CHAPTER I

ELLIPTIC COMPLEX EQUATIONS OF FIRST ORDER

In this chapter, we mainly discuss the discontinuous Riemann-Hilbert boundary value problem for some degenerate elliptic systems of first order equations. Firstly we reduce the above systems to a class of complex equations with singular coefficients, give the representations and a priori estimates of solutions of the boundary value problem for the class of degenerate elliptic complex equations, and then prove the existence and uniqueness of solutions for the boundary value problem.

1 The Discontinuous Riemann-Hilbert Problem for Nonlinear Uniformly Elliptic Complex Equations of First Order

First of all, we reduce general uniformly elliptic systems of first order equations with certain conditions to the complex equations, and then give estimates of solutions of the discontinuous Riemann-Hilbert problem for the complex equations, finally we verify the solvability of the boundary value problem.

1.1 Reduction of general uniformly elliptic systems of first order equations to standard complex form

Let D be a bounded simply connected domain in $\mathbf{R}^2$ with the boundary ∂D. Without loss of generality we can assume that ∂D is a smooth closed curve, because the requirement can be realized through a conformal mapping. We first consider the linear uniformly elliptic system of first order equations

$$\begin{aligned} &a_{11}u_x + a_{12}u_y + b_{11}v_x + b_{12}v_y = a_1u + b_1v + c_1,\\ &a_{21}u_x + a_{22}u_y + b_{21}v_x + b_{22}v_y = a_2u + b_2v + c_2, \end{aligned} \tag{1.1}$$

where the coefficients $a_{jk}, b_{jk}, a_j, b_j, c_j (j, k = 1, 2)$ are known real bounded measurable functions of $(x, y) \in D$. The uniform ellipticity condition in D

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is as follows

$$\begin{aligned} J &= 4K_1K_4 - (K_2+K_3)^2 \\ &= 4K_5K_6 - (K_2-K_3)^2 \geq J_0 > 0,\ K_1 > 0 \text{ in } D, \end{aligned} \tag{1.2}$$

in which J_0 is a positive constant and

$$K_1 = \begin{vmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{vmatrix},\ K_2 = \begin{vmatrix} a_{11} & b_{12} \\ a_{21} & b_{22} \end{vmatrix},\ K_3 = \begin{vmatrix} a_{12} & b_{11} \\ a_{22} & b_{21} \end{vmatrix},$$
$$K_4 = \begin{vmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{vmatrix},\ K_5 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},\ K_6 = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}.$$

From $J > 0$ it follows that

$$K_1K_6 > 0, \text{ or } K_1K_6 < 0, \text{ i.e. } K_1 > 0, K_6 \neq 0.$$

We can assume that $K_6 > 0$. Hence from the elliptic system (1.1), we can solve v_x, v_y and obtain the system of equations

$$\begin{aligned} v_y &= au_x + bu_y + a_0u + b_0v + f_0, \\ -v_x &= du_x + cu_y + c_0u + d_0v + g_0, \end{aligned} \tag{1.3}$$

where $a = K_1/K_6$, $b = K_3/K_6$, $c = K_4/K_6$, $d = K_2/K_6$, and the uniform ellipticity condition (1.2) is transformed into the condition

$$\Delta = \frac{J}{4K_6^2} = ac - \frac{1}{4}(b+d)^2 \geq \Delta_0 > 0,\ a > 0, \tag{1.4}$$

here Δ_0 is a positive constant and a, b, c, d are bounded for almost every point in D. Noting that

$$\begin{aligned} &z = x+iy,\ w = u+iv,\ w_z = \frac{1}{2}(w_x - iw_y),\ w_{\bar z} = \frac{1}{2}(w_x + iw_y), \\ &u_x = \frac{1}{2}(w_z + \overline{w}_{\bar z} + w_{\bar z} + \overline{w}_z),\ u_y = \frac{i}{2}(w_z - \overline{w}_{\bar z} - w_{\bar z} + \overline{w}_z), \\ &v_x = \frac{i}{2}(-w_z + \overline{w}_{\bar z} - w_{\bar z} + \overline{w}_z),\ v_y = \frac{1}{2}(w_z + \overline{w}_{\bar z} - w_{\bar z} - \overline{w}_z), \end{aligned}$$

the system (1.3) can be written in the complex form

$$w_{\bar z} = Q_1(z)w_z + Q_2(z)\overline{w}_{\bar z} + A_1(z)w + A_2(z)\overline{w} + A_3(z), \tag{1.5}$$

where

$$Q_1(z) = \frac{-2q_2}{|q_1+1|^2-|q_2|^2},\ Q_2(z)=\frac{|q_2|^2-(q_1-1)(\overline{q_1}+1)}{|q_1+1|^2-|q_2|^2},$$

$$q_1(z) = \frac{1}{2}[a+c+i(d-b)],\ q_2(z) = \frac{1}{2}[a-c+i(d+b)].$$

On the basis of

$$|q_1+1|^2-|q_2|^2 = \frac{1}{4}[(2+a+c)^2+(d-b)^2]$$
$$-\frac{1}{4}[(a-c)^2+(d+b)^2]=1+a+c+\left(\frac{d-b}{2}\right)^2+\Delta\geq 1+\Delta,$$

the uniform ellipticity condition (1.4) can be written in the complex form

$$|Q_1(z)|+|Q_2(z)|\leq q_0<1, \tag{1.6}$$

in which q_0 is a non-negative constant. If the coefficients $a_{jk}, b_{jk} \in W_p^1(D)$, $p > 2$, $j,k = 1,2$, then the following function $\eta(z)$ can be extended in $D_R = \{|z|\leq R\}$ $(\supset D,\ 0<R<\infty)$, such that $\eta(z)\in W_p^1(D_R)$, thus the Beltrami equation

$$\zeta_{\bar z}-\eta(z)\zeta_z=0,$$
$$\eta(z)=\frac{2Q_1(z)}{1+|Q_1|^2-|Q_2|^2+\sqrt{(1+|Q_1|^2-|Q_2(z)|^2)^2-4|Q_1|^2}} \tag{1.7}$$

has a homeomorphic solution $\zeta(z)\,(\in W_{p_0}^2(D_R))$, and its inverse function $z(\zeta)\in W_{p_0}^2(G_R)$, herein $G_R=\zeta(D_R)$ and $p_0\,(2<p_0\leq p)$ is a positive constant. Setting $w=w[z(\zeta)]$, the complex equation (1.5) is reduced to the complex equation

$$w_{\bar\zeta}=Q(\zeta)\bar w_{\bar\zeta}+B_1(\zeta)w+B_2(\zeta)\bar w+B_3(\zeta), \tag{1.8}$$

in which

$$Q(\zeta) = \frac{Q_2[z(\zeta)]}{1-\eta[z(\zeta)]\overline{Q_1[z(\zeta)]}},$$

$$B_1(\zeta)=\{A_1[z(\zeta)]+\overline{A_2[z(\zeta)]}Q(\zeta)\eta[z(\zeta)]\}\bar z_{\bar\zeta},$$

$$B_2(\zeta)=\{A_2[z(\zeta)]+\overline{A_1[z(\zeta)]}Q(\zeta)\eta[z(\zeta)]\}\bar z_{\bar\zeta},$$

$$B_3(\zeta)=\{A_3[z(\zeta)]+\overline{A_3[z(\zeta)]}Q(\zeta)\eta[z(\zeta)]\}\bar z_{\bar\zeta}.$$

Setting $W(\zeta)=w(\zeta)-Q(\zeta)\overline{w(\zeta)}$, the complex equation (1.8) can be transformed into the complex equation

$$W_{\bar\zeta}=C_1(\zeta)W+C_2(\zeta)\overline{W}+C_3(\zeta), \tag{1.9}$$

in which

$$C_1(\zeta)=\frac{B_1+(B_2-Q_{\bar\zeta})\overline{Q}}{1-|Q|^2},\ C_2(\zeta)=\frac{B_1Q+B_2-Q_{\bar\zeta}}{1-|Q|^2},\ C_3(\zeta)=B_3,$$

(see [86]9), [87]1)). This is a standard complex form of the uniformly elliptic system (1.1), which is called the nonhomogeneous generalized Cauchy-Riemann system, and the solution of homogeneous generalized Cauchy-Riemann system in D is called the pseudoanalytic function (see [9]1)) or the generalized analytic function (see [81]1)).

