1 Physics Background for the Equations of Ferromagnetic Spin Chain

1.1.1 Motion Equations for Magnetization

Studying the dispersive theory of magnetization of ferromagnets, Landau–Lifshitz proposed the following motion equation of magnetization

$$\vec{S}_t = \lambda_1 \vec{S} \times \vec{H}^e - \lambda_2 \vec{S} \times (\vec{S} \times \vec{H}^e), \tag{1.1.1}$$

where $\vec{S} = (S_1, S_2, S_3)$ is the vector of magnetization, $\vec{H}^e$ is the effective magnetic field applied to magnetic moment,

$$\vec{H}^e = \frac{\partial}{\partial \vec{S}} e_{\text{mag}}(\vec{S}). \tag{1.1.2}$$

Here $e_{\text{mag}}(\vec{S})$ denotes the density of the total magnetic energy, $\vec{H}^e$ is also related to Maxwell equations, λ_1, λ_2 are constants, $\lambda_2 > 0$.

If a bounded multiply connected domain $\Omega \subset R^3$ is occupied by a ferromagnet under the constant temperature below Curie temperature, and if mechanics effects are not taking into account, one can consider the following magnetic energy functional:

1. *Anisotropic energy*

Anisotropic energy reads as

$$\mathcal{E}_{\text{an}} = \int_\omega \Phi(\vec{S}) dx, \tag{1.1.3}$$

in which $\Phi : R^3 \to R^+$ is a convex function and depends on the crystal structure of the materials.

Near the Curie temperature, taking first order approximation, one has

$$\Phi(\vec{S}) := \sum_{lm} b_{lm} S_l S_m, \tag{1.1.4}$$

where $\{b_{lm}\}$ is a symmetric, positively definite tensor. $\Phi(\vec{S}) = k|\vec{S}|^2(\sin\theta)^2$ for example for the uniaxial crystal, where θ is the angle between $\vec{S}$ and the direction of "easy" magnetization, k is a positive constant.

2. *Exchange energy*

The behavior of ferromagnets is that the quantum force makes the magnetic field of molecules arrange in order. the most important quantity is the exchange energy

$$\mathcal{E}_{\text{ex}} := \frac{1}{2}\sum_{l,m} a_{lm}\int_{\Omega}\frac{\partial\vec{S}}{\partial x_l}\frac{\partial\vec{S}}{\partial x_m}dx, \tag{1.1.5}$$

in which $\{a_{lm}\}$ is a symmetric, positive definite tensor.

3. *The energy to the magnetic field $\vec{H}$*

The energy of magnetic field $\vec{H}$ is:

$$\mathcal{E}_{\text{H}}(\vec{S}) := \frac{1}{8\pi}\int_{R^3} H^2 dx, \tag{1.1.6}$$

where $\vec{H}$ and $\vec{S}$ being given by Maxwell's equations. Note that here the integral is extended to the whole space R^3 since $\vec{H}$ does not vanish outside the domain Ω.

The total magnetic energy has the form

$$\mathcal{E}_{\text{mag}}(\vec{S}) := \mathcal{E}_{\text{an}}(\vec{S}) + \mathcal{E}_{\text{ex}}(\vec{S}) + \mathcal{E}_{\text{H}}(\vec{S}) \tag{1.1.7}$$

and at equilibrium state $\mathcal{E}_{\text{mag}}$ attains an absolute minimum. The magnetostatic Maxwell equation is:

$$\nabla\cdot(\vec{H} + 4\pi\vec{S}) = 0, \quad \text{in} \ \ R^3, \tag{1.1.8}$$

$$\nabla\times\vec{H} = 0, \quad \text{in} \ \ R^3, \tag{1.1.9}$$

where $\nabla\cdot$ = div, $\nabla\times$ = curl, "$\cdot$" denotes the inner product and "$\times$" denotes the vector product. $\vec{S}$ satisfies the non-convex condition:

$$|\vec{S}(x)| = S_0, \quad \text{in} \ \ \Omega. \tag{1.1.10}$$

1.1.2 Landau–Lifshitz Equations

One model of dynamical system is the Landau–Lifshitz equation:

$$\frac{\partial\vec{S}}{\partial t} = \lambda_1\vec{S}\times\vec{H}^e - \lambda_2\vec{S}\times(\vec{S}\times\vec{H}^e), \qquad \text{in} \ \ \Omega\times(0,T), \tag{1.1.11}$$

$$\vec{H}^e := -\frac{\partial\Phi(\vec{S})}{\partial\vec{S}} + \sum_{l,m} a_{lm}\frac{\partial^2\vec{S}}{\partial x_l\partial x_m} + \vec{H}, \quad \text{in} \ \ \Omega\times(0,T), \tag{1.1.12}$$

in which λ_1, λ_2 are constants in physics, $\lambda_2 > 0$. The first term on the right-hand side of (1.1.11) which is not dissipative but resulted from the motion of $\vec{S}$ around $\vec{H}^e$, has a constant angle; the second term expresses the ordered arrangement of $\vec{S}$ according to $\vec{H}^e$ and it is due to the viscosity and then dissipative.

The initial condition is as follows

$$\vec{S}(x,0) = \vec{S}_0(x), \quad x \in \Omega. \tag{1.1.13}$$

It follows from (1.1.11) that

$$\frac{\partial}{\partial t}|\vec{S}|^2 = 2\vec{S}\frac{\partial \vec{S}}{\partial t} = 0. \tag{1.1.14}$$

Then, if $\vec{S}_0(x)$ satisfies (1.1.10), one has $|\vec{S}(x,t)| = S_0$, $x \in \Omega$, $t \geq 0$; if $\vec{S} \times \vec{H}^e \neq 0$, $\vec{S} \times \vec{H}^e$ and $\vec{S} \times (\vec{S} \times \vec{H}^e)$ is an orthogonal base which are on the tangential plane of the sphere $|\vec{S}| = S_0$. Hence, (1.1.11) is a dissipative nonlinear evolutionary equation of $\vec{S}$ on the surface of sphere.

The fields $\vec{H}$ and $\vec{S}$ solve the Maxwell equation

$$\nabla \times \vec{H} = \frac{\varepsilon}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}, \qquad \text{in } R^3 \times [0,T], \tag{1.1.15}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\mu_0 \vec{H} + 4\pi \vec{S}), \quad \text{in } R^3 \times [0,T]. \tag{1.1.16}$$

It follows from Ohm law that:

$$\vec{J} = \sigma(\vec{E} + \vec{f}\,), \tag{1.1.17}$$

where $\vec{E}$ represents the electric field, $\vec{J}$ the density of current, σ the conductivity, ε magnetization rate of the electric medium, c the speed of light and $\vec{f}$ is the given non-induction electric force. Usually there holds: $\vec{B} := \mu_0 \vec{H} + 4\pi \vec{S}$.

The initial data are

$$\vec{E}(x,0) = \vec{E}_0(x), \tag{1.1.18}$$

$$\vec{B}(x,0) = \vec{B}_0(x), \quad \nabla \cdot \vec{B}_0 = 0. \tag{1.1.19}$$

1.2 A Simple Derivation of Landau–Lifshitz Equation

1.2.1 Magnetically Ordered Crystals

Many crystals have an ordered magnetic structure. This means that in the absence of an external magnetic field, the mean magnetic moment of at least one of the atoms in each unit of cell of the crystal is non-zero. In the simplest type of magnetically ordered crystals, i.e. ferromagnets such as Fe, Ni, Co and Dy, the mean magnetic moments of all the atoms have the same orientation provided that the temperature of the ferromagnet does not exceed a critical value, i.e. the Curie temperature. For this reason ferromagnets have spontaneous magnetic moments, i.e. non-zero macroscopic magnetic moments, even in the absence of an external magnetic field.

In antiferromagnets, these include carbonates, anhydrous sulphates, oxides and fluorides of transition metals Mn, Ni, Co and Fe, the mean atomic magnetic moments compensate each other within each unit cell (in zero external magnetic field). In other words, an antiferromagnet consists of a set of sublattices (called magnetic sublattices), each of which has a non-zero mean moment. This type of magnetic order occurs if the temperature of the antiferromagnets is less than a critical temperature, known as the Neel temperature.

Finally, there is one further type of magnetically ordered crystal — that of the ferrites — which consists of a number of magnetic sublattices whose magnetic moments are uncompensated (in contrast to antiferromagnets); thus ferrites exhibit spontaneous magnetic moments. Examples of this type are compounds of transition metals such as the salts $MnO.Fe_2O_3, 3Y_2O_3.5Fe_2O_3$.

1.2.2 The Wave Function and Spin Operator for the System of Two Electrons

Let us consider a simple molecule model. Assume the molecule of Argon has two electrons and two protons between which there is no interaction with each other since the mass of the protons is much larger. The interaction of this system is as in Figure 1.2.1 in which a, b denote protons, 1, 2 represent electrons provided that there is Coulomb force between them.

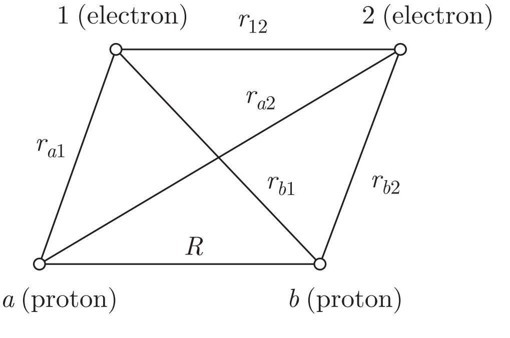


Figure 1.2.1. *Interaction of oxygen molecular system.*

1. *Wave function of electrons*

Consider two-body problem:

$$\left(-\frac{\hbar^2}{2m}\nabla_1^2 - \frac{e^2}{r_{a_1}}\right)\varphi(r_{a_1}) = E_0\varphi(r_{a_1}), \tag{1.2.1}$$

$$\left(-\frac{\hbar^2}{2m}\nabla_2^2 - \frac{e^2}{r_{b_2}}\right)\varphi(r_{b_2}) = E_0\varphi(r_{b_2}), \tag{1.2.2}$$

where $\varphi(r_{a_1})$ is the wave function of electron "1", $\varphi(r_{b_2})$ is the wave function of electron "2", $\hbar$ is the Planck constant, m is the mass of the electron and e is the charge. The Hamiltonian for this system is $\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_1$ where:

$$\hat{\mathcal{H}}_0 = -\frac{\hbar^2}{2m}(\nabla_1^2 + \nabla_2^2) - \frac{e^2}{r_{a_1}} - \frac{e^2}{r_{b_1}}, \tag{1.2.3}$$

$$\hat{\mathcal{H}}_1 = \frac{e^2}{R} + \frac{e^2}{r} - \frac{e^2}{r_{a_2}} - \frac{e^2}{r_{b_2}}, \tag{1.2.4}$$

$\hat{\mathcal{H}}_1$ is a small perturbation. To find the eigenvalue E such that $\hat{\mathcal{H}}\Psi = E\Psi$, we take single electron approximation:

$$\varphi_1 = \varphi(r_{a_1})\varphi(r_{b_2}), \quad \varphi_2 = \varphi(r_{a_2})\varphi(r_{b_1}), \tag{1.2.5}$$

which is the wave function of the system. Let

$$\psi = C_1\varphi_1 + C_2\varphi_2, \tag{1.2.6}$$

and assume that φ_1, φ_2 are normalized. It follows from the identical principle that electrons "1" and "2" are identical or are invariant after two exchanges. Thus we have

$$\begin{aligned} \int \varphi_1^* \hat{\mathcal{H}}\psi dx &= \int E\varphi_1^*\psi dx \\ &= E\int \varphi_1^*(C_1\varphi_1 + C_2\varphi_2)dx \\ &= C_1E_0 + C_2E_0\nu^2 \,, \end{aligned} \tag{1.2.7}$$

where

$$\nu^2 = \int \varphi^*(r_{b_2})\varphi^*(r_{a_1})\varphi(r_{a_2})\varphi(r_{b_1})dr_1dr_2 \,.$$

The left-hand side of (1.2.7) is

$$\int \varphi_1^*\hat{\mathcal{H}}\psi dx = C_1\int \varphi_1^*\hat{\mathcal{H}}\varphi_1 dx + C_2\int \varphi_1^*\hat{\mathcal{H}}\varphi_2 dx = C_1\alpha_{11} + C_2\alpha_{12}, \tag{1.2.8}$$

with

$$\begin{aligned} \alpha_{11} &= \int \varphi^*(r_{b_2})\varphi^*(r_{a_1})\hat{\mathcal{H}}_0\varphi(r_{a_1})\varphi(r_{b_2})dr_1dr_2 \\ &\quad + \int \varphi^*(r_{b_2})\varphi^*(r_{a_1})\frac{e^2}{R}\varphi(r_{a_1})\varphi(r_{b_2})dr_1dr_2 \\ &\quad + \int \varphi^*(r_{b_2})\varphi^*(r_{a_1})\left(\frac{e^2}{r} - \frac{e^2}{r_{a_2}} - \frac{e^2}{r_{b_1}}\right)\varphi(r_{a_1})\varphi(r_{b_2})dr_1dr_2 \\ &= 2E_0 + \frac{e^2}{R} + A(r), \end{aligned} \tag{1.2.9}$$

here

$$A(r) = \int \varphi^*(r_{b_2})\varphi^*(r_{a_1})\left(\frac{e^2}{r} - \frac{e^2}{r_{a_2}} - \frac{e^2}{r_{b_1}}\right)\varphi(r_{a_1})\varphi(r_{b_2})dr_1dr_2, \tag{1.2.10}$$

and

$$\begin{aligned}\alpha_{12} &= \int \varphi^*(r_{b_2})\varphi^*(r_{a_1})\hat{\mathcal{H}}_0\varphi(r_{a_2})\varphi(r_{b_1})dr_1dr_2\\ &\quad+\frac{e^2}{R}\int \varphi^*(r_{b_2})\varphi^*(r_{a_2})\varphi(r_{a_2})\varphi(r_{b_1})dr_1dr_2\\ &\quad+\int \varphi^*(r_{b_2})\varphi^*(r_{a_1})\left(\frac{e^2}{r}-\frac{e^2}{r_{a_2}}-\frac{e^2}{r_{b_1}}\right)\varphi(r_{b_1})\varphi(r_{b_2})dr_1dr_2\\ &=\left(2E_0+\frac{e^2}{R}\right)\nu^2+B(r), \end{aligned}\tag{1.2.11}$$

here

$$B(r)=\int \varphi^*(r_{b_2})\varphi^*(r_{a_1})\left(\frac{e^2}{r}-\frac{e^2}{r_{a_2}}-\frac{e^2}{r_{b_1}}\right)\varphi(r_{b_1})\varphi(r_{b_2})dr_1dr_2. \tag{1.2.12}$$

