Preface

In studying the dispersive theory of magnetization of ferromagnets in 1935, Landau–Lifshitz [102] proposed the equations of ferromagnetic spin chain which are important magnetization equations, called Landau–Lifshitz equations now. Later on, such equations were also found in the condensed matter physics. In the 1960s, Soviet physicists A. Z. Akhiezer, V. G. Beryahltar, S. V. Peletninskii studied spin wave, the equations of ferromagnetic chain and the traveling wave solutions in detail in their book "Spin Waves"[3]. In 1974, K. Nakamura, T. Sasada [122] first observed that there is a soliton solution to the one-dimensional Landau–Lifshitz equations without Gilbert damping. Then, many mathematicians and physicists studied the soliton theory of Landau–Lifshitz equations using the approaches including inverse scattering method, infinite many conservation laws, geometry expression method and gauge equivalence of nonlinear Schrödinger equations and so on. Early in 1957, Suhl [131] had studied the infinite dimensional dynamic system of the Landau–Lifshitz equations with Gilbert damping term. A series of further studies on the theory of dynamics and numerical results have appeared since then. In recent years, the ferromagnetic materials have been widely applied in the video and recording apparatus. This is one of the applications for Landau–Lifshitz equations.

From 1982, mathematicians began their studies on the well-posedness for Landau–Lifshitz equations. In China, a group headed by Yulin Zhou and Boling Guo proved the existence of the global weak solutions to the initial value problems and initial boundary value problems for Landau–Lifshitz equations from one dimension to multi-dimensions [150–157]. Alouges and Soyeur [4] proved similar results by penalty method in 1992. We also refer the readers to the result by P L. Sulem, C. Sulem and C. Bardos [132]. Since then, many other results on the global existence were obtained [20–22]. However, the regularity and the uniqueness were unsolved in the 1980s due to the complexity of Landau–Lifshitz equations.

However, in 1991, Zhou, Guo and Tan [158] obtained the existence and uniqueness of global smooth solution to one-dimensional Landau–Lifshitz equations with or without Gilbert damping by using a mobile frame on S^2 and some fine *a priori* estimates.

In 1993, Guo and Hong began the studies on two-dimensional Landau–Lifshitz equations. They established in [77] the relations between two-dimensional Landau–Lifshitz equations and harmonic maps and applied the approaches studying harmonic maps to get the global existence and uniqueness of partially regular weak solution.

This conclusion has been cited by many others up to now and gives rise to many successive works (see, for example, [47–49, 52, 89, 113, 115, 143]). Later on, in 1998, Chen, Ding and Guo [29] further proved that all the weak solutions with finite energy must be the Chen–Struwe solutions [34]. The uniqueness was also given. This says that the weak solution with finite energy is globally smooth with exception of finitely many singular points at most.

From 1998 to 2001, Guo and Ding discussed many other Landau–Lifshitz equations such as inhomogeneous equations, unsaturated equations and compressible equations [50, 51, 75, 108].

From the beginning of the new century, more and more mathematicians are interested in the researches of Landau–Lifshitz equations. We refer the readers to the works by Guo, Su, Carbou and Harpes, *et al.* [20–23, 80–84, 89] on Landau–Lifshitz equations and Landau–Lifshitz–Maxwell equations.

A natural question is the regularity of weak solutions to the higher dimensional Landau–Lifshitz equations. In this aspect, in 2004, Liu [109] proved that the "stationary" weak solutions of higher dimensional Landau–Lifshitz equations are partially regular. The Hausdorff dimensions and the Hausdorff measures of the singular set were estimated. These extend the results on harmonic map heat flow by Feldman [58] to Landau–Lifshitz equations. At the same time, Moser [115] obtained the similar results for lower dimensional Landau–Lifshitz equations by different methods.

We know that the "stationary" conditions are hard to verify. So, in 2005, Melcher [113] proved the partial regularity for the weak solutions to the initial value problems of Landau–Lifshitz equations. However, as stated by Melcher, his method does not fit the other dimensional problems and, the partial regularity of weak solutions to the boundary value problems are still unsolved.

This attracted the attention of Changyou Wang at the University of Kentucky. Wang [143], using the method of [142], proved the partial regularity for the weak solutions of the initial value problems and initial boundary value problems on three- and four-dimensional manifolds.

However, all the results on the partial regularity only answered the questions on the singular set such as how many points there are in the set or how large the set is, provided that the singularity does exist. But, the existence of finite time singularity of weak solutions is not answered. For the harmonic map heat flow, the similar questions were answered by Chen and Ding [30] in 1990 ($n \geq 3$) and by Chang, Ding and Ye [26] ($n = 2$) in 1992 (see also [39] and many others).

Does the weak solution of Landau–Lifshitz equations really blow-up at finite time? Pistella and Valente [123] in 2002, Bartels, Ko and Prohl [13] in 2005, gave positive answers respectively by numerical analysis.

In 2007, Ding and Wang [52] rigorously proved that in three and four dimensions, some Dirichlet problems and Neumann problems for Landau–Lifshitz equations indeed admit finite time blow-up solutions. Comparing with the similar proofs for harmonic map heat flows, we do not have the monotonicity inequality and the Bochner identity.

Unfortunately, our method does not apply to two-dimensional problems and higher dimensional problems. We do not know the types of the singularities either.

In recent years, there are many papers discussing the other problems such as domain wall, energy concentrations and vortices. We refer to [44–46, 98–100, 116, 117] and references therein.

The aim of this book is to introduce the readers the key works of the group headed by Yulin Zhou and Boling Guo in China from 1980s. There is a bibliography comment at the end of every chapter to introduce other mathematicians' works in this field and briefly state the development in recent years. However, it is not possible to include all the works and achievements throughout the world in such a short comment behind every chapter.

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