For the nonlinear uniformly elliptic system of first order equations

$$F_j(x,y,u,v,u_x,v_x,u_y,v_y)=0 \text{ in } D,\ \ j=1,2, \tag{1.10}$$

under certain conditions, we can transform the system into the complex form

$$w_{\bar z}=F(z,w,w_z),\ F=Q_1w_z+Q_2\overline{w}_{\bar z}+A_1w+A_2\bar w+A_3,\ z\in D, \tag{1.11}$$

where $Q_j=Q_j(z,w,w_z), j=1,2,\ A_j=A_j(z,w),\ j=1,2,3$ (see [86]9), [87]1)). We assume that equation (1.11) satisfy the following conditions.

Condition C :

1) $Q_j(z,w,U)\,(j=1,2),\ A_j(z,w)\,(j=1,2,3)$ are measurable in $z\in D$ for all continuous functions $w(z)$ in $D^*=\bar D\backslash Z$ and all measurable functions $U(z)\in L_{p_0}(D^*)$, and satisfy

$$L_p[A_j,\overline{D}]\le k_0,\ j=1,2,\ L_p[A_3,\overline{D}]\le k_1, \tag{1.12}$$

where $Z=\{z_1,...,z_m\}$, $z_1,...,z_m$ are different points on the boundary ∂D arranged according to the positive direction successively, $U(z)\in L_{p_0}(D^*)$ means $U(z)\in L_{p_0}(\tilde D^*)$, $\tilde D^*$ is any closed subset in D^*, and $p_0,p\,(2<p_0\le p),k_0,k_1$ are non-negative constants.

2) The above functions are continuous in $w\in \mathbf{C}$ for almost every point $z\in D,U\in\mathbf{C}$, and $Q_j=0\,(j=1,2),\ A_j=0\,(j=1,2,3)$ for $z\notin D$.

3) The complex equation (1.11) satisfies the uniform ellipticity condition

$$|F(z,w,U_1)-F(z,w,U_2)|\le q_0|U_1-U_2|, \tag{1.13}$$

for almost every point $z\in D$, in which $w,U_1,U_2\in\mathbf{C}$ and $q_0\,(<1)$ is a non-negative constant.

1.2 Representation of solutions of discontinuous Riemann-Hilbert problem for elliptic complex equations

Let D be a bounded domain in $\mathbf{C}$ with the smooth boundary $\partial D = \Gamma$. Now we formulate the discontinuous Riemann-Hilbert problem for equation (1.11).

Problem A The discontinuous Riemann-Hilbert boundary value problem for (1.11) is to find a continuous solution $w(z)$ in D^* satisfying the boundary condition:

$$\mathrm{Re}[\overline{\lambda(z)}w(z)] = r(z),\ z \in \Gamma^* = \partial D \backslash Z, \tag{1.14}$$

where $\lambda(z), r(z)$ satisfy the conditions

$$\begin{gathered} C_\alpha[\lambda(z), \Gamma_j] = \sup_{\Gamma_j} |\lambda(z)| + \sup_{z_1 \neq z_2} \frac{|\lambda(z_1) - \lambda(z_2)|}{|z_1 - z_2|^\alpha} \leq k_0 \\ C_\alpha[R_j(z)r(z), \Gamma_j] \leq k_2,\ j = 1, ..., m, \end{gathered} \tag{1.15}$$

in which $\lambda(z) = a(z) + ib(z), |\lambda(z)| = 1$ on ∂D, and $Z = \{z_1, ..., z_m\}$ are the first kind of discontinuous points of $\lambda(z)$ on ∂D, Γ_j is an arc from the point z_{j-1} to z_j on ∂D, and does not include the end points z_{j-1}, z_j $(j = 1, 2, ..., m)$, herein $z_0 = z_m$, $R_j(z) = |z - z_{j-1}|^{\beta_{j-1}} |z - z_j|^{\beta_j}$, $\alpha\,(1/2 < \alpha \leq 1), k_0, k_2, \beta = \min(\alpha, 1 - 2/p_0), \beta_j (0 < \beta_j < 1), \gamma_j$ are non-negative constants and satisfy the conditions

$$\beta_j + \gamma_j < \beta,\ j = 1, ..., m, \tag{1.16}$$

where $\gamma_j (j = 1, ..., m)$ are as stated in (1.17) below. Problem A with $A_3(z) = 0$ in D, $r(z) = 0$ on Γ^* is called Problem A_0.

Denote by $\lambda(z_j - 0)$ and $\lambda(z_j + 0)$ the left limit and right limit of $\lambda(z)$ as $z \to z_j$ $(j = 1, 2, ..., m)$ on ∂D, and

$$\begin{gathered} e^{i\phi_j} = \frac{\lambda(z_j - 0)}{\lambda(z_j + 0)},\ \gamma_j = \frac{1}{\pi i} \ln \frac{\lambda(z_j - 0)}{\lambda(z_j + 0)} = \frac{\phi_j}{\pi} - K_j, \\ K_j = \left[\frac{\phi_j}{\pi}\right] + J_j, \quad J_j = 0 \text{ or } 1, \quad j = 1, ..., m, \end{gathered} \tag{1.17}$$

in which $0 \leq \gamma_j < 1$ when $J_j = 0$, and $-1 < \gamma_j < 0$ when $J_j = 1$, $j = 1, ..., m$. The index K of Problems A and A_0 is defined as follows:

$$K = \frac{1}{2}(K_1 + \cdots + K_m) = \sum_{j=1}^{m} \left[\frac{\phi_j}{2\pi} - \frac{\gamma_j}{2}\right]. \tag{1.18}$$

If $\lambda(x)$ on Γ is continuous, then $K = \Delta_\Gamma \arg \lambda(x)/2\pi$ is a unique integer. Now the function $\lambda(x)$ on Γ is not continuous, we can choose $J_j = 0$ or 1, hence the index K is not unique. If we choose $K = -1/2$, then the solution of Problem A is unique.

In order to prove the solvability of Problem A for the complex equation (1.11), we need to give a representation theorem for Problem A.

Theorem 1.1 *Suppose that the complex equation* (1.11) *satisfies Condition C, and $w(z)$ is a solution of Problem A for* (1.11). *Then $w(z)$ is representable by*

$$w(z) = \Phi[\zeta(z)]e^{\phi(z)} + \psi(z), \tag{1.19}$$

where $\zeta(z)$ is a homeomorphism in $\bar{D}$, which quasiconformally maps D onto the unit disk $G = \{|\zeta| < 1\}$ with boundary $L = \{|\zeta| = 1\}$, such that three points on Γ are mapped onto three points on L respectively, $\Phi(\zeta)$ is an analytic function in G, $\psi(z), \phi(z), \zeta(z)$ and its inverse function $z(\zeta)$ satisfy the estimates

$$C_\beta[\psi, \bar{D}] \le k_3, C_\beta[\phi, \bar{D}] \le k_3, C_\beta[\zeta(z), \bar{D}] \le k_3, C_\beta[z(\zeta), \bar{G}] \le k_3, \tag{1.20}$$

$$L_{p_0}[|\psi_{\bar{z}}| + |\psi_z|, \bar{D}] \le k_3,\ L_{p_0}[|\phi_{\bar{z}}| + |\phi_z|, \bar{D}] \le k_3, \tag{1.21}$$

$$C_\beta[z(\zeta), \bar{G}] \le k_3,\ L_{p_0}[|\chi_{\bar{z}}| + |\chi_z|, \bar{D}] \le k_4, \tag{1.22}$$

in which $\chi(z)$ is as stated in (1.27) *below,* $\beta = \min(\alpha, 1-2/p_0)$, $p_0\,(2 < p_0 \le p)$, $k_j = k_j(q_0, p_0, \beta, k_0, k_1, D)\,(j = 3, 4)$ *are non-negative constants dependent on $q_0, p_0, \beta, k_0, k_1, D$. Moreover, if the coefficients $Q_j(z) = 0\,(j = 1, 2)$ of the complex equation* (1.11) *in D, then the representation* (1.19) *becomes the form*

$$w(z) = \Phi(z)e^{\phi(z)} + \psi(z), \tag{1.23}$$

and when $K < 0$, $\Phi(z)$ satisfies the estimate

$$C_\delta[X(z)\Phi(z), \bar{D}] \le M_1 = M_1(p_0, \delta, k, D) < \infty, \tag{1.24}$$

in which

$$X(z) = \prod_{j=1}^{m} |z - z_j|^{\eta_j},\ \eta_j = \begin{cases} |\gamma_j| + \tau,\ \gamma_j < 0,\ \beta_j \le |\gamma_j|, \\ |\beta_j| + \tau,\ \ for\ \ other\ \ case, \end{cases} \tag{1.25}$$

here $\gamma_j\,(j = 1, ..., m)$ are real constants as stated in (1.17) *and $\tau, \delta\,(0 < \delta <$* $\min(\beta, \tau))$ *are sufficiently small positive constants, $k = (k_0, k_1, k_2)$, and M_1 is a non-negative constant dependent on p_0, δ, k, D.*