Hence we have from (1.2.7) and (1.2.8) that

$$C_1\alpha_{11}+C_2\alpha_{12}=C_1E_0+C_2E_0\nu^2. \tag{1.2.13}$$

Similarly it follows from $\varphi_2^*\hat{\mathcal{H}}\psi=E_0\varphi_2^*\psi$ that

$$C_1\alpha_{21}+C_2\alpha_{22}=C_1E_0\nu^2+C_2E_0, \tag{1.2.14}$$

where

$$\begin{cases}\alpha_{11}=(2E_0+\frac{e^2}{R})\nu^2+B(r),\\ \alpha_{22}=2E_0+\frac{e^2}{R}+A(r).\end{cases} \tag{1.2.15}$$

We have from (1.2.13) and (1.2.14) that

$$\begin{aligned}&C_1(E_0-\alpha_{11})+C_2(E_0\nu^2-\alpha_{12})=0,\\ &C_1\left\{\left[E_0-\left(2E_0+\frac{e^2}{R}\right)\right]\nu^2+B(r)\right\}+C_2\left\{\left[E_0-\left(2E_0+\frac{e^2}{R}\right)\right]+A(r)\right\}=0.\end{aligned}$$

If $C_1,\, C_2 \not\equiv 0$, one has

$$\begin{vmatrix} E_0-\alpha_{11} & E_0\nu^2-\alpha_{12}\\ E_0\nu^2-\alpha_{21} & E_0-\alpha_{22}\end{vmatrix}=0 \tag{1.2.16}$$

and the corresponding eigenfunctions are

$$\begin{aligned}&(1)\qquad C_1=-C_2=C_a,\\ &\psi_a=C_a[\varphi(r_{a_1})\varphi(r_{b_2})-\varphi(r_{a_2})\varphi(r_{b_1})]. \end{aligned}\tag{1.2.17}$$

$$\begin{aligned}&(2)\qquad C_1=C_2=C_s,\\ &\psi_s=C_s[\varphi(r_{a_1})\varphi(r_{b_2})+\varphi(r_{a_2})\varphi(r_{b_1})]. \end{aligned}\tag{1.2.18}$$

Since $\int \psi_a \psi_a^* = \int \psi_s \psi_s^* = 1$, we have

$$C_a = \frac{1}{\sqrt{2(1-\gamma^2)}}, \quad C_a = \frac{1}{\sqrt{2(1+\gamma^2)}}, \tag{1.2.19}$$

where

$$\gamma = \int \varphi(r_{a_1})\varphi(r_{b_1})dr_1. \tag{1.2.20}$$

2. *Spin wave function*

The wave function ψ for the system of two electrons can be written as a product of the space and the spin wave functions

$$\psi(r_1\sigma_1, r_2\sigma_2) = \varphi(r_1, r_2)\chi(\sigma_1, \sigma_2), \tag{1.2.21}$$

where σ_1, σ_2 are the projections of the electron spins along a given axis. In accordance with the Pauli principle, the wave function ψ must be antisymmetric with respect to the simultaneous interchange of the coordinates and of the spin variables of electrons. This means that an antisymmetric space function will be associated with symmetric spin function, and conversely, a symmetric space function will be associated with an antisymmetric spin function.

The function χ will be symmetric if the resultant spin S of the two electrons is equal to unity ($S = 1$) and antisymmetric if $S = 0$. Therefore the space wave function will be antisymmetric for $S = 1$ and symmetric for $S = 0$. We shall denote these wave functions by φ_a (for $S = 1$) and φ_s (for $S = 0$):

$$\psi_a = C_a[\varphi(r_{a_1})\varphi(r_{b_2}) - \varphi(r_{a_2})\varphi(r_{b_1})], \quad S = 1. \tag{1.2.22}$$

$$\psi_s = C_s[\varphi(r_{a_1})\varphi(r_{b_2}) + \varphi(r_{a_2})\varphi(r_{b_1})], \quad S = 0. \tag{1.2.23}$$

The energies of the molecules in states corresponding to $S = 1$ and $S = 0$ are related to the functions φ_a and φ_s by

$$E_{\uparrow\uparrow}(r) = \int \varphi_a(r_1, r_2)\hat{\mathcal{H}}\varphi_a(r_1, r_2)dr_1 dr_2, \tag{1.2.24}$$

$$E_{\uparrow\downarrow}(r) = \int \varphi_s(r_1, r_2)\hat{\mathcal{H}}_1\varphi_s(r_1, r_2)dr_1 dr_2 \tag{1.2.25}$$

in which we have omitted the symbol indicating complex conjugate in the integrals, since φ_a and φ_s are real functions (the atoms are assumed to be in the ground states). Substituting the expressions of ψ_a and ψ_s into (1.2.24) and (1.2.25), we find that

$$E_{\uparrow\uparrow}(r) = 2E_0 + \frac{e^2}{R} + \frac{A(r) - B(r)}{1-\gamma^2}, \quad S = 1, \tag{1.2.26}$$

$$E_{\uparrow\downarrow}(r) = 2E_0 + \frac{e^2}{R} + \frac{A(r) + B(r)}{1+\gamma^2}, \quad S = 0. \tag{1.2.27}$$

Choosing wave functions such that $\gamma = 0$, we have

$$E_{\uparrow\uparrow}(r) = 2E_0 + \frac{e^2}{R} + (A(r) - B(r)), \quad S = 1, \tag{1.2.28}$$

$$E_{\uparrow\downarrow}(r) = 2E_0 + \frac{e^2}{R} + (A(r) + B(r)), \quad S = 0. \tag{1.2.29}$$

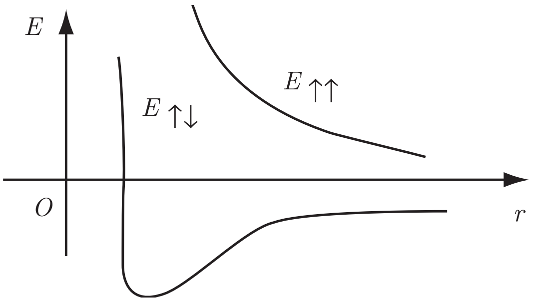


Figure 1.2.2. *Total energy of a system of two electrons.*

As shown in Figure 1.2.2, the total energy of the system can be represented by

$$E = \hat{\mathcal{H}} = 2E_0 + \frac{e^2}{R} + A - \frac{B}{2} - 2B\vec{S}_1 \cdot \vec{S}_2$$
$$= \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{ex}, \tag{1.2.30}$$

where $\hat{\mathcal{H}}_{ex} = -2B\vec{S}_1 \cdot \vec{S}_2$ is called exchange Hamiltonian which was first obtained by Dirac in 1929. $\vec{S}_1$ and $\vec{S}_2$ are electron spin operators. We want to prove that (1.2.30) is identical to (1.2.28) and (1.2.29).

In fact,

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\vec{S}^2 - \frac{1}{2}(\vec{S}_1^2 + \vec{S}_2^2)$$
$$= \frac{1}{2}S(S+1) - \frac{3}{4}$$
$$= \begin{cases} -\frac{3}{4}, & S = 0 \\ \frac{1}{4}, & S = 1, \end{cases} \tag{1.2.31}$$

therefore we get $E_{\uparrow\uparrow}$ $(S = 1)$ and $E_{\uparrow\downarrow}$ $(S = 0)$. We see that the term $-2B\vec{S}_1 \cdot \vec{S}_2$ denote the multibody effect.

For any operator $\hat{F}$ we can obtain the following motion equation:

$$\frac{d\hat{F}}{dt} = \frac{\partial \hat{F}}{\partial t} + \frac{1}{h}[\hat{\mathcal{H}}, \hat{\mathcal{H}}], \tag{1.2.32}$$

where $\hat{\mathcal{H}}$ is the Hamiltonian operator, $[\hat{\mathcal{H}}, \hat{F}] = \hat{\mathcal{H}}\hat{F} - \hat{F}\hat{\mathcal{H}}$. For the spin operator $\vec{S}_1$, we have

$$\frac{d\vec{S}_1}{dt} = \frac{\partial \vec{S}_1}{\partial t} + \frac{1}{ih}[\hat{\mathcal{H}}, \vec{S}_1] = [\hat{\mathcal{H}}, \vec{S}_1]$$
$$= \frac{1}{ih}\left[-\frac{h^2}{2m}(\nabla_1^2 + \nabla_2^2) - A - \frac{1}{2}B - 2B\vec{S}_1 \cdot \vec{S}_2, \vec{S}_1\right]$$
$$= \frac{1}{ih}[-2B\vec{S}_1 \cdot \vec{S}_2, \vec{S}_1] = \frac{-2B}{ih}[\vec{S}_1 \cdot \vec{S}_2, \vec{S}_1],$$
$$[\vec{S}_1 \cdot \vec{S}_2, \vec{S}_1] = [\vec{S}_1 \cdot \vec{S}_2, \vec{S}_{1x}]\vec{i} + [\vec{S}_1 \cdot \vec{S}_2, \vec{S}_{1y}]\vec{j} + [\vec{S}_1 \cdot \vec{S}_2, \vec{S}_{1z}]\vec{k},$$

where

$$\begin{aligned}[\vec{S}_1 \cdot \vec{S}_2, \vec{S}_{1x}] &= [\vec{S}_{1x} \cdot \vec{S}_{2x} + \vec{S}_{1y} \cdot \vec{S}_{2y} + \vec{S}_{1z} \cdot \vec{S}_{2z}, \vec{S}_{1x}] \\ &= [\vec{S}_{1y} \cdot \vec{S}_{2y} + \vec{S}_{1z} \cdot \vec{S}_{2z}, \vec{S}_{1x}] \\ &= [\vec{S}_{1y}, \vec{S}_{1x}]\vec{S}_{2y} + [\vec{S}_{1z}, \vec{S}_{1x}]\vec{S}_{2z} \\ &= ih(\vec{S}_{1y}\vec{S}_{2z} - \vec{S}_{1z}\vec{S}_{2y})\end{aligned}$$

and the other terms can be derived in the similar manner. Hence we finally obtain

$$\frac{d\vec{S}_1}{dt} = \vec{S}_1 \times (-2B\vec{S}_2) = \vec{S}_1 \times \vec{H}_{\text{eff}}, \tag{1.2.33}$$

where

$$\hat{\mathcal{H}}_e = -2B\vec{S}_1 \cdot \vec{S}_2 = \vec{S}_1(-2B\vec{S}_2) = \vec{S}_1 \cdot \vec{H}_{\text{eff}}. \tag{1.2.34}$$

1.2.3 Multi-electron Wave Function and Spin Operator

1. *Equation for the isotropic ferromagnetic chain*

Now we consider the spin operator for multi-electron system, i.e. one-dimensional homogeneous Heisenberg model (see Figure 1.2.3): Assume that the ferromagnets are isotropic.

$$\hat{H}_e = -2B\vec{S}_i \cdot \sum_{j=1}^{k} \vec{S}_j = -2B\vec{S}_i(\vec{S}_{i-1} + \vec{S}_{i+1}). \tag{1.2.35}$$

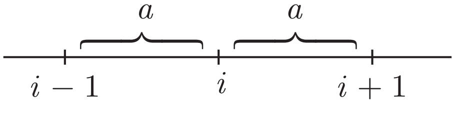


Figure 1.2.3. *One-dimensional homogeneous Heisenberg model.*

$$\begin{aligned}\vec{S}_{i+1} &= \vec{S}_i + \frac{\partial \vec{S}_i}{\partial x}a + \frac{\partial^2 \vec{S}_i}{\partial x^2}a^2, \\ \vec{S}_{i-1} &= \vec{S}_i - \frac{\partial \vec{S}_i}{\partial x}a + \frac{\partial^2 \vec{S}_i}{\partial x^2}a^2,\end{aligned}$$

therefore

$$\begin{aligned}\frac{d\vec{S}_i}{dt} &= \vec{S}_i \times \hat{H}_e = \vec{S}_i \times (-2B)\left(2\vec{S}_i + a^2\frac{\partial^2 \vec{S}_i}{\partial x^2}\right) \\ &= -2Ba^2\vec{S}_i \cdot \vec{S}_i \times \frac{\partial^2 \vec{S}_i}{\partial x^2}.\end{aligned} \tag{1.2.36}$$

Passing to the continuous case: $\vec{S}_i \to \vec{S}(x,t)$ and setting the total energy of the system to be

$$\mathcal{H} = \frac{1}{2}\int \left(\frac{\partial \vec{S}}{\partial x}\right)^2 dx, \tag{1.2.37}$$

we get the magnetization spin motion equation as follows:

$$\frac{\partial \vec{S}}{\partial t} = \vec{S} \times H_{\text{eff}}, \tag{1.2.38}$$

$$\vec{H}_{\text{eff}} = -\frac{\partial \mathcal{H}}{\partial S}. \tag{1.2.39}$$

For the nonhomogeneous isotropic Heisenberg chain, the nonhomogeneous isotropic Heisenberg exchange Hamiltonian is

$$\mathcal{H} = -J \sum_{i=1}^{N-1} f_i \vec{S}_i \cdot \vec{S}_{i+1}, \tag{1.2.40}$$

The motion equation of $\vec{S}_i$ is

$$\frac{\partial \vec{S}_i}{\partial t} = J f_i (\vec{S}_i \times \vec{S}_{i+1}) + J f_{i-1} (\vec{S}_i \times \vec{S}_{i-1}). \tag{1.2.41}$$

Considering the continuous situation: $\vec{S}_i \to \vec{S}(x,t)$, $f_i \to f(x,t)$ and $\vec{S}_i$, f_i varies slowly in the same lattices (with length a), taking Taylor expansions for $\vec{S}(x+a,t)$, $f(x-a,t)$, we have from (1.2.41) that $\vec{S}(x,t)$ meets the motion equation

$$\vec{S}_t(x,t) = f(x)(\vec{S} \times \vec{S}_{xx}) + f_x(x)(\vec{S} \times \vec{S}_x), \tag{1.2.42}$$

where the variable of times has the scaling factor Ja^2.