Proof We substitute the solution $w(z)$ of Problem A into the coefficients of equation (1.11) and consider the following system

$$\begin{aligned} &\psi_{\bar z}=Q\psi_z+A_1\psi+A_2\bar\psi+A_3, Q=\begin{cases} Q_1+Q_2\dfrac{\overline{w_z}}{w_z} \text{ for } w_z\neq 0,\\ 0 \text{ for } w_z=0 \text{ or } z\notin D,\end{cases}\\ &\phi_{\bar z}=Q\phi_z+A,\ A=\begin{cases} A_1+A_2\dfrac{\overline{w}-\overline{\psi}}{w-\psi} \text{ for } w(z)-\psi(z)\neq 0,\\ 0 \text{ for } w(z)-\psi(z)=0 \text{ or } z\notin D,\end{cases}\\ &W_{\bar z}=QW_z,\ \ W(z)=\Phi[\zeta(z)]. \end{aligned} \tag{1.26}$$

By using the continuity method and the principle of contracting mapping, we can find the solution

$$\begin{gathered} \psi(z)=Tf=-\frac{1}{\pi}\iint_D\frac{f(\zeta)}{\zeta-z}d\sigma_\zeta,\\ \phi(z)=Tg,\ \zeta(z)=\Psi[\chi(z)],\ \chi(z)=z+Th \end{gathered} \tag{1.27}$$

of (1.26), where $f(z),g(z),h(z)\in L_{p_0}(\overline{D})$, $2<p_0\leq p$, $\chi(z)$ is a homeomorphism in $\bar D$, $\Psi(\chi)$ is a univalent analytic function, which conformally maps $E=\chi(D)$ onto the unit disk G (see [81]1)), and $\Phi(\zeta)$ is an analytic function in G. We can verify that $\psi(z),\phi(z),\zeta(z)$ satisfy the estimates (1.20) and (1.21). It remains to prove that $z=z(\zeta)$ satisfies the estimate (1.22). In fact, we can find a homeomorphic solution of the last equation in (1.26) in the form $\chi(z)=z+Th$ such that $[\chi(z)]_z,[\chi(z)]_{\bar z}\in L_{p_0}(\bar D)$ (see [87]1)). Next, we find a univalent analytic function $\zeta=\Psi(\chi)$, which maps $\chi(D)$ onto G, hence $\zeta=\zeta(z)=\Psi[\chi(z)]$. By the result on conformal mappings, applying the method of Lemma 2.1, Chapter II in [87]1), we can prove that (1.22) is true. When $Q_j(z)=0$ in D, $j=1,2$, then we can choose $\chi(z)=z$ in (1.27), in this case $\Phi[\zeta(z)]$ can be replaced by the analytic function $\Phi(z)$, herein $\Psi(z),\zeta(z)$ are as stated in (1.27), it is clear that the representation (1.19) becomes the form (1.23). Thus the analytic function $\Phi(z)$ satisfies the boundary conditions

$$\mathrm{Re}[\overline{\lambda(z)}e^{\phi(z)}\Phi(z)]=r(z)-\mathrm{Re}[\overline{\lambda(z)}\psi(z)],\ z\in\Gamma^*. \tag{1.28}$$

On the basis of the estimate (1.20), by using the methods in the proof of Theorems 1.1 and 1.8, Chapter IV in [87]1), we can prove that $\Phi(z)$ satisfies the estimate (1.24).

1.3 Existence of solutions of discontinuous Riemann-Hilbert problem for nonlinear complex equations in upper half-unit disk

We first consider a special domain, i.e. D is an upper half-unit disk with the boundary $\Gamma'=\Gamma\cup\gamma$, where $\Gamma=\{|z|=1, \mathrm{Im}z>0\}$ and $\gamma=\{-1<x<1, y=0\}$.

Theorem 1.2 *Under the same conditions as in Theorem* 1.1 *for the above domain D, the following statements hold.*

(1) *If the index $K\geq 0$, then Problem A for* (1.11) *is solvable, and the general solution includes $2K+1$ arbitrary real constants.*

(2) *If $K<0$, then Problem A has $-2K-1$ solvability conditions.*

Proof Let us introduce a closed, convex and bounded subset B_1 in the Banach space $B=L_{p_0}(\bar{D})\times L_{p_0}(\bar{D})\times L_{p_0}(\bar{D})\,(2<p_0\leq p)$, whose elements are systems of functions $q=[Q(z),\,f(z),\,g(z)]$ with the norm $||\,q\,||=L_{p_0}(Q,\bar{D})\;+L_{p_0}(f,\bar{D})+L_{p_0}(g,\bar{D})$, which satisfy the conditions

$$|Q(z)|\leq q_0<1\,(z\in D),\ L_{p_0}[f(z),\bar{D}]\leq k_3,\ L_{p_0}[g(z),\bar{D}]\leq k_3, \qquad (1.29)$$

where q_0,k_3 are non-negative constants as stated in (1.13) and (1.21). Moreover introduce a closed and bounded subset B_2 in B, the elements of which are systems of functions $\omega=[f(z),g(z),h(z)]$ satisfying the condition

$$L_{p_0}[f(z),\bar{D}]\leq k_4,\ L_{p_0}[g(z),\bar{D}]\leq k_4,\ |h(z)|\leq q_0|1+\Pi h|, \qquad (1.30)$$

where $\Pi h=-\frac{1}{\pi}\iint_D[h(\zeta)/(\zeta-z)^2]d\sigma_\zeta$.

We arbitrarily select $q=[Q(z),f(z),g(z)]\in B_1$, and using the principle of contracting mapping, a unique solution $h(z)\in L_{p_0}(\bar{D})$ of the integral equation

$$h(z)=Q(z)[1+\Pi h] \qquad (1.31)$$

can be found, which satisfies the third inequality in (1.30). Moreover, $\chi(z)=z+Th$ is a homeomorphism in $\bar{D}$. Now, we find a univalent analytic function $\zeta=\Psi(\chi)$, which maps $\chi(D)$ onto the unit disk G as stated in Theorem 1.1. Moreover, we find an analytic function $\Phi(\zeta)$ in G satisfying the boundary condition in the form

$$\mathrm{Re}[\overline{\Lambda(\zeta)}\Phi(\zeta)]=R(\zeta),\ \zeta\in L^*=\zeta(\Gamma^*), \qquad (1.32)$$

in which $\zeta(z)=\Psi[\chi(z)]$, $z(\zeta)$ is its inverse function, $\psi(z)=Tf$, $\phi(z)=Tg$, $\Lambda(\zeta)=\lambda[z(\zeta)]\exp[\overline{\phi(z(\zeta))}]$, $R(\zeta)=r[z(\zeta)]-\mathrm{Re}[\overline{\lambda(z(\zeta))}\psi(z(\zeta))]$, where

$\Lambda(\zeta)$, $R(\zeta)$ on L^* satisfy the conditions similar to those of $\lambda(z)$, $r(z)$ in (1.15) and the index of $\Lambda(\zeta)$ on L^* is K. In the following, we first consider the case $K \geq 0$. By using Theorem 1.1, we can find the analytic function $\Phi(\zeta)$ in the form (1.73), Chapter I, [87]1), here $2K+1$ arbitrary real constants can be chosen. Thus the function $w(z) = \Phi[\zeta(z)]e^{\phi(z)} + \psi(z)$ is determined. Afterwards, we find out the solution $[f^*(z), g^*(z), h^*(z), Q^*(z)]$ of the system of integral equations

$$f^*(z) = F(z,w,\Pi f^*) - F(z,w,0) + A_1(z,w)Tf^* + A_2(z,w)\overline{Tf^*} + A_3(z,w), \tag{1.33}$$

$$Wg^*(z) = F(z,w,W\Pi g^* + \Pi f^*) - F(z,w,\Pi f^*) + A_1(z,w)W + A_2(z,w)\overline{W}, \tag{1.34}$$

$$\begin{aligned} S'(\chi)h^*(z)e^{\phi(z)} = F[z,w,S'(\chi)(1+\Pi h^*)e^{\phi(z)} + W\Pi g^* + \Pi f^*] \\ - F(z,w,W\Pi g^* + \Pi f^*), \end{aligned} \tag{1.35}$$