2. *Anisotropic ferromagnetic chain equations*

Now let us consider the continuous anisotropic chain equation for the nonhomogeneous ferromagnets:

$$\mathcal{H} = \frac{1}{2} \int \left[\left(\frac{\partial \vec{S}}{\partial x} \right)^2 - J_1 S_1^2 - J_2 S_2^2 - J_3 S_3^2 \right]. \tag{1.2.43}$$

It follows from this that

$$\frac{\partial \vec{S}}{\partial t} = \vec{S} \times \vec{H}_{\text{eff}}, \tag{1.2.44}$$

in which

$$\vec{H}_{\text{eff}} = -\frac{\partial \mathcal{H}}{\partial \vec{S}} = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix} \cdot \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}. \tag{1.2.45}$$

1.3 Equations for the Antiferromagnets

1.3.1 Antiferromagnetic Moments and Magnetic Energy

Some ferromagnets exhibit the so-called antiferromagnetic property if placed under the temperature much less than the Curie temperature. These ferromagnets are called

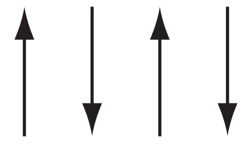


Figure 1.3.1. *Orientation of electron spin.*

antiferromagnets. Their electron spins are in different directions and compensate with each other as shown in Figure 1.3.1.

We can divide antiferromagnets into two systems: moment densities $\mathcal{M}_1(r,t)$ and $\mathcal{M}_2(r,t)$, and $|\mathcal{M}_1(r,t)| = |\mathcal{M}_2(r,t)|$. The energy of magnetic dipole interaction W_m and the energy W_H of the ferromagnet in the external magnetic field are given by the following formulae

$$W_m = \frac{-1}{2}\int_V (\mathcal{M}_1 + \mathcal{M}_2) H^{(m)} d\vec{r}, \tag{1.3.1}$$

$$W_H = -\int_V (\mathcal{M}_1 + \mathcal{M}_2) H_0^{(e)} d\vec{r}, \tag{1.3.2}$$

where $H^{(m)}$ represents the magnetic field due to the magnetic moment of the atoms in the antiferromagnets and the integrals are evaluated over the volume of the ferromagnet.

1.3.2 Equations for the Antiferromagnets

The density of the exchange energy W_e of an antiferromagnet is therefore of the form

$$W_e = f(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_1^2, \mathcal{M}_2^2) + \frac{1}{2}\alpha_{ik}\left(\frac{\partial \mathcal{M}_1}{\partial x_i}\frac{\partial \mathcal{M}_1}{\partial x_k} + \frac{\partial \mathcal{M}_2}{\partial x_i}\frac{\partial \mathcal{M}_2}{\partial x_k}\right) + \alpha'_{ik}\frac{\partial \mathcal{M}_1}{\partial x_i}\frac{\partial \mathcal{M}_2}{\partial x_k}, \tag{1.3.3}$$

where f is a symmetric function of the magnetic moments $\mathcal{M}_1, \mathcal{M}_2$ and α_{ik}, α'_{ik} are tensors. The first term in this expression represents the exchange energy density of uniformly magnetized sublattices, and the second and the third terms represent the exchange energy density connected with the non-uniformity of the magnetic moments. At the same time, the second term describes the exchange interaction in each of the sublattices, and the third term the exchange interaction between the sublattices. At sufficiently low temperatures the squares of the magnetic moment densities are practically constant and the function f can be regarded as depending only on the single variable $\mathcal{M}_1 \cdot \mathcal{M}_2$. In the simplest model of an antiferromagnet it is assumed that

$$f(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_1^2, \mathcal{M}_2^2) = \delta \mathcal{M}.\mathcal{M}_2, \quad \delta > 0. \tag{1.3.4}$$

The total exchange energy of an antiferromagnet is therefore of the form

$$\begin{aligned} W_e &= \int_V w_e dr \\ &= \int_V \left\{ \delta\mathcal{M}\cdot\mathcal{M}_2 + \frac{1}{2}\alpha_{ik}\left(\frac{\partial\mathcal{M}_1}{\partial x_i}\frac{\partial\mathcal{M}_1}{\partial x_k} + \frac{\partial\mathcal{M}_2}{\partial x_i}\frac{\partial\mathcal{M}_2}{\partial x_k}\right) + \alpha'_{ik}\frac{\partial\mathcal{M}_1}{\partial x_i}\frac{\partial\mathcal{M}_2}{\partial x_k}\right\} dr, \end{aligned} \tag{1.3.5}$$

and the total exchange energy of an isotropic antiferromagnet is

$$\begin{aligned} \mathcal{H} = \int_V d^3x \bigg[&\frac{1}{2}\alpha_{ik}\left(\frac{\partial\mathcal{M}_1}{\partial x_i}\frac{\partial\mathcal{M}_1}{\partial x_k} + \frac{\partial\mathcal{M}_2}{\partial x_i}\frac{\partial\mathcal{M}_2}{\partial x_k}\right) + \alpha'_{ik}\frac{\partial\mathcal{M}_1}{\partial x_i}\frac{\partial\mathcal{M}_2}{\partial x_k} \\ &+ \delta\mathcal{M}_1\cdot\mathcal{M}_2 - \frac{1}{2}(\mathcal{M}_1+\mathcal{M}_2)H^{(m)} - (\mathcal{M}_1+\mathcal{M}_2)H_0^{(e)}\bigg], \end{aligned} \tag{1.3.6}$$

In some simplified situations, we have

$$\mathcal{H} = \int_V d^3x[k_1|\nabla\mathcal{M}_1|^2 + k_{12}|\nabla\mathcal{M}_2|^2 + k_{12}\nabla\mathcal{M}_1\cdot\nabla\mathcal{M}_2]. \tag{1.3.7}$$

In this case the motion equation is of the form

$$\begin{cases} \dfrac{\partial\mathcal{M}_1}{\partial t} = \mathcal{M}_1\times[2k_1\nabla^2\mathcal{M}_1 + k_{12}\nabla^2\mathcal{M}_2], \\ \dfrac{\partial\mathcal{M}_2}{\partial t} = \mathcal{M}_2\times[2k_2\nabla^2\mathcal{M}_2 + k_{12}\nabla^2\mathcal{M}_1]. \end{cases} \tag{1.3.8}$$

1.4 Spin Waves in Ferromagnets

1.4.1 Equilibrium State of Ferromagnets

1. *Equilibrium state conditions of ferromagnets*

Consider a uniaxial ferromagnet. Assume that

$$\mathcal{M}(r,t) = \mathcal{M}_0(r,t) + \vec{m}(r,t), \tag{1.4.1}$$

$$\vec{H}^{(i)}(r,t) = \vec{H}_0^{(i)} + \vec{h}(r,t), \tag{1.4.2}$$

where $\vec{m}, \vec{h}$ are small derivations from $\mathcal{M}_0$ and $\vec{H}_0^{(i)}$, $\mathcal{M}_0$ is the equilibrium magnetization, $\vec{H}_0^{(i)}$ denotes the magnetic field inside the ferromagnet. According to the equilibrium conditions one has

$$\vec{H}_0^{(i)} + \beta\vec{n}(\mathcal{M}_0\cdot\vec{n}) - 2\mathcal{M}_0 f'(\mathcal{M}_0^2) = 0, \tag{1.4.3}$$

or

$$\vec{H}_0^{(e)} + \beta\vec{n}(\mathcal{M}_0\cdot\vec{n}) - 4\pi\hat{N}\cdot\mathcal{M}_0 - 2\mathcal{M}_0 f'(\mathcal{M}_0^2) = 0, \tag{1.4.4}$$

where β is a constant, $\vec{n}$ is a unit vector along the anisotropy axis, $\hat{N} = \hat{N}(r)$ is the demagnetization tensor with elements

$$N_{ik}(r) = \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_k} \int_V \frac{dr'}{|r - r'|}. \tag{1.4.5}$$

When the ferromagnet is an ellipsoid, $\mathcal{M} =$ constant, and then

$$\int_V \frac{dr'}{|r - r'|} = \pi abc \int_0^\infty \frac{ds}{R_s} \left(1 - \frac{x^2}{a^2 + s} - \frac{y^2}{b^2 + s} - \frac{z^2}{c^2 + s}\right), \tag{1.4.6}$$

where $R_s = \sqrt{(a^2 + s)(-b^2 + s)(c^2 + s)}$, a, b, c are the semi-axes of the ellipsoid, x, y, z are the projection of the radial vector $\vec{r}$ in any point of the ellipsoid onto the main axses. The elements on the diagonal of the tensor $\vec{N}$ are

$$\begin{cases} N_1 = \frac{1}{2} abc \int_0^\infty \frac{ds}{(a^2 + s)R_s}, \\ N_2 = \frac{1}{2} abc \int_0^\infty \frac{ds}{(b^2 + s)R_s}, \\ N_3 = \frac{1}{2} abc \int_0^\infty \frac{ds}{(c^2 + s)R_s}. \end{cases} \tag{1.4.7}$$

It is clear that $N_1 + N_2 + N_3 = 1$. If the ferromagnet occupies a sphere, then $N_1 = N_2 = N_3 = 1/3$. If it occupies a cylinder ($a = \infty, b = c$), $N_1 = 0$, $N_2 = N_3 = 1/2$. What we are concerned are the equilibrium states when (1.4.7) admits solution:

$$N_3 > N_2 > N_1, \text{ if } \beta > 0; \quad N_3 < N_2 < N_1, \text{ if } \beta < 0,$$

see Figures 1.4.1 and 1.4.2.

2. *The equation of motion for the magnetization*

The expression of effective magnetic field related to (1.4.1) and (1.4.2) is

$$\begin{aligned} \mathcal{H} = \vec{h} + \alpha_{ik} \frac{\partial^2 \vec{m}}{\partial x_i \partial x_k} - \frac{1}{\mathcal{M}_0^2} \{\mathcal{M}_0 \cdot H_0^{(i)} + \beta(\mathcal{M}_0 \cdot \vec{n})^2\} \vec{m} \\ + \beta \vec{n}(\vec{m} \cdot \vec{n}) - 4\mathcal{M}_0 f''(\mathcal{M}_0^2)(\mathcal{M}_0 \cdot \vec{m}). \end{aligned} \tag{1.4.8}$$

Therefore the equation of motion for the magnetization in the case of small departures from the equilibrium value, which we shall refer to as the linearized equation of motion, will be of the form

$$\frac{\partial \vec{m}}{\partial t} = g\left[\mathcal{M}_0, \vec{h} + \alpha_{ik} \frac{\partial^2 \vec{m}}{\partial x_i \partial x_k} + \beta \vec{n}(\vec{m} \cdot \vec{n}) - \frac{1}{\mathcal{M}_0^2} \{\mathcal{M}_0 \cdot H_0^{(i)} + \beta(\mathcal{M}_0 \cdot \vec{n})^2\} \vec{m}\right].$$

If the ferromagnet exhibits magnetic anisotropy of the "easy axis" type ($\beta > 0$) and $H_0^{(i)}$ is parallel to $\vec{n}$, the linearized effective magnetic field is given by

$$\mathcal{H} = \vec{h} - \left(\beta + \frac{H_0^{(i)}}{\mathcal{M}_0}\right) \vec{m} + \alpha_{ik} \frac{\partial^2 \vec{m}}{\partial x_i \partial x_k} + (\beta - 4\mathcal{M}_0 f''(\mathcal{M}_0^2))(\vec{m} \cdot \vec{n}) n. \tag{1.4.9}$$

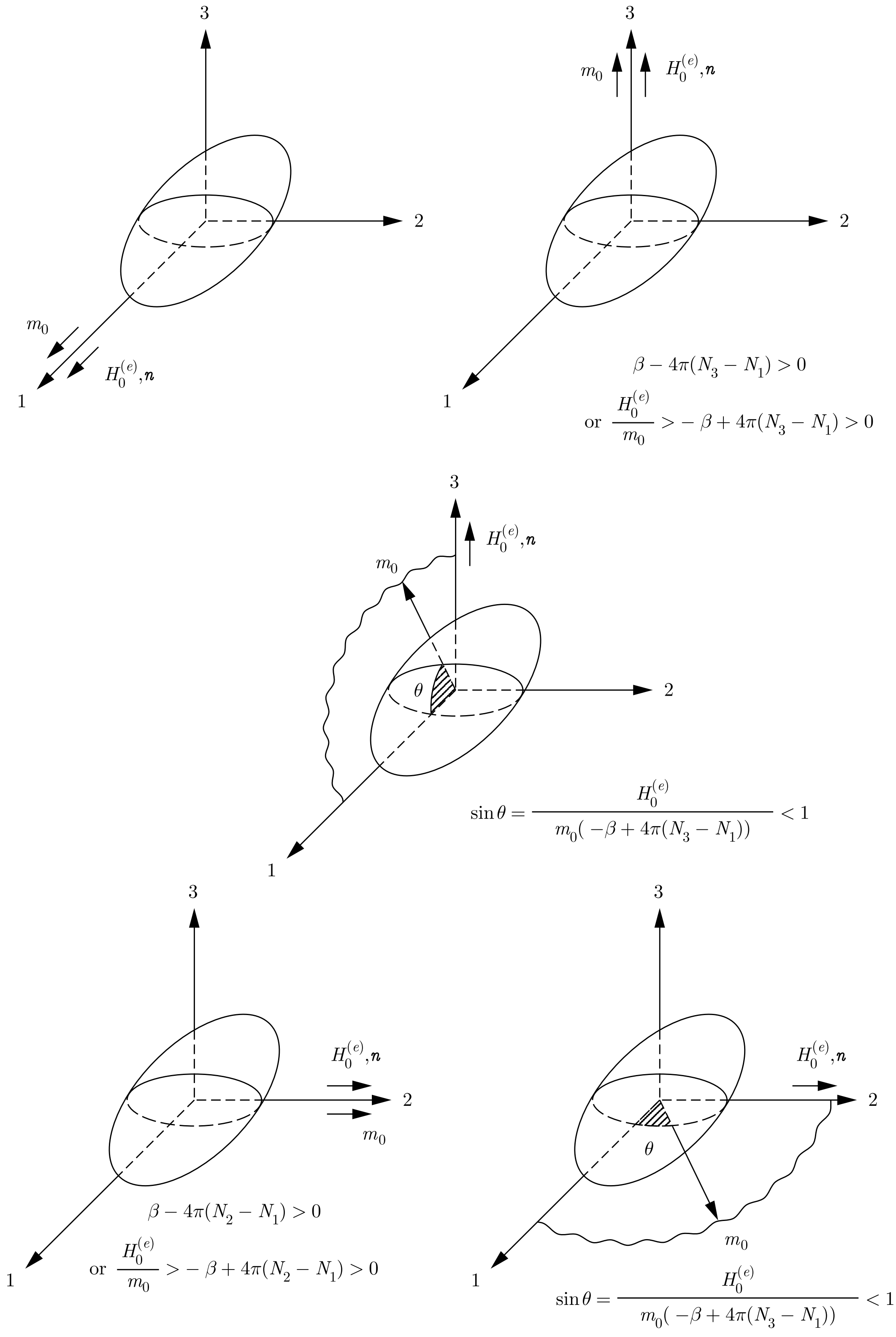


Figure 1.4.1. *Equilibrium states of a ferromagnet-*1 $N_3 > N_2 > N_1$, *if* $\beta > 0$.

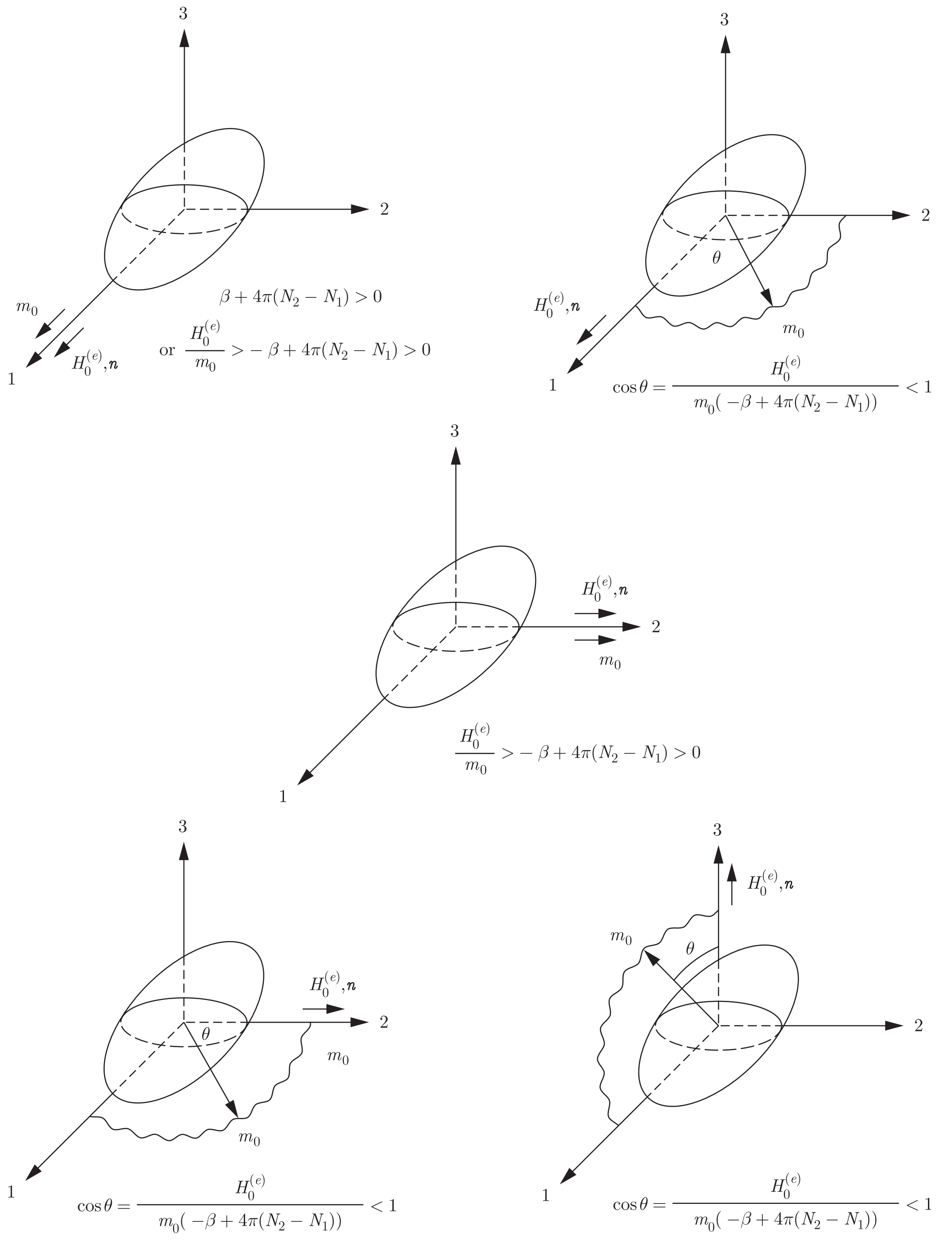


Figure 1.4.2. *Equilibrium states of a ferromagnet-2* $N_3 < N_2 < N_1$, *if* $\beta < 0$.

If the ferromagnet exhibits magnetic anisotropy of the "easy plane" type ($\beta < 0$) and $H_0^{(i)}$ is perpendicular to $\vec{n}$, the linearized effective magnetic field is given by

$$\mathcal{H} = \left(\vec{h} - \frac{H_0^{(i)}}{\mathcal{M}_0}\right)\vec{m} + \alpha_{ik}\frac{\partial^2 \vec{m}}{\partial x_i \partial x_k} + \beta\vec{n}(\vec{m}\cdot\vec{n}) - 4\mathcal{M}_0 f''(\mathcal{M}_0^2)(\mathcal{M}_0\cdot\vec{m}). \tag{1.4.10}$$

The linearized equation of motion for the magnetization (1.4.8) must be augmented by the boundary conditions for the magnetization. The general boundary condition for the magnetization was formulated as

$$\frac{\partial \mathcal{F}}{\partial(\partial\mathcal{M}/\partial x_k)}\nu_k\bigg|_S = 0, \tag{1.4.11}$$

where $\mathcal{F}$ is the energy density of the ferromagnet:

$$\mathcal{F}\left(\mathcal{M}, \frac{\partial \mathcal{M}_i}{\partial x_k}\right) = \frac{1}{2}\alpha_{ik,lm}(\mathcal{M})\frac{\partial \mathcal{M}_i}{\partial x_k}\frac{\partial \mathcal{M}_k}{\partial x_m} + \gamma_{ik}(\mathcal{M})\frac{\partial \mathcal{M}_i}{\partial x_k} + \frac{1}{2}\beta_{ik}\mathcal{M}_i\cdot\mathcal{M}_k + f(\mathcal{M}^2), \tag{1.4.12}$$

where ν is the unit vector along the outward normal to the surface of the ferromagnet.

Since we are interested in small departures of the magnetic moment from the equilibrium value and small magnetization gradients, we have

$$\frac{\partial \mathcal{F}}{\partial(\partial\mathcal{M}_i/\partial x_k)} = \alpha_{kj}\frac{\partial \vec{m}_i}{\partial x_j} + \gamma_{ik,l}\vec{m}_l, \qquad \gamma_{ik,l} = \left(\frac{\partial \gamma_{ik}}{\partial \mathcal{M}_l}\right)\bigg|_{\mathcal{M}=\mathcal{M}_0},$$

and the boundary condition assumes the form

$$\left(\alpha_{kj}\frac{\partial m_i}{\partial x_j} + \gamma_{ik,l}m_l\right)\nu_k\bigg|_S = 0\,. \tag{1.4.13}$$

The linearized equation of motion for the magnetization is then of the form

$$\frac{\partial \vec{m}}{\partial t} = g[\mathcal{M}_0, \mathcal{H}], \tag{1.4.14}$$

where

$$\mathcal{H} = \vec{h} - \beta(z)\vec{m} + \alpha\Delta\vec{m}. \tag{1.4.15}$$

Assuming that the function $\beta(z)$ increases rapidly in the thin layer of thickness δ in which $\vec{m}$ and $\vec{h}$ are practically constant, we obtain after integrating this equation with respect to z between 0 and δ,

$$\delta\frac{\partial \vec{m}}{\partial t} = g\left[\mathcal{M}_0, \vec{h}\cdot\delta - \vec{m}\int_0^\delta \beta(z)dz + \alpha\frac{\partial \vec{m}}{\partial z}\bigg|_{z=\delta} + \alpha\left(\frac{\partial^2 \vec{m}}{\partial x^2} + \frac{\partial^2 \vec{m}}{\partial y^2}\right)\delta\right], \tag{1.4.16}$$

where we have taken into account the fact that

$$\left.\frac{\partial \vec{m}}{\partial z}\right|_{z=0} = 0. \tag{1.4.17}$$

Assuming further that as $\delta \to 0$, the integral

$$\int_0^\delta \beta(z)dz < \infty, \tag{1.4.18}$$

we obtain from the last equation the following effective boundary condition:

$$d\frac{\partial \vec{m}}{\partial z} - \vec{m}|_{z=0} = 0, \tag{1.4.19}$$

where

$$d = \frac{\alpha}{\int_0^\delta \beta(z)dz}. \tag{1.4.20}$$

In order to be able to neglect in (1.4.16) terms proportional to δ, it is clearly necessary that the frequency ω of the variation in the magnetization, the wavelength λ and the quantity $\beta(0)$ must satisfy the conditions:

$$\omega \ll g\mathcal{M}_0\beta(0), \quad \lambda \gg \sqrt{\frac{\alpha}{\beta(0)}}, \quad \beta(0) \gg 1. \tag{1.4.21}$$

Then

$$\alpha\frac{\partial^2 \vec{m}}{\partial z^2} - \beta(0)\vec{m} = 0,$$

it can readily be concluded that

$$\delta \ll \sqrt{\frac{\alpha}{\beta(0)}}$$

or

$$\delta \ll d.$$

If the wavelength λ satisfies the inequalities $\sqrt{\delta}d \ll \lambda \ll d$, the effective boundary condition assumes the form

$$\left.\frac{\partial \vec{m}}{\partial z}\right|_{z=0} = 0, \quad \sqrt{\delta}d \ll \lambda \ll d. \tag{1.4.22}$$

If on the other hand $\lambda \gg d$, then

$$\vec{m}|_{z=0} = 0, \qquad \lambda \gg d. \tag{1.4.23}$$

1.4.2 Spin Waves in Ferromagnets

1. *Motion of spin waves in ferromagnets*

Now we consider the propagations of spin waves in ferromagnets. Using the Fourier representations

$$\begin{aligned}\vec{m}(r,t) &= \int \vec{m}(\vec{k},\omega)\exp[i(\vec{k}\cdot\vec{r}-\omega t)]d\vec{k}d\omega\,,\\ \vec{h}(r,t) &= \int \vec{h}(\vec{k},\omega)\exp[i(\vec{k}\cdot\vec{r}-\omega t)]d\vec{k}d\omega\,,\end{aligned} \tag{1.4.24}$$

we have from (1.4.8) that

$$\begin{aligned}-i\omega\vec{m}(\vec{k},\omega) = g\bigg[\mathcal{M}_0, h(\vec{k},\omega) - \left\{\alpha_{ij}k_ik_j + \frac{\mathcal{M}_0\cdot H_0^{(i)}}{\mathcal{M}_0^2} + \beta\frac{(\mathcal{M}_0\cdot\vec{n})^2}{\mathcal{M}_0^2}\right\}\vec{m}(\vec{k},\omega)\\ + \beta\vec{n}(\vec{n}\cdot\vec{m}(\vec{k},\omega))\bigg].\end{aligned} \tag{1.4.25}$$

This equation gives the relationship between the Fourier components $\vec{m}(\vec{k},\omega)$ and $h(\vec{k},\omega)$ which we shall write in the form

$$m_i(\vec{k},\omega) = \chi_{ij}(\vec{k},\omega)h_j(\vec{k},\omega), \tag{1.4.26}$$

where

$$\chi_{ij}(\vec{k},\omega) = \begin{pmatrix}\chi_{xx} & \chi_{xy} & 0\\ \chi_{yx} & \chi_{yy} & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{1.4.27}$$

$$\chi_{xx} = \frac{g\mathcal{M}_0\Omega_1}{\Omega_1\Omega_2-\omega^2},\quad \chi_{yy} = \frac{g\mathcal{M}_0\Omega_2}{\Omega_1\Omega_2-\omega^2},\quad \chi_{xy} = -\chi_{yx} = \frac{i\omega g\mathcal{M}_0}{\Omega_1\Omega_2-\omega^2},$$

$$\begin{aligned}\Omega_1 &= g\mathcal{M}_0\left(\alpha_{ij}k_ik_j + \frac{\mathcal{M}_0\cdot H_0^{(i)}}{\mathcal{M}_0^2} + \beta\cos^2\psi\right),\\ \Omega_2 &= g\mathcal{M}_0\left(\alpha_{ij}k_ik_j + \frac{m_0\cdot H_0^{(i)}}{\mathcal{M}_0^2} + \beta\cos 2\psi\right),\end{aligned} \tag{1.4.28}$$

and ψ is the angle between the anisotropy axis $\vec{n}$ and the vector $\mathcal{M}_0$; the z-axis lies along $\mathcal{M}_0$ and the x-axis lies in the plane containing the vector $\vec{n}$ and $\mathcal{M}_0$. The quantities $\chi_{ij}(\vec{k},\omega)$ form a tensor called the high-frequency magnetic susceptibility tensor of the ferromagnet.