$$Q^*(z) = h^*(z)/[1+\Pi h^*],\ S'(\chi) = [\Phi(\Psi(\chi))]_\chi, \tag{1.36}$$

and denote by $q^* = E(q)$ the mapping from $q = (Q,f,g)$ to $q^* = (Q^*, f^*, g^*)$. According to Lemma 5.5, Chapter III, [87]1), we can prove that $q^* = E(q)$ continuously maps B_1 onto a compact subset in B_1. By means of the Schauder fixed-point theorem, there exists a system $q = (Q,f,g) \in B_1$, such that $q = E(q)$. Applying the above method, from $q = (Q,f,g)$, we can construct a function $w(z) = \Phi[\zeta(z)]e^{\phi(z)} + \psi(z)$, which is just a solution of Problem A for (1.11). As for the case $K < 0$, it can be similarly discussed, but we first permit that the function $\Phi(\zeta)$ satisfying the boundary condition (1.32) has a pole of order $|[K+1]|$ at $\zeta = 0$, if $-2K$ is an even integer, then we need to add a point condition: $\mathrm{Im}[\overline{\lambda(z_0')}w(z_0')] = b_0$, z_0' is a fixed point on $\Gamma\backslash Z$, b_0 is a real constant, and then find the solution of the nonlinear complex equation (1.11) in this case. From the representation $w(z) = \Phi[\zeta(z)]e^{\phi(z)} + \psi(z)$, we can derive the $-2K-1$ solvability conditions of Problem A for (1.11).

Besides, we can discuss the solvability of the discontinuous Riemann-Hilbert boundary value problem for the complex equation (1.11) in the upper half-plane and the zone domain. For some problems in nonlinear mechanics as stated in [61]2),[91], it can be solved by the results in Theorem 1.2.

1.4 The discontinuous Riemann-Hilbert problem for nonlinear complex equations in general domains

In this subsection, let D' be a general simply connected domain with the boundary $\Gamma' = \Gamma_1' \cup \Gamma_2'$, herein $\Gamma_1', \Gamma_2' \in C_\mu^1\ (0 < \mu < 1)$ and their intersection points z', z'' with the inner angles $\alpha_1\pi, \alpha_2\pi\,(0<\alpha_1,\alpha_2<1)$ respectively. We discuss the nonlinear uniformly elliptic complex equation

$$w_{\bar z}=F(z,w,w_z),\ F=Q_1w_z+Q_2\overline{w}_{\bar z}+A_1w+A_2\bar w+A_3,\ z\in D', \tag{1.37}$$

in which $F(z,w,U)$ satisfies Condition C in D'. There exist m point $Z=\{z_1=z',...,z_n=z'',...,z_m=z_0\}$ on Γ' arranged according to the positive direction successively. Denote by Γ_j the curve on Γ' from z_{j-1} to $z_j\,(j=1,2,...,m)$, and Γ_j does not include the end points $z_{j-1}\,(j=1,...,m)$.

Problem A' The discontinuous Riemann-Hilbert boundary value problem for (1.37) is to find a continuous solution $w(z)$ in $D^*=\overline{D'}\backslash Z$ satisfying the boundary condition:

$$\begin{aligned}&\mathrm{Re}[\overline{\lambda(z)}w(z)]=r(z),\ x\in\Gamma^*=\Gamma'\backslash Z,\\&\mathrm{Im}[\overline{\lambda(z_j')}w(z_j')]=b_j,\ j=1,...,2K+1,\end{aligned}\tag{1.38}$$

where $z_1',...,z_{2K+1}'\,(\notin Z)$ are distinct points on Γ' and $b_j\,(j=1,...,2K+1)$ are real constants, and $\lambda(z)$, $r(z)$, b_j $(j=1,...,2K+1)$ are given functions satisfying

$$\begin{aligned}&C_\alpha[\lambda(z),\Gamma_j]\le k_0,\ C_\alpha[R_j(z)r(z),\Gamma_j]\le k_2,\ j=1,...,m,\\&|b_j|\le k_2,\ j=1,...,2K+1,\end{aligned}\tag{1.39}$$

in which $\alpha\,(1/2<\alpha<1)$, k_0, k_2 are non-negative constants, $R_j(z)=|z-z_{j-1}|^{\beta_{j-1}}|z-z_j|^{\beta_j}$, and assume that $\beta_j+\gamma_j<\beta=\alpha_0\min(\alpha,1-2/p_0)$, $\gamma_j,\beta_j(j=1,...,m)$ are similar to those in (1.16) and (1.17), $\alpha_0=\min(\alpha_1,\alpha_2)$, and $K(\ge -1/2)$ is the index of $\lambda(z)$ on Γ', which is defined as in (1.18).

In order to give the uniqueness result of solutions of Problem A' for equation (1.37), we need to add one condition: For any complex functions $w_j(z)\in C(D^*), U_j(z)\in L_{p_0}(D^*)\,(2<p_0\le p, j=1,2)$, the following equality holds:

$$F(z,w_1,U_1)-F(z,w_1,U_2)=Q(U_1-U_2)+A(w_1-w_2)\ \text{in}\ D', \tag{1.40}$$

in which $|Q(z,w_1,w_2,U_1,U_2)|\le q_0$, $A(z,w_1,w_2)\in L_{p_0}(\overline{D'})$. Especially, if (1.37) is a linear equation, then the condition (1.40) obviously holds.

Applying a similar method as stated in the proof of Theorem 1.1, we can prove the following theorem.

Theorem 1.3 *If the complex equation* (1.37) *in* D' *satisfies Condition C, then Problem* A' *for* (1.37) *is solvable. If Condition C and the condition* (1.40) *hold, then the solution of Problem* A' *is unique. Moreover the solution* $w(z)$ *can be expressed as* (1.19) *satisfying the estimates* (1.20) – (1.22), *in which* $\beta = \alpha_0 \min(\alpha, 1-2/p_0)$. *If* $Q_j(z) = 0\,(j = 1, 2)$ *in* D' *in* (1.37), *then the representation* (1.19) *becomes the form*

$$w(z) = \Phi(z)e^{\phi(z)} + \psi(z), \tag{1.41}$$

and $w(z)$ *satisfies the estimate*

$$C_\delta[X(z)w(z), \overline{D'}] \le M_2 = M_2(p_0, \delta, k, D') < \infty, \tag{1.42}$$

in which

$$\begin{aligned} &X(z)= \prod_{j=1,j\neq 1,n}^{m} |z-z_j|^{\eta_j}|z-z_1|^{\eta_1/\alpha_1}|z-z_n|^{\eta_n/\alpha_2},\\ &\eta_j = \begin{cases} |\gamma_j| + \tau, \text{ if } \gamma_j < 0, \ \beta_j \le |\gamma_j|, \\ |\beta_j|+\tau, \text{ if } \gamma_j \ge 0, \text{ and } \gamma_j < 0, \beta_j > |\gamma_j|, \end{cases} \end{aligned} \tag{1.43}$$

here $\gamma_j (j = 1, ..., m)$ *are real constants as stated in* (1.17), $\tau, \delta\,(0 < \delta < \min(\beta, \tau))$ *are sufficiently small positive constants, and* $M_2 = M_2(p_0, \delta, k, D')$ *is a non-negative constant dependent on* p_0, δ, k, D' (see [86]33),[92]6)).

2 The Riemann-Hilbert Problem for Linear Degenerate Elliptic Complex Equations of First Order

In this section we discuss the Riemann-Hilbert Problem for linear degenerate elliptic systems of first order equations in a simply connected domain. We first give the representation of solutions of the boundary value problem for the systems, and then prove the uniqueness and existence of solutions for the problem.