If the ferromagnet exhibits anisotropy of the "easy axis" type ($\beta > 0$) and m_0, $\vec{n}$, $H_0^{(i)}$ are parallel with each other, then

$$\Omega_1 = \Omega_2 = \Omega, \tag{1.4.29}$$

where

$$\Omega = g\mathcal{M}_0\bigg(\alpha_{ij}k_ik_j + \frac{H_0^{(i)}}{\mathcal{M}_0} + \beta\bigg). \tag{1.4.30}$$

The formulae corresponding to this case are valid also for ferromagnets with cubic symmetry. All that is required is to replace β by $2\beta'\mathcal{M}_0^2$, if the easy magnetization axis lies along the edge of the cube, and by $\frac{4}{3}|\beta'|\mathcal{M}_0^2$, if the easy magnetization axis lies along the space diagonal of the cube, where β' is the anisotropy constant.

If the ferromagnet exhibits magnetic anisotropy of the "easy plane" type ($\beta < 0$) and $\mathcal{M}_0 \perp \vec{n}$, $\mathcal{M}_0$ is parallel to $H_0^{(i)}$,

$$\begin{cases} \Omega_1 = g\mathcal{M}_0\bigg(\alpha_{ij}k_ik_j + \dfrac{H_0^{(i)}}{\mathcal{M}_0}\bigg), \\ \Omega_2 = g\mathcal{M}_0\bigg(\alpha_{ij}k_ik_j + \dfrac{H_0^{(i)}}{\mathcal{M}_0} + |\beta|\bigg). \end{cases} \tag{1.4.31}$$

2. *Dispersion of spin waves*

We must now establish the dependence of the frequency ω of the spin wave on its wave vector $\vec{k}$. This requires the use of both the equation of motion for the magnetic moment and the Maxwell equations. However, spin waves are low-frequency magnetic waves, so that the electric field can be neglected and the magnetic field may be assumed to be irrotational. In other words, spin waves can be treated in terms of the magnetostatic approximation, i.e. we can assume that $\vec{m}(r,t)$ and $\vec{h}(r,t)$ satisfy the equations

$$\begin{cases} \operatorname{rot}\vec{h}(r,t) = 0, \\ \operatorname{div}\vec{h}(r,t) = 4\pi \operatorname{div}\vec{m}(r,t). \end{cases} \tag{1.4.32}$$

Transforming to the Fourier components in these equations, we obtain

$$\begin{cases} [\vec{k}, \vec{h}(\vec{k},\omega)] = 0, \\ \vec{k}\cdot\vec{h}(\vec{k},\omega) = 4\pi\vec{k}\cdot\vec{m}(\vec{k},\omega). \end{cases} \tag{1.4.33}$$

From the first equation it follows that the magnetic field $\vec{h}(\vec{k},w)$ is parallel to the wave vector $\vec{k}$:

$$\vec{h}(\vec{k},\omega) = -i\vec{k}\cdot\varphi(\vec{k},\omega),$$

where $\varphi(\vec{k},\omega)$ is the Fourier component of the magnetic potential. Since

$$\vec{m}(\vec{k},\omega) = \chi(\vec{k},\omega)\vec{h}(\vec{k},\omega),$$

we can write the second equation in (1.4.33) in the form

$$(k^2 + 4\pi k_ik_j\chi_{ij}(\vec{k},\omega))\varphi(\vec{k},\omega) = 0$$

and hence

$$k^2 + 4\pi k_ik_j\chi_{ij}(\vec{k},\omega) = 0. \tag{1.4.34}$$

This equation relates ω and the wave vector $\vec{k}$ of the spin wave (it is called the dispersion relation). Using (1.4.27) we can reduce the dispersion relation (1.4.34) to the form

$$1+\frac{4\pi g\mathcal{M}_0\Omega_1}{\Omega_1\Omega_2-\omega^2k^2}\frac{k_x^2}{k^2}+\frac{4\pi g\mathcal{M}_0\Omega_2}{\Omega_1\Omega_2-\omega^2k^2}\frac{k_y^2}{k^2}=0$$

and hence

$$\omega_s(\vec{k})=\sqrt{\Omega_1\Omega_2+4\pi g\mathcal{M}_0(\Omega_1\cos^2\varphi_k+\Omega_2\sin^2\varphi_k)\sin^2\theta_k}, \quad (1.4.35)$$

where θ_k and φ_k are the polar and azimuthal angles of the wave vector $\vec{k}$. We recall that the z-axis lies along the vector $\mathcal{M}_0$ and the x-axis lies in the plane containing $\mathcal{M}_0$ and $\vec{n}$. For wave vectors with $\alpha k^2 \gg 1$, the equation for the frequency of the spin wave becomes much simpler:

$$\omega_S(\vec{k})=g\mathcal{M}_0\alpha_{ij}k_ik_j. \quad (1.4.36)$$

In the isotropic case this formula assumes the form

$$\omega_S(\vec{k})=\frac{\theta_C}{\hbar}(ak)^2, \quad (1.4.37)$$

where $\theta_C=\hbar g\mathcal{M}_0\alpha/a^2$ (θ_C is of the order of the Curie temperature). It follows that for large wave vectors ($\alpha k^2 \gg 1$), the spin wave frequency is proportional to the square of the wave vector.

1.4.3 Damping of Spin Waves

1. *Expression of the damping of spin waves*

Now we consider the damping of spin waves. Damping is due to the interaction of spin waves with each other and also with lattice vibrations and conduction electrons. The phenomenological description can be obtained from the equation of motion for the magnetization, containing the relaxation term R:

$$R=\frac{1}{\tau_2}\mathcal{H}-\frac{1}{\tau_1}[\vec{n},[\vec{n},\mathcal{H}]], \quad (1.4.38)$$

where $\vec{n}=\frac{\mathcal{M}_0}{|\mathcal{M}_0|}$ and τ_1,τ_2 are constants which have the dimensions of time, ($\frac{1}{\tau_2}>0$, $\frac{1}{\tau_1}+\frac{1}{\tau_2}>0$). Then the equation for $\mathcal{M}$ will be the form

$$\frac{\partial\mathcal{M}}{\partial t}=g[\mathcal{M},\mathcal{H}]+\frac{1}{\tau_2}\mathcal{H}-\frac{1}{\tau_1}[\vec{n},[\vec{n},\mathcal{H}]], \quad (1.4.39)$$

the effective magnetic field is

$$\mathcal{H}=\vec{h}-\left(\beta+\frac{H_0^{(i)}}{\mathcal{M}_0}\right)\vec{m}+\alpha_{ik}\frac{\partial^2\vec{m}}{\partial x_i\partial x_k}+(\beta-4\mathcal{M}_0f''(\mathcal{M}_0^2))(\vec{m}\cdot\vec{n})\vec{n}, \quad (1.4.40)$$

here we mainly focus on the uniaxial ferromagnet and assume that the field $H_0^{(i)}$ is parallel to the easy magnetization axis. The quantity $f''(\mathcal{M}_0^2)$ in the expression for $\mathcal{H}$ can readily be related to the static susceptibility of the ferromagnet, $\chi_{zz}^0 = \partial\mathcal{M}_0/\partial H_0^{(i)}$. We have seen that in the state of equilibrium

$$H_0^{(i)} + \beta\mathcal{M}_0 - 2\mathcal{M}_0 f'(\mathcal{M}_0^2) = 0 \tag{1.4.41}$$

and hence

$$(2\mathcal{M}_0)^2 f'(\mathcal{M}_0^2) = \frac{1}{\chi_{zz}^0} - \frac{H_0^{(i)}}{\mathcal{M}_0}. \tag{1.4.42}$$

From these formulae we obtain the following expression for the high-frequency susceptibility tensor χ:

$$\chi(\vec{k},\omega) = \begin{pmatrix} \chi_{xx} & \chi_{xy} & 0 \\ \chi_{yx} & \chi_{yy} & 0 \\ 0 & 0 & \chi_{zz} \end{pmatrix}, \tag{1.4.43}$$

in which

$$\begin{aligned} \chi_{xx} = \chi_{yy} &= \frac{g\mathcal{M}_0\Omega - (i\omega/\tau) + (\Omega/g\mathcal{M}_0\tau^2)}{\Omega^2 - (\omega + i\Omega/g\mathcal{M}_0\tau)^2}, \\ \chi_{zz} &= \frac{\chi_{zz}^0}{1 + \chi_{zz}^0(\alpha_{ij}k_ik_j - i\omega\tau_2)}, \\ \chi_{xy} = -\chi_{yx} &= \frac{i\omega g\mathcal{M}_0}{\Omega^2 - (\omega + i\Omega/g\mathcal{M}_0\tau)^2}, \\ \frac{1}{\tau} = \frac{1}{\tau_1} + \frac{1}{\tau_2}, \quad \Omega &= g\mathcal{M}_0\left(\alpha_{ij}k_ik_j + \frac{H_0^{(i)}}{\mathcal{M}_0} + \beta\right). \end{aligned} \tag{1.4.44}$$

We note that $1/g\mathcal{M}_0\tau \ll 1$. We also note that the component χ_{zz} on the tensor $\chi(\vec{k},\omega)$ is not zero, whereas it does vanish when $\tau_2 = \infty$. Substituting (1.4.44) into (1.4.34) we obtain the equation for the frequency of the spin wave as a function of its wave vector. When $R \neq 0$, this equation has complex roots whose real part determines the frequencies of the spin waves and whose imaginary part determines the damping rate.

2. *Damping rate*

Since $\Omega/g\mathcal{M}_0\tau \ll |\omega|$, the damping rate $\gamma_S(\vec{k})$ is given by

$$\gamma_S(\vec{k}) = \frac{1}{g\mathcal{M}_0\tau}(\Omega + 2\mathcal{M}_0\sin^2\theta_k). \tag{1.4.45}$$

If $\alpha k^2 \ll 1$, then

$$\gamma_S(\vec{k}) = \frac{\beta^2}{24\pi} g\mathcal{M}_0 \frac{\mu_0\mathcal{M}_0}{\theta_C}\left(\frac{T}{\theta_C}\right)^2; \tag{1.4.46}$$

if, on the other hand $\alpha k^2 \gg 1$, then

$$\gamma_S(\vec{k}) \approx \frac{\theta_C}{\hbar}(ak)^3\left(\frac{T}{\theta_C}\right), \quad \hbar\omega_S(\vec{k}) \gg T. \tag{1.4.47}$$

Substitute

$$\mathcal{M} = \mathcal{M}_0 + \vec{m}$$

into (1.4.39), here $\vec{m}$ is a small addition to the equilibrium magnetization $\mathcal{M}_0$, which we shall assume that $\mathcal{M}_0 = \mathcal{M}_0(t)$ is independent of r. The effective magnetic field is given by, to within terms linear in $\vec{m}$,

$$\mathcal{H} = -\left(\frac{4\pi}{3} + \beta + \frac{H_0^{(e)}}{\mathcal{M}_{\;0}}\right)\vec{m} + \left(\beta - \frac{4\pi}{3} + \frac{H_0^{(e)}}{\mathcal{M}_{\;0}} - \frac{1}{\chi_{zz}^0}\right)(\vec{m}\cdot\vec{n})\vec{n},$$

and the solution of (1.4.39) is of the form

$$\begin{cases} m_z = m_{z0}\exp(-\gamma_z t), \\ m_x + im_y = (m_{x0} + im_{y0})\exp(-\gamma_\perp t)\exp(i\omega_0 t), \end{cases} \tag{1.4.48}$$

where

$$\gamma_z = \frac{1}{\tau_2}\left(\frac{4\pi}{3} + \frac{1}{\chi_{zz}^0}\right), \quad \gamma_\perp = \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\left(\beta + \frac{H_0^{(e)}}{\mathcal{M}_0}\right), \tag{1.4.49}$$

and m_{z0} and $m_{x0} + im_{y0}$ are the initial values of the longitudinal and transverse (relative to $\mathcal{M}_0$) components of the deviation of the magnetization $\vec{m}$ from the equilibrium value $\mathcal{M}_0$.

The quantities

$$\tau_z = \frac{1}{\gamma_z}, \quad \tau_\perp = \frac{1}{\gamma_\perp}, \tag{1.4.50}$$

are the relaxation times for the longitudinal and transverse components of the magnetic moment. Since $\mathcal{M}_0$ is only slightly dependent on $H_0^{(i)}$, it follows that $\chi_{zz}^0 \ll 1$ and consequently $\gamma_z \approx 1/\tau_2\chi_{zz}^0$. If $\beta + H_0^{(e)}/\mathcal{M}_0 \gg 1$, we have

$$\gamma_z = \frac{1}{\tau_2\chi_{zz}^0}, \quad \gamma_\perp = \left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\left(\beta + \frac{H_0^{(e)}}{\mathcal{M}_0}\right). \tag{1.4.51}$$

1.5 Spin Waves in Antiferromagnets

1.5.1 Equilibrium States of Antiferromagnets

1. *Motion equations of magnetic moments*

If we do not take dissipation of energy into account, the equation of motion for the magnetizations $\mathcal{M}_1(r,t)$ and $\mathcal{M}_2(r,t)$ of the two magnetic sublattices will be of the form

$$\begin{cases} \dfrac{\partial \mathcal{M}_1}{\partial t} = g[\mathcal{M}_1, \mathcal{H}_1], \\ \dfrac{\partial \mathcal{M}_2}{\partial t} = g[\mathcal{M}_2, \mathcal{H}_2], \end{cases} \tag{1.5.1}$$

where g is the gyromagnetic ratio and $\mathcal{H}_1$ and $\mathcal{H}_2$ are the effective magnetic fields acting on the moments $\mathcal{M}_1(r,t)$ and $\mathcal{M}_2(r,t)$

$$\mathcal{H}_1 = \frac{\delta W}{\delta \mathcal{M}_1}, \quad \mathcal{H}_2 = \frac{\delta W}{\delta \mathcal{M}_2}, \tag{1.5.2}$$

W is the energy of the antiferromagnets. Using the expression of W, we obtain

$$\begin{cases} \mathcal{H}_1 = H^{(i)} - \dfrac{\partial \mathcal{F}}{\partial \mathcal{M}_1} + \dfrac{\partial}{\partial x_k} \dfrac{\partial \mathcal{F}}{\partial(\partial \mathcal{M}_1/\partial x_k)}, \\ \mathcal{H}_2 = H^{(i)} - \dfrac{\partial \mathcal{F}}{\partial \mathcal{M}_2} + \dfrac{\partial}{\partial x_k} \dfrac{\partial \mathcal{F}}{\partial(\partial \mathcal{M}_2/\partial x_k)}, \end{cases} \tag{1.5.3}$$

where $H^{(i)}$ is the magnetic field inside the antiferromagnets.

We note that the equations of motion for the magnetizations are consistent with the conservation of energy and lead to the following expression for the energy flux density in a ferromagnet

$$\begin{aligned} \pi_i = {} & \frac{c}{4\pi}[\vec{E}, H^{(m)}]_i - \alpha_{ij}\left(\frac{\partial \mathcal{M}_1}{\partial x_j}\frac{\partial \mathcal{M}_1}{\partial t} + \frac{\partial \mathcal{M}_2}{\partial x_j}\frac{\partial \mathcal{M}_2}{\partial t}\right) \\ & - \alpha'_{ij}\left(\frac{\partial \mathcal{M}_1}{\partial x_j}\frac{\partial \mathcal{M}_2}{\partial t} + \frac{\partial \mathcal{M}_2}{\partial x_j}\frac{\partial \mathcal{M}_1}{\partial t}\right). \end{aligned} \tag{1.5.4}$$

2. *Equilibrium states of antiferromagnets*

Consider the equilibrium values of the magnetizations $\mathcal{M}_{10}, \mathcal{M}_{20}$ of the two sublattices about which the oscillations of the magnetizations $\mathcal{M}_1$ and $\mathcal{M}_2$ take place. To do this we must equate the effective magnetic fields to zero. The equilibrium state of the antiferromagnets corresponding to the direction of magnetic moments of sublattices for which the energy density of the antiferromagnets

$$\begin{aligned} w = {} & \delta \mathcal{M}_1 \cdot \mathcal{M}_2 - \frac{1}{2}\beta[(\mathcal{M}_1 \cdot \vec{n})^2 + (\mathcal{M}_2 \cdot \vec{n})^2] \\ & - \beta'(\mathcal{M}_1 \cdot \vec{n}) \cdot (\mathcal{M}_2 \cdot \vec{n}) - H_0^{(e)}(\mathcal{M}_1 + \mathcal{M}_2) \end{aligned}$$

reaches a minimum. We have neglected in this expression the energy w_d which is responsible for the appearance of weak ferromagnetism and the energy

$$2\pi(\mathcal{M}_1 - \mathcal{M}_2)\hat{N}(\mathcal{M}_1 + \mathcal{M}_2)$$

which depends on the shape of the body. Let us assume that the external magnetic field is zero. If at the same time, $\beta - \beta' > 0$, it is readily to verify that the minimum of w is reached when the magnetic moments of the sublattices lie along the anisotropy axis and $\mathcal{M}_{10}+\mathcal{M}_{20}=0$. Such antiferromagnets are said to have magnetic anisotropy of the "easy axis" type.

When $\beta-\beta'<0$, the minimum of w is reached when the magnetic moments of the sublattices are perpendicular to the anisotropy axis and $\mathcal{M}_{10}+\mathcal{M}_{20}=0$. In this case it is said that the antiferromagnets has a magnetic anisotropy of "easy plane" type. Examples of the "easy axis" anisotropy are $CuCl_2.2H_2O, Cr_2O_3$ and $FeCO_3$; antiferromagnets with magnetic anisotropy of the "easy plane" type are hematite in its low temperature phase, the carbonates and fluorides of transition metals.

The equilibrium directions of $\mathcal{M}_{10}$ and $\mathcal{M}_{20}$ are readily determined even in the presence of an external magnetic field $H_0^{(e)}$. The results of the corresponding calculations are given in Figures 1.5.1 and 1.5.2.

1.5.2 Spin Waves in Antiferromagnets

1. *Motion of spin waves in antiferromagnets*

Consider the spin waves in antiferromagnets. Substitute

$$\begin{cases} \mathcal{M}_1(r,t)=\mathcal{M}_{10}+\vec{m}_1(r,t), \\ \mathcal{M}_2(r,t)=\mathcal{M}_{20}+\vec{m}_2(r,t), \\ H^{(i)}(r,t)=H_0^{(i)}+\vec{h}(r,t). \end{cases} \tag{1.5.5}$$

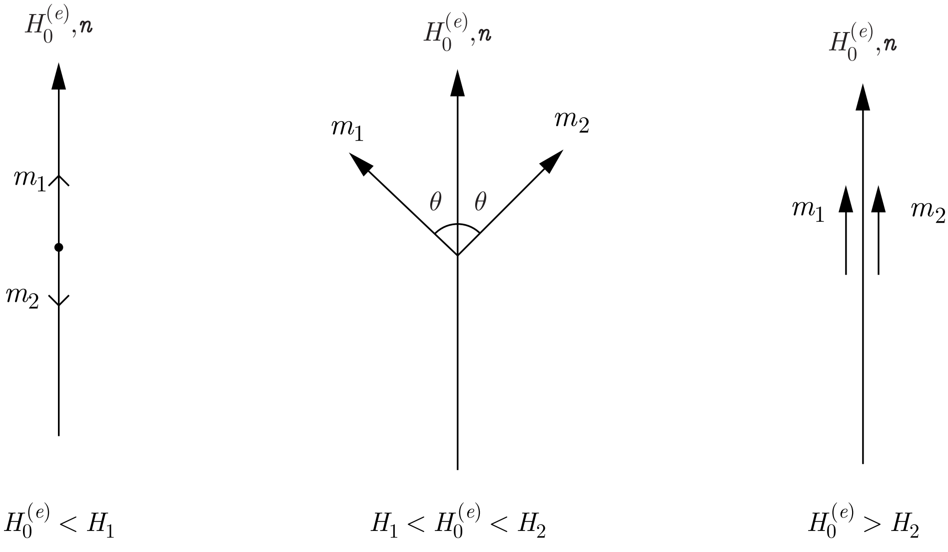
into (1.5.1) where $\vec{m}_1,\vec{m}_2$ and $\vec{h}$ are small deviations from the equilibrium values $\mathcal{M}_{10}$, $\mathcal{M}_{20}$ and $H_0^{(i)}$, and then linearize these equations. This results in a set of two linear differential equations for $\vec{m}_1$ and $\vec{m}_2$. We can now express the Fourier component of the deviations of the magnetic moments by $\vec{m}_1(\vec{k},\omega)$, $\vec{m}_2(\vec{k},\omega)$ and $\vec{h}(\vec{k},\omega)$

$$\vec{m}(\vec{k},\omega)=\vec{m}_1(\vec{k},\omega)+\vec{m}_2(\vec{k},\omega)=\chi(\vec{k},\omega)\vec{h}(\vec{k},\omega), \tag{1.5.6}$$

where $\chi(\vec{k},\omega)$ is a tensor which depends on $\vec{k}$ and ω and on the quantities characterizing the equilibrium state of the antiferromagnets. If the magnetic field $H_0^{(e)}$ is parallel to the anisotropy axis and $H_0^{(e)}<H_1$, $H_1=\mathcal{M}_0\sqrt{2\delta(\beta-\beta')}$, then the linearized equations for $\vec{m}_1(\vec{k},\omega)$ and $\vec{m}_2(\vec{k},\omega)$ are of the form

$$\begin{cases} -i\omega\vec{m}_1(\vec{k},\omega)=g\Big\{\mathcal{M}_{10},\vec{h}(\vec{k},\omega)-\left(\delta+\dfrac{H_0^{(e)}}{\mathcal{M}_0}+\beta-\beta'+\alpha_{ij}k_ik_j\right)\cdot\vec{m}_1(\vec{k},\omega) \\ \qquad\qquad -(\delta+\alpha'_{ij}k_ik_j)\vec{m}_2(\vec{k},\omega)\Big\}, \\ -i\omega\vec{m}_2(\vec{k},\omega)=g\Big\{\mathcal{M}_{20},\vec{h}(\vec{k},\omega)-\left(\delta+\dfrac{H_0^{(e)}}{\mathcal{M}_0}+\beta-\beta'+\alpha_{ij}k_ik_j\right)\cdot\vec{m}_2(\vec{k},\omega) \\ \qquad\qquad -(\delta+\alpha'_{ij}k_ik_j)\vec{m}_1(\vec{k},\omega)\Big\}, \end{cases} \tag{1.5.7}$$

$$\beta - \beta' > 0$$



$$H_1 = m_0\sqrt{2\delta(\beta - \beta')}, \quad H_2 = 2\delta m_0\,, \quad \cos\theta = \frac{H_0^{(e)}}{H_2}$$

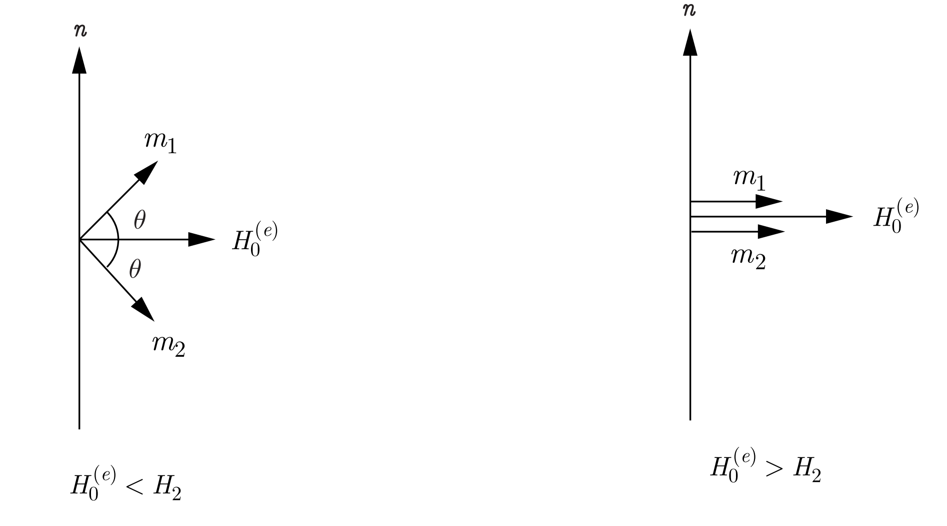


Figure 1.5.1. *Numerical results for the equilibrium states of an antiferromagnet* (1).