2.1 Formulation of the Riemann-Hilbert problem for degenerate elliptic complex equations

Let D be a domain in the upper half-plane with the boundary ∂D, which consists of $\gamma = \{-1 < x < 1, y = 0\}$ and a curve $\Gamma(\in C^1_\mu, 0 < \mu < 1)$

with the end points $-1, 1$ in the upper half-plane. We consider the linear degenerate elliptic equation of first order

$$\begin{cases} H(y)u_x - v_y = a_1 u + b_1 v + c_1 \\ H(y)v_x + u_y = a_2 u + b_2 v + c_2 \end{cases} \text{ in } D, \tag{2.1}$$

where $H(y) = \sqrt{K(y)}$, $G(y) = \int_0^y H(t)dt$, $G'(y) = H(y)$, $K(y) = y^m h(y)$ is continuous in $\overline{D}$, here m is a positive number and $h(y)$ is a continuously differentiable positive function in $\overline{D}$, and a_j, b_j, c_j $(j = 1, 2)$ are functions of $z\,(\in D)$. The following degenerate elliptic system is a special case of system (2.1) with $H(y) = y^{m/2}$:

$$\begin{cases} y^{m/2}u_x - v_y = a_1 u + b_1 v + c_1 \\ y^{m/2}v_x + u_y = a_2 u + b_2 v + c_2 \end{cases} \text{ in } D. \tag{2.2}$$

For convenience, we mainly discuss equation (2.2), and equation (2.1) can be similarly discussed. From the ellipticity condition in (1.2), namely

$$J = 4K_1K_4 - (K_2 + K_3)^2 = 4H^2(y) > 0 \text{ in } \overline{D}\backslash\gamma \tag{2.3}$$

and $J = 0$ on $\gamma = \{-1 < x < 1, y = 0\}$, hence system (2.1) or (2.2) is elliptic system of first order equations in D with the parabolic degenerate line $\gamma = (-1, 1)$ on the x-axis. Setting $Y = G(y) = \int_0^y H(t)dt$, $Z = x + iY$ in $\overline{D}$, if $H(y) = y^{m/2}$, $Y = \int_0^y H(t)dt = 2y^{(m+2)/2}/(m+2)$, then its inverse function is $y = [(m+2)Y/2]^{2/(m+2)} = JY^{2/(m+2)}$. Denote

$$\begin{aligned} &w(z) = u + iv,\ w_{\overline{z}} = \frac{1}{2}[H(y)w_x + iw_y] \\ &= \frac{H(y)}{2}[w_x + iw_Y] = H(y)w_{x-iY} = H(y)w_{\overline{Z}}, \end{aligned} \tag{2.4}$$

then the system (2.1) can be written in the complex form

$$\begin{gathered} w_{\overline{z}} = H(y)w_{\overline{Z}} = A_1(z)w + A_2(z)\overline{w} + A_3(z) = g(Z) \text{ in } D_Z, \\ A_1 = \frac{1}{4}[a_1 + ia_2 - ib_1 + b_2], A_2 = \frac{1}{4}[a_1 + ia_2 + ib_1 - b_2], A_3 = \frac{1}{2}[c_1 + ic_2], \end{gathered} \tag{2.5}$$

in which D_Z is the image domain of D with respect to the mapping $Z = Z(z) = x + iY = x + iG(y)$ in D. If the slopes of the Γ at $z = \mp 1$ are satisfied the conditions $-\infty < \partial y/\partial x \le 0$, $0 \le \partial y/\partial x < \infty$ respectively, then $\partial Y/\partial x = (\partial Y/\partial y)(\partial y/\partial x) = H(y)\partial y/\partial x = 0$ at $z = \mp 1$ respectively, i.e.

the inner angles of ∂D_Z are equal to π in D_Z at $Z=\mp 1$; if the slopes of the Γ at $z=\mp 1$ are satisfied the conditions $0\le \partial y/\partial x<\infty$, $-\infty<\partial y/\partial x\le 0$ respectively, then $\partial Y/\partial x=(\partial Y/\partial y)(\partial y/\partial x)=H(y)\partial y/\partial x=0$ at $z=\mp 1$ respectively, i.e. the inner angles of ∂D_Z are equal to 0 in D_Z at $Z=\mp 1$. If the boundary $\partial D\backslash\gamma(\in C^1_\mu)$ is a curve with the form $x=G(y)/\alpha_1-1\,(\alpha_1\ne \pm 1)$ and $x=1-G(y)/\alpha_2\,(\alpha_2\ne\pm 1)$ near the points $z=-1,1$ respectively, then the inner angles of the boundary ∂D_Z in $Z-$plane at $Z=-1,1$ are equal to $\tan^{-1}\alpha_1(\alpha_1\ge 0),\pi-\tan^{-1}(-\alpha_1)(\alpha_1\le 0)$ and $\tan^{-1}(-\alpha_2)(\alpha_2\le 0),\pi-\tan^{-1}\alpha_2(\alpha_2\ge 0)$ respectively, especially if $\alpha_1=1,\alpha_2=-1$, then the inner angles are equal to $\pi/4$. If $Y_x=Y_y y_x=H(y)/x_y=\pm\infty$ at $Z=\pm 1$, which include $x_y=0$ and $\pm H^2(y)$ at $Z=\pm 1$, in this case the inner angles of the curve $\tilde\Gamma=Z(\Gamma)$ and $\tilde\gamma=Z(\gamma)$ in $Z=x+iY-$plane at $Z=\pm 1$ are equal to $\pi/2$. For equations (2.5), we can give a conformal mapping $\zeta=\zeta(Z)$, which maps the domain D_Z onto D_ζ, such that line segment $\gamma=(-1,1)$ and boundary points $-1,1$ are mapped onto themselves respectively, and the boundary $\partial D_\zeta\backslash\gamma(\in C^1_\mu)$ is a curve with the form $\mathrm{Re}\,\zeta=G(\mathrm{Im}\zeta)-1$ and $\mathrm{Re}\,\zeta=1-G(\mathrm{Im}\zeta)$ near the points $\zeta=-1,1$ respectively. Denote by $Z=Z(\zeta)$ the inverse function of $\zeta=\zeta(Z)$, thus equation (2.5) is reduced to

$$\begin{gathered}w_{\overline{\zeta}}=g[Z(\zeta)]\overline{Z'(\zeta)}/H(y),\ \text{i.e.}\\ w_{\overline{\zeta}}=[A_1(z)w+A_2(z)\overline{w}+A_3(z)]\overline{Z'(\zeta)}/H(y)\ \text{ in } \overline{D_\zeta}.\end{gathered}\tag{2.6}$$

In this section, there is no harm in assuming that the boundary Γ is a curve with the form $x=G(y)-1$ and $x=1-G(y)$ near the points $z=-1,1$ respectively.

Suppose that equation (2.5) satisfies the following conditions: **Condition** C

The coefficients $A_j[z(Z)]\,(j=1,2,3)$ in (2.5) satisfy

$$L_\infty[A_j(z(Z)),\overline{D_Z}]\le k_0,\ j=1,2,\ L_\infty[A_3(z(Z)),\overline{D_Z}]\le k_1,\tag{2.7}$$

where $z(Z)$ is the inverse function of $Z(z)$, and k_0, k_1 are non-negative constants.

Now we formulate the Riemann-Hilbert boundary value problem as follows:

Problem A Find a solution $w(z)$ of (2.5) in D, which is continuous in $D^*=\overline{D}\backslash\{-1,1\}$ and satisfies the boundary conditions

$$\mathrm{Re}[\overline{\lambda(z)}w(z)]=r(z)\ \text{ on } \partial D^*=\partial D\backslash\{-1,1\},\ \mathrm{Im}[\overline{\lambda(z_0)}w(z_0)]=b_0,\tag{2.8}$$

where $\lambda(z) = a(x) + ib(x)\,(|\lambda(z)| = 1)$, b_0 is a real constants, $z_0\,(\in \Gamma\backslash \{-1,1\})$ is a point, and $\lambda(z)\,r(z)$, b_0 satisfy the conditions

$$\begin{gathered} C_\alpha[\lambda(z),\Gamma] \le k_0,\ \ C_\alpha[\lambda(z),\gamma] \le k_0, \\ C_\alpha[r(z),\Gamma] \le k_2,\ C_\alpha[r(z),\gamma] \le k_2,\ |b_0| \le k_2, \end{gathered} \tag{2.9}$$

in which $\alpha\,(0 < \alpha < 1), k_0, k_2$ are non-negative constants. In particular, if $\lambda(z) = a(x) + ib(x) = 1$, then Problem A is the Dirichlet boundary value problem, which will be called Problem D. Denote by $\lambda(z_j - 0)$ and $\lambda(z_j + 0)$ the left limit and right limit of $\lambda(z)$ as $z \to z_j (j = 1, 2)$ on ∂D^*, and

$$\begin{gathered} e^{i\phi_j} = \frac{\lambda(z_j-0)}{\lambda(z_j+0)}, \gamma_j = \frac{1}{\pi i} \ln\left[\frac{\lambda(z_j-0)}{\lambda(z_j+0)}\right] = \frac{\phi_j}{\pi} - K_j, \\ K_j = \left[\frac{\phi_j}{\pi}\right] + J_j,\ J_j = 0 \ \text{ or } \ 1,\ j = 1, 2, \end{gathered} \tag{2.10}$$

in which $z_1 = -1, z_2 = 1$, $0 \le \gamma_j < 1$ when $J_j = 0$, and $-1 < \gamma_j < 0$ when $J_j = 1$, $1 \le j \le 2$, and

$$K = \frac{1}{2}(K_1 + K_2) = \frac{1}{2}\sum_{j=1}^{2}[\frac{\phi_j}{\pi} - \gamma_j]$$

is called the index of Problem A. If $\lambda(z)$ on ∂D is continuous, then $K = \Delta_\Gamma \arg \lambda(z)/2\pi$ is a unique integer. If the function $\lambda(z)$ on ∂D is not continuous, we can choose $J_j = 0$ or 1, hence the index K is not unique. We shall only discuss the case $K = 0$ later on, and the other cases for instance $K = -1/2$, the last point condition in (2.8) should be cancelled, we can similarly discussed.