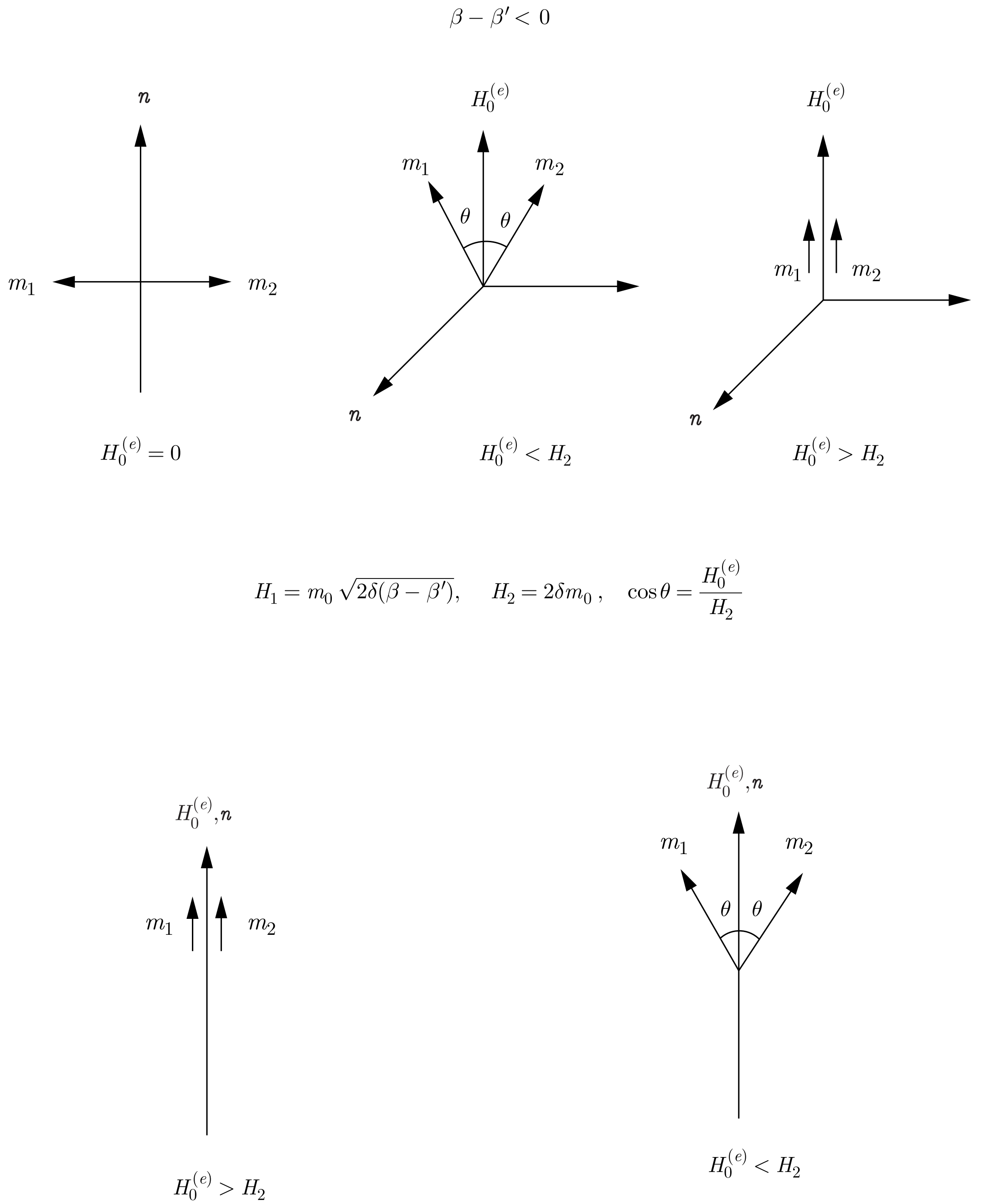


Figure 1.5.2. *Computational results for the equilibrium states of an antiferromagnet* (2).

$$\hat{\chi}(\vec{k},\omega)=\begin{pmatrix}\chi_{xx} & \chi_{xy} & 0\\ \chi_{yx} & \chi_{yy} & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{1.5.8}$$

where

$$\begin{cases}\chi_{xx}=\chi_{yy}=\dfrac{1}{2}\chi_0\left(\dfrac{\Omega_+(\Omega_+-gH_0^{(e)})}{\Omega_+^2-\omega^2}+\dfrac{\Omega_-(\Omega_-+gH_0^{(e)})}{\Omega_-^2-\omega^2}\right),\\ \chi_{xy}=-\chi_{yx}=i\omega\chi_0\left(\dfrac{(\Omega_+-gH_0^{(e)})}{\Omega_+^2-\omega^2}+\dfrac{(\Omega_-+gH_0^{(e)})}{\Omega_-^2-\omega^2}\right),\\ \Omega_\pm=g\mathcal{M}_0\sqrt{2\delta(\alpha_{ij}-\alpha_{ij'})k_ik_j+(H_1/\mathcal{M}_0)^2}\pm gH_0^{(e)},\end{cases} \tag{1.5.9}$$

in which $\chi_0=\delta^{-1}$.

Proceeding in an analogous fashion, we can determine the high-frequency magnetic susceptibility tensor of an antiferromagnet when the magnetic field $H_0^{(i)}$ is at right angles to the anisotropy axis:

$$\hat{\chi}(\vec{k},\omega)=\begin{pmatrix}\chi_{xx} & 0 & 0\\ 0 & \chi_{yy} & \chi_{yz}\\ 0 & \chi_{zy} & \chi_{zz}\end{pmatrix}, \tag{1.5.10}$$

where

$$\begin{cases}\chi_{xx}=\chi_0\dfrac{\Omega_2^2}{\Omega_2^2-\omega^2}, \quad \chi_{yy}=\chi_0\dfrac{\Omega_1^2}{\Omega_1^2-\omega^2},\\ \chi_{zz}=\chi_0\dfrac{(gH_0^{(e)})^2}{\Omega_1^2-\omega^2}, \quad \chi_{yz}=-\chi_{zy}=i\omega\chi_0\dfrac{gH_0^{(e)}}{\Omega_1^2-\omega^2},\\ \Omega_1=g\mathcal{M}_0\sqrt{2\delta(\alpha_{ij}-\alpha_{ij'})k_ik_j+(H_1/\mathcal{M}_0)^2+(H_0^{(e)}/\mathcal{M}_0)^2},\\ \Omega_2=g\mathcal{M}_0\sqrt{2\delta(\alpha_{ij}-\alpha_{ij'})k_ik_j+(H_1/\mathcal{M}_0)^2}.\end{cases} \tag{1.5.11}$$

It is assumed that $H_0^{(e)}\ll H_2$ and $H_2=2\delta\mathcal{M}_0$ (the z-axis lies along the anisotropy axis and the x-axis lies along $H_0^{(e)}$).

Consider now an antiferromagnet with magnetic anisotropy of the "easy plane" type. If the field $H_0^{(e)}$ is perpendicular to the anisotropy axis and $H_0^{(e)}\ll H_2$, the high-frequency magnetic susceptibility tensor is given by

$$\hat{\chi}(\vec{k},\omega)=\begin{pmatrix}\chi_{xx} & 0 & 0\\ 0 & \chi_{yy} & \chi_{yz}\\ 0 & \chi_{zy} & \chi_{zz}\end{pmatrix}, \tag{1.5.12}$$

where

$$\begin{cases} \chi_{xx} = \chi_0 \dfrac{\Omega_2'^2}{\Omega_2'^2 - \omega^2}\,, \quad \chi_{yy} = \chi_0 \dfrac{(gH_0^{(e)})^2}{\Omega_1'^2 - \omega^2}\,, \\ \chi_{zz} = \chi_0 \dfrac{\Omega_1'^2}{\Omega_1'^2 - \omega^2}\,, \quad \chi_{yz} = -\chi_{zy} = -i\omega\chi_0 \dfrac{gH_0^{(e)}}{\Omega_1'^2 - \omega^2}\,, \\ \Omega_1' = g\mathcal{M}_0 \sqrt{2\delta(\alpha_{ij} - \alpha_{ij'})k_i k_j + (H_0^{(e)}/\mathcal{M}_0)^2}, \\ \Omega_2' = g\mathcal{M}_0 \sqrt{2\delta(\alpha_{ij} - \alpha_{ij'})k_i k_j + (H_1/\mathcal{M}_0)^2} \end{cases} \tag{1.5.13}$$

(the z-axis lies along the anisotropy axis and the x-axis lies along the magnetic field $H_0^{(e)}$).

If the magnetic field $H_0^{(e)}$ is parallel to the anisotropy axis, then

$$\hat{\chi}(\vec{k}, \omega) = \begin{pmatrix} \chi_{xx} & \chi_{xy} & 0 \\ \chi_{yx} & \chi_{yy} & 0 \\ 0 & 0 & \chi_{zz} \end{pmatrix}, \tag{1.5.14}$$

where

$$\begin{cases} \chi_{xx} = \chi_0 \dfrac{(gH_0^{(e)})^2}{\Omega_1''^2 - \omega^2}\,, \quad \chi_{yy} = \chi_0 \dfrac{\Omega_1''^2}{\Omega_1''^2 - \omega^2}\,, \\ \chi_{zz} = \chi_0 \dfrac{\Omega_2''^2}{\Omega_2''^2 - \omega^2}\,, \quad \chi_{yz} = -\chi_{zy} = i\omega\chi_0 \dfrac{gH_0^{(e)}}{\Omega_1''^2 - \omega^2}, \\ \Omega_1'' = g\mathcal{M}_0 \sqrt{2\delta(\alpha_{ij} - \alpha_{ij'})k_i k_j + (H_1/\mathcal{M}_0)^2 + (H_0^{(e)}/\mathcal{M}_0)^2}, \\ \Omega_2'' = g\mathcal{M}_0 \sqrt{2\delta(\alpha_{ij} - \alpha_{ij'})k_i k_j (1 - (H_1/\mathcal{M}_0)^2)}. \end{cases} \tag{1.5.15}$$

Here the z-axis lies along the anisotropy axis and the x-axis lies in the plane of the magnetic moments of the sublattices.

2. *The spin-wave spectrum in antiferromagnets*

If we know the high-frequency magnetic susceptibility tensor of an antiferromagnet, we can readily find its spin-wave spectrum. To do this we must use the general dispersion relation (1.4.42) which determines the spin-wave spectrum in the magnetostatic approximation of both ferromagnets and antiferromagnets,

$$k^2 + 4\pi k_i k_j \chi_{ij}(\vec{k}, \omega) = 0. \tag{1.5.16}$$

However, in the case of an antiferromagnet, this equation needs not be solved since the components of the tensor $\chi_{ij}(\vec{k}, \omega)$ for an antiferromagnet are proportional to a small parameter χ_0 and therefore to within terms of the order of $g\mathcal{M}_0\chi_0$, the spin-wave frequencies must coincide with the poles of the tensor $\hat{\chi}(\vec{k}, \omega)$. Hence it follows, for example, that in the case of antiferromagnets with magnetic anisotropy of the "easy axis" type, the spin-wave frequencies are given by

$$\begin{cases} \omega_{S1} \equiv \Omega_+ = g\mathcal{M}_0\sqrt{2\delta(\alpha_{ij} - \alpha'_{ij})k_ik_j + (H_1/\mathcal{M}_0)^2} + gH_0^{(e)}, \\ \omega_{S2} \equiv \Omega_- = g\mathcal{M}_0\sqrt{2\delta(\alpha_{ij} - \alpha'_{ij})k_ik_j + (H_1/\mathcal{M}_0)^2} - gH_0^{(e)}, \end{cases} \tag{1.5.17}$$

if the field $H_0^{(e)}$ is parallel to the anisotropy axis and $H_0^{(e)} < H_1$, and by

$$\begin{cases} \omega_{S1} \equiv \Omega_1 = g\mathcal{M}_0\sqrt{2\delta(\alpha_{ij} - \alpha'_{ij})k_ik_j + (H_1/\mathcal{M}_0)^2 + (H_0^{(e)}/\mathcal{M}_0)^2}, \\ \omega_{S2} \equiv \Omega_2 = g\mathcal{M}_0\sqrt{2\delta(\alpha_{ij} - \alpha'_{ij})k_ik_j + (H_1/\mathcal{M}_0)^2}, \end{cases} \tag{1.5.18}$$

if the field $H_0^{(e)}$ is perpendicular to the anisotropy axis.

In the case of antiferromagnets with magnetic anisotropy of the "easy plane" type, the spin-wave frequencies are given by

$$\begin{cases} \omega_{S1} \equiv \Omega''_1 = g\mathcal{M}_0\sqrt{2\delta(\alpha_{ij} - \alpha'_{ij})k_ik_j + (H_1/\mathcal{M}_0)^2 + (H_0^{(e)}/\mathcal{M}_0)^2}, \\ \omega_{S2} \equiv \Omega''_2 = g\mathcal{M}_0\sqrt{2\delta(\alpha_{ij} - \alpha'_{ij})k_ik_j(1 - (H_1/\mathcal{M}_0)^2)}, \end{cases} \tag{1.5.19}$$

if the magnetic field $H_0^{(e)}$ is parallel to the anisotropy axis, and by

$$\omega_{S1} \equiv \Omega'_1 = g\mathcal{M}_0\sqrt{2\delta(\alpha_{ij} - \alpha'_{ij})k_ik_j + (H_0^{(e)}/\mathcal{M}_0)^2},$$
$$\omega_{S2} \equiv \Omega'_2 = g\mathcal{M}_0\sqrt{2\delta(\alpha_{ij} - \alpha'_{ij})k_ik_j + (H_1/\mathcal{M}_0)^2}$$

if the magnetic field $H_0^{(e)}$ is perpendicular to the anisotropy axis.

1.5.3 Electromagnetic Waves in Magnetically Ordered Crystals

1. *Dispersion relation for electromagnetic waves*

First of all we discuss the dispersion relation for electromagnetic waves. Assume that the magnetic moments have magnetostatic oscillations. This leads to

$$\vec{m}(\vec{k},\omega) = \hat{\chi}(\vec{k},\omega)\cdot\vec{h}(\vec{k},\omega), \tag{1.5.20}$$

where $\vec{m}(\vec{k},\omega)$ and $\vec{h}(\vec{k},\omega)$ are the amplitudes of the oscillating components of the magnetization and the the magnetic field, and $\hat{\chi}(\vec{k},\omega)$ is the high-frequency magnetic susceptibility tensor of a ferromagnet or an antiferromagnets.