2.2 Representations and estimates of solutions of Riemann-Hilbert problem for elliptic complex equations

It is clear that the complex equation

$$w_{\overline{Z}} = 0 \ \text{ in } \ \overline{D_Z} \tag{2.11}$$

is a special case of equation (2.5). On the basis of Theorem 1.3, we can find a unique solution of Problem A for equation (2.11) in $\overline{D_Z}$.

Now we consider the function $g(Z) \in L_\infty(D_Z)$, and first extend the function $g(Z)$ to the exterior of $\overline{D_Z}$ in $\mathbf{C}$, i.e. set $g(Z)=0$ in $\mathbf{C}\backslash\overline{D_Z}$, hence we can only discuss the domain $D_0=\{|x|<R_0\}\cap\{\mathrm{Im}Y\geq 0\}\supset\overline{D_Z}$, here $Z=x+iY$, R_0 is a positive number. In the following we shall verify that the integral

$$\Psi(Z)=Tg/H=-\frac{1}{\pi}\iint_{D_0}\frac{g(t)/H(\mathrm{Im}t)}{t-Z}d\sigma_t \text{ in } D_0,$$
$$L_\infty[g(Z),D_0]\leq k_3, \tag{2.12}$$

satisfies the estimate (2.13) below, where $H(y)=y^{m/2}$, m ia a positive number. It is clear that the function $g(Z)/H(y)$ belongs to the space $L_1(D_0)$ and in general is not belonging to the space $L_p(D_0)\,(p>2,m\geq 2)$, and the integral $\Psi(Z_0)$ is definite when $\mathrm{Im}Z_0>0$. If $Z_0\in D_0$ and $\mathrm{Im}Z_0=0$, we can define the integral $\Psi(Z_0)$ as the limit of the corresponding integral over $D_0\cap\{|\mathrm{Re}t-\mathrm{Re}Z_0|\geq\varepsilon\}\cap\{|\mathrm{Im}t-\mathrm{Im}Z_0|\geq\varepsilon\}$ as $\varepsilon\to 0$, where ε is a sufficiently small positive number. The Hölder continuity of the integral will be proved by the following method.

Lemma 2.1 *If the function $g(Z)$ in D_Z satisfies the condition in* (2.12), *and $H(y)=y^{m/2}$, where m is a positive number, then the integral in* (2.12) *satisfies the estimate*

$$C_\beta[\Psi(Z),\overline{D_Z}]\leq M_1, \tag{2.13}$$

where $\beta=2/(m+2)-\delta$, δ is a sufficiently small positive constant, and $M_1=M_1(\beta,\,k_3,H,D_Z)$ is a positive constant.

Proof We first verify the boundedness of the integral in (2.12), as stated before, if $H(y)=y^{m/2}$, then $H(y)=J^{m/2}Y^{m/(m+2)}$. For any two points $Z_0=x_0\in\gamma=(-1,1)$ on $x-$axis and $Z_1=x_1+iY_1(Y_1>0)\in D_0$ satisfying the condition $2\mathrm{Im}Z_1/\sqrt{3}\leq|Z_1-Z_0|\leq 2\mathrm{Im}Z_1$, this means that the inner angle at Z_0 of the triangle $Z_0Z_1Z_2\,(Z_2=x_0+iY_1\in D_0)$ is not less than $\pi/6$ and not greater than $\pi/3$, choose a sufficiently large positive number q, from the Hölder inequality, we have $L_1[\Psi(Z),D_0]\leq L_q[g(Z),D_0]L_p[1/H(\mathrm{Im}t)(t-Z),D_0]$, where $p=q/(q-1)\,(>1)$ is close to 1. In fact we can derive as follows

$$|\Psi(Z_0)|\leq|\frac{1}{\pi}\iint_{D_0}\frac{g(t)/H(\mathrm{Im}t)}{t-Z_0}d\sigma_t|\leq\frac{1}{J^{m/2}\pi}L_q[g(Z),D_0]$$
$$\times[\iint_{D_0}|\frac{1}{t^{m/(m+2)}(t-Z_0)}|^p d\sigma_t]^{1/p}=\frac{1}{J^{m/2}\pi}L_q[g(Z),D_0]\,J_1^{1/p}, \tag{2.14}$$

where

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$$
\begin{aligned}
J_1 &= \iint_{D_0} \left|\frac{1}{t^{m/(m+2)}(t-Z_0)}\right|^p d\sigma_t \\
&\le \iint_{D_0} \frac{1}{|t|^{pm/(m+2)}|\mathrm{Im}(t-Z_0)|^{p\beta_0}|\mathrm{Re}(t-Z_0)|^{p(1-\beta_0)}} d\sigma_t \\
&\le \left| \int_0^{d_0} \frac{1}{Y^{pm/(m+2)}|Y-Y_0|^{p\beta_0}} dY \int_{d_1}^{d_2} \frac{1}{|x-x_0|^{p(1-\beta_0)}} dx \right| \le k_4,
\end{aligned}
$$

in which $d_0 = \max_{Z\in\overline{D_0}} \mathrm{Im}Z$, $d_1 = \min_{Z\in\overline{D_0}} \mathrm{Re}Z$, $d_2 = \max_{Z\in\overline{D_0}} \mathrm{Re}Z$, $\beta_0 = 2/(m+2) - \varepsilon$, $\varepsilon\,(< 1/p - m/(m+2))$ is a sufficiently small positive constant, we can choose $\varepsilon = 2(p-1)/p\,(< 2/(m+2))$ such that $p(1-\beta_0/2) < 1$ and $p[m/(m+2) + \beta_0] < 1$, and $k_4 = k_4(\beta, k_3, H, D_0)$ is a non-negative constant.

Next we estimate the Hölder continuity of the integral $\Psi(Z)$ in $\overline{D_0}$, i.e.

$$
\begin{aligned}
|\Psi(Z_1)-\Psi(Z_0)| &\le \frac{|Z_1-Z_0|}{\pi}\left|\iint_{D_0}\frac{g(t)/H(\mathrm{Im}t)}{(t-Z_0)(t-Z_1)}d\sigma_t\right| \\
&\le \frac{|Z_1-Z_0|}{J^{m/2}\pi} L_q[g(Z), D_0]\left[\iint_{D_0}\left|\frac{1}{t^{m/(m+2)}(t-Z_0)(t-Z_1)}\right|^p d\sigma_t\right]^{1/p},
\end{aligned}
\tag{2.15}
$$

and

$$
\begin{aligned}
J_2 &= \iint_{D_0}\left|\frac{1}{t^{m/(m+2)}(t-Z_0)(t-Z_1)}\right|^p d\sigma_t \\
&\le \iint_{D_0}\frac{|\mathrm{Re}(t-Z_0)|^{p(\beta_0/2-1)}|\mathrm{Re}(t-Z_1)|^{p(\beta_0/2-1)}}{|t|^{pm/(m+2)}|\mathrm{Im}(t-Z_0)|^{p\beta_0/2}|\mathrm{Im}(t-Z_1)|^{p\beta_0/2}} d\sigma_t \\
&\le \int_0^{d_0}\frac{1}{Y^{pm/(m+2)}|\mathrm{Im}(Y-Z_0)|^{p\beta_0/2}|\mathrm{Im}(Y-Z_1)|^{p\beta_0/2}} dY \\
&\times \int_{d_1}^{d_2}\frac{1}{|\mathrm{Re}(t-Z_0)|^{p(1-\beta_0/2)}|\mathrm{Re}(t-Z_1)|^{p(1-\beta_0/2)}} d\mathrm{Re}t \\
&\le k_5\int_{d_1}^{d_2}\frac{1}{|x-x_0|^{p(1-\beta_0/2)}|x-x_1|^{p(1-\beta_0/2)}} dx,
\end{aligned}
$$

where $\beta_0 = 2/(m+2) - \varepsilon$ is chosen as before and

$$
k_5 = \max_{Z_0, Z_1\in D_0}\int_0^{d_0}[Y^{pm/(m+2)}|\mathrm{Im}(Y-Z_0)|^{p\beta_0/2}|\mathrm{Im}(Y-Z_1)|^{p\beta_0/2}]^{-1} dY.
$$