Maxwell's equations for plane waves with allowance for (1.5.20) are of the form

$$[k,e] = \frac{\omega}{c}b, \quad [k,h] = -\frac{\omega}{c}d, \tag{1.5.21}$$

where $b = \hat{\mu}\vec{h}$ and $d = \hat{\epsilon}e$ are the amplitudes of the oscillating components of the magnetic and electric induction, $\hat{\mu}(\vec{k},\omega) = 1 + 4\pi\hat{\chi}(\vec{k},\omega)$ is the magnetic permeability tensor and $\hat{\epsilon}$ is the permittivity tensor. We shall assume for simplicity that $\epsilon_{ij} = \epsilon\delta_{ij}$ and that ϵ is independent of $\vec{k}$ and ω. Equating the determinant of (1.5.21) to zero, we obtain the dispersion relation connecting the frequency and the wave vectors of the electromagnetic waves in magnetically ordered crystals

$$D(\vec{k},\omega) = A(\vec{k},\omega)n^4 + B(\vec{k},\omega)n^2 + C(\vec{k},\omega), \tag{1.5.22}$$

where $n = ck/\omega\sqrt{\epsilon}$ is the refractive index,

$$\begin{aligned} A(\vec{k},\omega) &= 1 + \frac{4\pi}{k^2}k_i k_j \chi_{ij}(\vec{k},\omega), \\ B(\vec{k},\omega) &= \left(\frac{k_i k_j}{k^2} - \delta_{ij}\right)\Delta_{ij}(\vec{k},\omega), \\ C(\vec{k},\omega) &= \det\mu_{ij}(\vec{k},\omega), \end{aligned} \tag{1.5.23}$$

and $\Delta_{ij}(\vec{k},\omega)$ are the minors of the determinant $|\mu_{ij}(\vec{k},\omega)|$. This dispersion relation will in general define not one but several frequencies for a given wave vector $\vec{k}$. For a ferromagnet there are three such frequencies, whereas for an antiferromagnet there are four. Different frequencies corresponding to the same $\vec{k}$ define different branches of the oscillations. In fact, dividing (1.5.22) by n^4, and allowing n to tend to infinity, we obtain

$$A(\vec{k},\omega) = 0, \tag{1.5.24}$$

which is the same as the dispersion relation

$$1 + \frac{4\pi}{k^2}k_i k_j \chi_{ij}(\vec{k},\omega) \tag{1.5.25}$$

for the spin waves.

Since the spin wave frequency $\omega_S(\vec{k})$ is of the order of $g\mathcal{M}_0$, it may be said that spin waves correspond to wave vector $\vec{k}$ satisfying the inequality

$$k \gg \frac{g\mathcal{M}_0}{c} \tag{1.5.26}$$

or in terms of wave length

$$\lambda \ll \frac{c}{g\mathcal{M}_0}.$$

When $k \gg \frac{g\mathcal{M}_0}{c}$, in addition to the spin wave there are also two proper electromagnetic waves characterized by the dispersion relation

$$\omega = \frac{ck}{\sqrt{\epsilon}}. \tag{1.5.27}$$

Our problem now is to determine the properties of the branches of electromagnetic oscillations for $k \ll \frac{g\mathcal{M}_0}{c}$.

Since $\frac{1}{\sqrt{\alpha}} \gg \frac{g\mathcal{M}_0}{c}$, it follows that in this region of wave vectors $\alpha k \ll 1$ and, consequently, the spatial dispersion of the high-frequency magnetic susceptibility tensor is unimportant. This means that the coefficients A, B and C in the dispersion relation (1.5.22) will depend only on the frequency in the direction of the wave vector when $k \ll \frac{g\mathcal{M}_0}{c}$, but not on its magnitude.

Under these conditions the dispersion relation can conveniently be looked upon as the equation for n or, what amounts to the same thing, as an equation for the modulus of the wave vector $\vec{k}$ for given frequency and directions of propagation.

The solution of this equation is of the form

$$n_{1,2}^2 = \left(\frac{ck_{1,2}}{\omega\sqrt{\epsilon}}\right)^2 = \frac{-B(\kappa,\omega) \pm \sqrt{B^2(\kappa,\omega) - 4A(\kappa,\omega)C(\kappa,\omega)}}{2A(\kappa,\omega)}, \tag{1.5.28}$$

where $\kappa = \frac{\vec{k}}{k}$.

Real refractive indices n_i correspond to waves propagating with phase velocity

$$v_i(\kappa,\omega) = \frac{c}{\sqrt{\epsilon}n_i(\kappa,\omega)}, \tag{1.5.29}$$

which is a function of κ and ω.

2. *Interaction between proper electromagnetic waves and spin waves*

Further study of the branches of electromagnetic oscillations for $k \leq \frac{g\mathcal{M}_0}{c}$ requires detailed knowledge of the high-frequency magnetic susceptibility tensor. Here we shall confine our attention to a uniaxial ferromagnet and will assume that the external magnetic field lies along the anisotropy axis. The tensor $\hat{\mu}(\vec{k},\omega)$ is then of the form

$$\hat{\mu} = \begin{Bmatrix} \mu & i\mu' & 0 \\ -i\mu' & \mu & 0 \\ 0 & 0 & 1 \end{Bmatrix}, \tag{1.5.30}$$

where

$$\begin{cases} \mu(\omega) = 1 + \dfrac{4\pi g\mathcal{M}_0\Omega_0}{\Omega_0 - \omega^2}, \\ \mu'(\omega) = 1 + \dfrac{4\pi g\mathcal{M}_0\omega}{\Omega_0 - \omega^2}, \\ \Omega_0 = g\mathcal{M}_0\left(\beta + \dfrac{H_0^{(i)}}{\mathcal{M}_0}\right), \end{cases} \tag{1.5.31}$$

and the z-axis lies along the anisotropy axis.

Using the unit vectors

$$\vec{J}_1 = \frac{[\vec{n},\vec{k}]}{|[\vec{n},\vec{k}]|}, \quad \vec{J}_2 = \frac{[\vec{k},[\vec{n},\vec{k}]]}{|[\vec{k},[\vec{n},\vec{k}]]|}, \quad \vec{J}_3 = \frac{\vec{k}}{k}, \tag{1.5.32}$$

where $\vec{n}$ is a unit vector along the anisotropy axis, we can write the vectors $\vec{b}, \vec{h}$ and $\vec{e}$ in the form

$$\begin{cases} \vec{b} = b_1\vec{J}_1 + b_2\vec{J}_2, \\ \vec{h} = h_1\vec{J}_1 + h_2\vec{J}_2 + h_3\vec{J}_3, \\ \vec{e} = e_1\vec{J}_1 + e_2\vec{J}_2. \end{cases} \tag{1.5.33}$$

Eliminating the vector $\vec{e}$ from (1.5.21) we obtain

$$\vec{k}(\vec{k}\cdot\vec{h}) - k^2\vec{h} = -\frac{\omega^2\epsilon}{c^2}\vec{b}, \tag{1.5.34}$$

and hence

$$b_1 = n^2 h_1, \quad b_2 = n^2 h_2. \tag{1.5.35}$$

Since $\vec{b} = \hat{\mu}\vec{h}$, it follows that

$$\begin{cases} \left\{n^2 - \dfrac{\mu\cos^2\theta_k + (\mu^2 - \mu'^2)\sin^2\theta_k}{\mu\sin^2\theta_k + \cos^2\theta_k}\right\} b_1 - i\dfrac{\mu'\cos\theta_k}{\mu\sin^2\theta_k + \cos^2\theta_k} b_2 = 0, \\ i\dfrac{\mu'\cos\theta_k}{\mu\sin^2\theta_k + \cos^2\theta_k} b_1 + \left\{n^2 - \dfrac{\mu}{\mu\sin^2\theta_k + \cos^2\theta_k}\right\} b_2 = 0, \end{cases} \tag{1.5.36}$$

where θ_k is the angle between the wave vector $\vec{k}$ and the anisotropy axis.

If we eliminate the amplitudes b_1 and b_2 from these equations, we obtain the following values for the refractive index

$$\begin{aligned} n_{1,2}^2 = {} & \frac{1}{2}(\mu\sin^2\theta_k + \cos^2\theta_k)^{-1}\{\mu(1+\cos^2\theta_k) + (\mu^2 - \mu'^2)\sin^2\theta_k \\ & \pm\sqrt{(\mu^2 - \mu'^2 - \mu)\sin^4\theta_k} + 4\mu'^2\cos^2\theta_k\}. \end{aligned} \tag{1.5.37}$$

These formulae are a special case of (1.4.79) for a uniaxial crystal. The value of b_1 and b_2 for waves with refractive index n_j are related by

$$\frac{b_1^{(j)}}{b_2^{(j)}} = i\rho_j, \tag{1.5.38}$$

where

$$\rho_j = \frac{n_j^2\cos^2\theta_k - \mu(1 - n_j^2\sin^2\theta_k)}{\mu'\cos\theta_k}. \tag{1.5.39}$$

We note that

$$\rho_1\rho_2 = -1 \tag{1.5.40}$$

and consequently

$$\frac{b_1^{(1)}}{b_2^{(1)}} = -\frac{b_2^{(2)}}{b_1^{(2)}} = i\rho_1. \tag{1.5.41}$$

It follows from (1.5.36) that

$$\frac{h_1^{(j)}}{h_2^{(j)}} = i\rho_j, \quad \frac{h_3^{(j)}}{h_2^{(j)}} = \frac{\mu' \sin\theta_k + (\mu - 1)\sin\theta_k\cos\theta_k)}{\mu\sin^2\theta_k + \cos^2\theta_k},$$

where as before the subscript j represents waves with refractive index n_j.

Let us now return to the expressions given by (1.5.37) for the refractive indices and find the values of ω for which the wave vector is zero. Since the right-hand sides of (1.5.37) have finite limits, which are equal to $\mu(0)$ and

$$\frac{\mu(0)}{\cos^2\theta_k + \mu(0)\sin^2\theta_k}, \tag{1.5.42}$$

it follows that k will be zero together with ω, and when $\omega \ll g\mathcal{M}_0$,

$$k_1 = \omega\frac{\sqrt{\epsilon\mu(0)}}{c}, \quad k_2 = \omega\frac{\sqrt{\epsilon\mu(0)}}{c}\frac{1}{\sqrt{\cos^2\theta_k + \mu(0)\sin^2\theta_k}}. \tag{1.5.43}$$

Moreover, the wave vector will vanish for a certain value of ω of the order of $g\mathcal{M}_0$ and, in particular, for

$$\omega = \omega_0 = g(H_0^{(i)} + 4\pi\mathcal{M}_0 + \beta\mathcal{M}_0). \tag{1.5.44}$$

3. *Properties of the branches of electromagnetic oscillations in ferromagnets*

The above results can be used to obtain a schematic representation of the properties of the branches of electromagnetic oscillations in uniaxial ferromagnets. In Figure 1.5.3 spin waves correspond to the broken curve $\omega = \omega_S(\vec{k})$, whilst the proper electromagnetic waves correspond to the broken curve $\omega = ck/\sqrt{\epsilon}$. This curve tends asymptotically to a part of branch I for $k \gg g\mathcal{M}_0/c$, and the straight line is the common asymptote of branches II and III (also for $k \gg g\mathcal{M}_0/c$). It is readily shown that the deviation of these curves from the straight line $\omega = ck/\sqrt{\epsilon}$ for $k \gg g\mathcal{M}_0/c$ is given by

$$\omega_{\rm II,III} = \frac{ck}{\sqrt{\epsilon}}\left(1 \pm \frac{2\pi g\mathcal{M}_0}{ck}\sqrt{\epsilon}\cos\theta_k\right). \tag{1.5.45}$$

Consider now the polarization properties of the above branches of electromagnetic oscillations. It is readily shown that as $k \to 0$, the induction vector $\vec{b}$ lies along the vector $\vec{J}_1$ for oscillations in branch I, and along the vector $\vec{J}_2$ for oscillations in branch II. Branch III is characterized by elliptical polarization for $k \to 0$, and

$$\frac{b_1^{\rm III}}{b_2^{\rm III}} = \frac{1}{\cos\theta_k}.$$

For large $k \gg g\mathcal{M}_0/$, branches II and II have circular polarizations.

For large k the magnetic field for branches II and III is transverse and for branch I longitudinal. For small k, on the other hand, there are both transverse and longitudinal magnetic field components, and both are of the same order of magnitude.

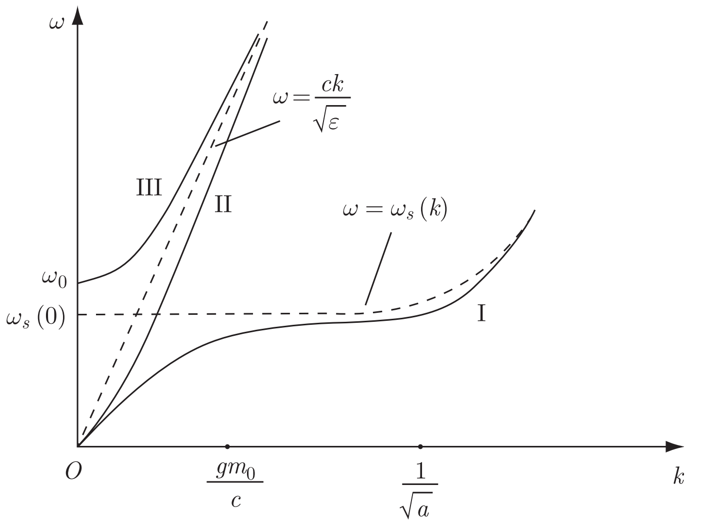


Figure 1.5.3. *Electromagnetic oscillating bifurcation property of a uni-axial ferromagnet.*

1.6 Bibliography Comments

In this chapter, we give the physics backgrounds and the derivations for the Landau–Lifshitz equations. There are two different approaches to derive Landau–Lifshitz equations. One is to consider the macroscopic motion of the ferromagnets and evaluate the total energy, then derive Landau–Lifshitz equations from the Hamiltonian (see the original paper by Landau and Lifshitz [102], [140]). The other is to begin with the microscopic aspects. Applying the quantum-mechanical spin theory, one can make out the Hamiltonian for the Heisenberg chain, and then get the Landau–Lifshitz equations and at the same time the equations of ferromagnets. For the ferromagnetic equations for multi-media and the propagations and interactions of spin waves in ferromagnets, we refer to the book "Spin waves" by Akhiezer *et al.* [3] and references therein.