Denote $\rho_0 = |\text{Re}(Z_1 - Z_0)| = |x_1 - x_0|$, $L_1 = D_0 \cap \{|x - x_0| \le 2\rho_0, Y = Y_0\}$ and $L_2 = D_0 \cap \{2\rho_0 < |x - x_0| \le 2\rho_1 < \infty, Y = Y_0\} \supset [d_1, d_2]\backslash L_1$, where ρ_1 is a sufficiently large positive number, we can derive

$$\begin{aligned}
&J_2 \le k_5[\int_{L_1} \frac{1}{|x-x_0|^{p(1-\beta_0/2)}|x-x_1|^{p(1-\beta_0/2)}}dx \\
&+ \int_{L_2} \frac{1}{|x-x_0|^{p(1-\beta_0/2)}|x-x_1|^{p(1-\beta_0/2)}}dx] \\
&\le k_5[|x_1-x_0|^{1-2p+p\beta_0} \int_{|\xi|\le 2} \frac{1}{|\xi|^{p(1-\beta_0/2)}|\xi \pm 1|^{p(1-\beta_0/2)}}d\xi \\
&+k_6|\int_{2\rho_0}^{2\rho_1} \rho^{p\beta_0-2p}d\rho|] \le k_7|x_1-x_0|^{1-p(2-\beta_0)} = \\
&= k_7|x_1-x_0|^{p(2/(m+2)-\varepsilon+1/p-2)},
\end{aligned}$$

in which we use $|x - x_0| = \xi|x_1 - x_0|$, $|x - x_1| = |x - x_0 - (x_1 - x_0)| = |\xi \pm 1||x_1 - x_0|$ if $x \in L_1$, $|x - x_0| = \rho \le 2|x - x_1|$ if $x \in L_2$, choose that $p(> 1)$ is close to 1 such that $1 - p(2 - \beta_0) < 0$, and $k_j = k_j(\beta, k_3, H, D_0)$ $(j = 6, 7)$ are non-negative constants. Thus we get

$$|\Psi(Z_1)-\Psi(Z_0)| \le k_7|Z_1-Z_0||x_1-x_0|^{2/(m+2)-\varepsilon+1/p-2} \le k_8|Z_1-Z_0|^{\beta}, \tag{2.16}$$

in which we use that the inner angle at Z_0 of the triangle $Z_0Z_1Z_2$ $(Z_2 = x_0 + iY_1 \in D_0)$ is not less than $\pi/6$ and not greater than $\pi/3$, and choose $\varepsilon = 2(p - 1)/p$, $\beta = 2/(m + 2) - \delta, \delta = 3(p - 1)/p$, $k_8 = k_8(\beta,\ k_3, H, D_0)$ is a non-negative constant. The above points $Z_0 = x_0$, $Z_1 = x_1 + iY_1$ can be replaced by $Z_0 = x_0 + iY_0$, $Z_1 = x_1 + iY_1 \in \overline{D_0}$, $0 < Y_0 < Y_1$ and $2(Y_1 - Y_0)/\sqrt{3} \le |Z_1 - Z_0| \le 2(Y_1 - Y_0)$.

Finally we consider any two points $Z_1 = x_1 + iY_1$, $Z_2 = x_2 + iY_1$ and $x_1 < x_2$, from the above estimates, the following estimate can be derived

$$\begin{aligned}
|\Psi(Z_1) - \Psi(Z_2)| &\le |\Psi(Z_1) - \Psi(Z_3)| + |\Psi(Z_3) - \Psi(Z_2)| \\
&\le k_8|Z_1-Z_3|^{\beta} + k_8|Z_3-Z_2|^{\beta} \le k_9|Z_1-Z_2|^{\beta},
\end{aligned} \tag{2.17}$$

where $Z_3 = (x_1 + x_2)/2 + i[Y_1 + (x_2 - x_1)/(2\sqrt{3})]$. If $Z_1 = x_1 + iY_1$, $Z_2 = x_1 + iY_2$, $Y_1 < Y_2$, and we choose $Z_3 = x_1 + (Y_2 - Y_1)/2\sqrt{3} + i(Y_2 + Y_1)/2$, and can also get (2.17). If $Z_1 = x_1 + iY_1$, $Z_2 = x_2 + iY_2$, $x_1 < x_2, Y_1 < Y_2$, and we choose $Z_3 = x_2 + iY_1$, obviously

$$|\Psi(Z_1) - \Psi(Z_2)| \le |\Psi(Z_1) - \Psi(Z_3)| + |\Psi(Z_3) - \Psi(Z_2)|, \tag{2.18}$$

and $|\Psi(Z_1)-\Psi(Z_3)|$, $|\Psi(Z_3)-\Psi(Z_2)|$ can be estimated by the above way, hence we can obtain the estimate of $|\Psi(Z_1)-\Psi(Z_2)|$. For other case, the similar estimate can be also derived. Hence we have the estimate (2.13).

Remark 2.1 If the condition $H(y)=y^{m/2}$ in Lemma 2.1 is replaced by $H(y)=y^{\eta}$, herein η is a positive constant satisfying the inequality $\eta<(m+2)/2$, then by the same method we can prove that the integral $\Psi(Z)=T(g/H)$ satisfies the estimate

$$C_{\beta}[\Psi(Z),D_Z]\le M_1,$$

where $\beta=1-2\eta/(m+2)-\delta$, δ is a sufficiently small positive constant, and $M_1=M_1(\beta,k_3,H,D_Z)$ is a positive constant. In particular if $H(y)=y$, i.e. $\eta=1$, then we can choose $\beta=m/(m+2)-\delta$, δ is a sufficiently small positive constant.

Now we give two representation theorems of solutions of Problem A for system (2.2) or equation (2.5).

Theorem 2.2 *Suppose that the equation* (2.5) *satisfies Condition C. Then any solution of Problem A for* (2.5) *can be expressed as*

$$w[z(Z)]=[\tilde{\Phi}(Z)+\tilde{\psi}(Z)]e^{\tilde{\phi}(Z)} \text{ in } D_Z, \tag{2.19}$$

where $\tilde{\psi}(Z)$, $\tilde{\phi}(Z)$ possess the form

$$\tilde{\phi}(Z)=T\tilde{h}=-\frac{1}{\pi}\iint_{D_0}\frac{\tilde{h}(t)}{t-Z}d\sigma_t \text{ in } D_Z,$$

$$\tilde{h}(Z)=\begin{cases}\dfrac{1}{H(y)}\{A_1[z(Z)]+A_2[z(Z)]\dfrac{\overline{w[z(Z)]}}{w[z(Z)]}\} \text{ if } w[z(Z)]\ne 0, Z\in D_Z,\\ 0 \text{ if } w[z(Z)]=0,\ Z\in D_Z, \text{ or } Z\in D_0\backslash D_Z,\end{cases}$$

$$\tilde{\psi}(Z)=T\tilde{f}=-\frac{1}{\pi}\iint_{D_0}\frac{\tilde{f}(t)}{t-Z}d\sigma_t, \tilde{f}(Z)=\frac{A_3[z(Z)]}{H(y)}e^{-\tilde{\phi}(Z)},$$

in which D_0 is as stated before, $\tilde{\phi}(Z)$, $\tilde{\psi}(Z)$ satisfy the estimate similar to that in (2.13), *$Z=x+iY=x+iG(y)$, and $\tilde{\Phi}(Z)$ is an analytic function in D_Z satisfying the estimate*

$$C_{\delta}[X(Z)\tilde{\Phi}(Z),\overline{D_Z}]\le M_2, \tag{2.20}$$

where $X(Z)=|Z-t_1|^{\eta_1}|Z-t_2|^{\eta_2}$, here $\eta_j=\max(-4\gamma_j,0)+8\delta$, $j=1,2$, γ_j $(j=1,2)$ are as stated in (2.10), *and $t_1=-1,t_2=1$, δ is a sufficiently* *small positive constant,* $k = (k_0, k_1, k_2)$, *and* $M_2 = M_2(\delta, k, H, D_Z)$ *is a non-negative constant..*

Proof On the basis of Lemma 2.1, we see that $\tilde{\phi}(Z), \tilde{\psi}(Z)$ in $\overline{D_Z}$ satisfy the similar estimate as in (2.13). Next it is easy to derive that

$$\tilde{\Phi}_{\overline{Z}} = [w_{\overline{Z}} - w(A_1 + A_2\overline{w}/w)/H - A_3/H]e^{-\tilde{\phi}(Z)} = 0 \text{ in } D_Z,$$

namely $\tilde{\Phi}(Z)$ is an analytic function in D_Z, which satisfies the boundary conditions

$$\begin{aligned} &\text{Re}[\overline{\lambda(z(Z))}e^{\tilde{\phi}(Z)}\Phi(Z)] = r[z(Z)] - \text{Re}[\overline{\lambda(z(Z))}e^{\tilde{\phi}(Z)}\tilde{\psi}(Z)] \text{ on } \partial D_Z^*, \\ &\text{Im}[\overline{\lambda(z_0)}e^{\tilde{\phi}(Z_0)}\tilde{\Phi}(Z_0)] = b_0 - \text{Im}[\overline{\lambda(z_0)}e^{\tilde{\phi}(Z_0)}\tilde{\psi}(Z_0)], \end{aligned} \tag{2.21}$$

in which $z(Z)$ is the inverse function of $Z(z)$, $Z_0 = Z(z_0)$, $\partial D_Z^* = \partial D_Z \backslash \{-1, 1\}$, and the index of $\lambda[z(Z)]\exp[\overline{\tilde{\phi}(Z)}]$ on ∂D_Z is $K = 0$. Hence according to the proof of Theorems 1.1 and 1.8, Chapter IV, [87]1), we can derive that $\tilde{\Phi}(Z)$ in $\overline{D_Z}$ satisfies the estimate (2.20). This completes the proof.

Theorem 2.3 *Suppose that the equation* (2.5) *satisfies Condition C. Then any solution of Problem A for* (2.5) *can be expressed as*

$$w[z(Z)] = \Phi(Z)e^{\phi(Z)} + \psi(Z) \text{ in } D_Z, \tag{2.22}$$

where $\psi(Z)$, $\phi(Z)$ *possess the form*

$$\begin{aligned} &\psi(Z) = Tf = -\frac{1}{\pi}\iint_{D_0} \frac{f(t)}{t-Z}d\sigma_t, L_\infty[f(Z)H(y), D_Z] < \infty \\ &\phi(Z) = Th = -\frac{1}{\pi}\iint_{D_0} \frac{h(t)}{t-Z}d\sigma_t \text{ in } D_Z, \\ &h(Z) = \begin{cases} \dfrac{1}{H(y)}\{A_1[z(Z)] + A_2[z(Z)]\dfrac{\overline{W(Z)}}{W(Z)}\} \text{ if } W(Z) \neq 0, Z \in D_Z, \\ 0 \text{ if } W(Z) = 0, \ Z \in D_Z \cup \{D_0 \backslash D_Z\}, \end{cases} \end{aligned}$$

in which $\psi(Z)$, $\phi(Z)$ *satisfy the estimate* (2.13), $W(Z) = w[z(Z)] - \psi(Z)$, $Z = x + iY = x + iG(y)$, *and* $\Phi[z(Z)]$ *is an analytic function in* D_Z.

Proof Firstly by using the method of parameter extension as stated in the proof of Theorem 2.5 below, Lemma 3.4, Chapter IV, [86]9), or Theorem

3.3, Chapter II, [87]1), we can find a solution of equation (2.5) in the form

$$\psi(Z) = -\frac{1}{\pi}\iint_{D_0}\frac{f(t)}{t-Z}d\sigma_t,\ H(y)f(Z)\in L_\infty(D_Z).$$

On the basis of Theorem 2.2, the solution of (2.5) in D_Z can be expressed by $\psi(Z)=\tilde\psi(Z)e^{\tilde\phi(Z)}$, where

$$\tilde\phi(Z)=T\tilde h=-\frac{1}{\pi}\iint_{D_0}\frac{\tilde h(t)}{t-Z}d\sigma_t \text{ in } D_Z,$$

$$\tilde h(Z)=\begin{cases}\dfrac{1}{H(y)}\{A_1[z(Z)]+A_2[z(Z)]\dfrac{\overline{\psi(Z)}}{\psi(Z)}\} \text{ if } \psi(Z)\neq 0, Z\in D_0,\\ 0 \text{ if } \psi(Z)=0,\ Z\in D_0,\end{cases}$$

$$\tilde\psi(Z)=T\tilde f=-\frac{1}{\pi}\iint_{D_0}\frac{\tilde f(t)}{t-Z}d\sigma_t, \tilde f(Z)=A_3[z(Z)]e^{-\tilde\phi(Z)},$$

it is clear that the functions $\tilde\phi(Z),\tilde\psi(Z)$ satisfy the estimate similar to (2.13).

Next let $w(z)$ be a solution of Problem A for equation (2.5), it is clear that $W(Z)=\Phi(Z)e^{\phi(Z)}=w[z(Z)]-\psi(Z)$ is a solution of the complex equation

$$W_{\overline{Z}}=A_1W(Z)+A_2\overline{W(Z)} \text{ in } D_Z,$$

where $\psi(Z)$ is as stated in (2.22), and we can verify that the function $\Phi(Z)$ is an analytic function in D_Z. Finally applying Theorem 1.3, we can find an analytic function $\Phi(Z)$ in D_Z satisfying the boundary conditions

$$\operatorname{Re}[\overline{\lambda(z(Z))}e^{i\operatorname{Im}\phi(Z)}\Phi(Z)]=\{r[z(Z)]-\operatorname{Re}[\overline{\lambda(z(Z))}\psi(Z)]\}e^{-\operatorname{Re}\phi(Z)}$$

$$\text{on } \partial D^*,\ \operatorname{Im}[\overline{\lambda(z_0)}e^{i\operatorname{Im}\phi(Z_0)}\Phi(Z_0)]=\{b_0-\operatorname{Im}[\overline{\lambda(z_0)}\psi(Z_0)]\}e^{-\operatorname{Re}\phi(Z_0)}, \tag{2.23}$$

herein $Z_0=Z(z_0)$, hence the function $w[z(Z)]=\Phi(Z)e^{\phi(Z)}+\psi(Z)$ in (2.22) is just the solution of Problem A in D_Z for equation (2.5).

On the basis of Lemma 2.1 and the above discussion, we can obtain the estimates of solutions of Problem A for equation (2.5), namely

Theorem 2.4 *Any solution $w[z(Z)]$ of Problem A for equation* (2.5) *satisfies the estimates*

$$\hat C_\delta[w(z),\overline{D}]=C_\delta[X(Z)w(z(Z)),\overline{D_Z}]\le M_3,\ \hat C_\delta[w(z),\overline{D}]\le M_4(k_1+k_2), \tag{2.24}$$

in which $X(Z) = |Z-t_1|^{\eta_1}|Z-t_2|^{\eta_2}$, *here* $\eta_j = \max(-4\gamma_j, 0)+8\delta$, $j = 1, 2$, $\gamma_j(j = 1, 2)$ *are as stated in* (2.10), *and* $t_1 = -1, t_2 = 1$, δ *is a sufficiently small positive constant,* $k = (k_0, k_1, k_2)$, *and* $M_3 = M_3(\delta, k, H, D)$, $M_4 = M_4(\delta, k_0, H, D)$ *are non-negative constants.*

Proof Noticing the conditions (2.7), and using Lemma 2.1 and Theorem 2.3, we see that the functions $\psi(Z)$, $\phi(Z)$ in (2.22) satisfy the estimates

$$C_\beta[\psi(Z), \overline{D_Z}] \le M_5,\ C_\beta[\phi(Z), \overline{D_Z}] \le M_5, \tag{2.25}$$

where $\beta = 2/(m+2)-\varepsilon$, ε is a sufficiently small positive constant, and $M_5 = M_5(\beta, k, H, D)$ is a non-negative constant. Due to the analytic function $\Phi(Z)$ satisfies the boundary condition (2.23), and from (2.20) and Theorem 2.3, we can get the representation and estimate of the analytic function $\Phi(Z)$ in D_Z similar to those in (2.22) and (2.20), thus the first estimate of (2.24) is derived. Moreover we verify the second estimate in (2.24). If $k = k_1+k_2 > 0$, then the function $w^*(z) = u^*(z)+iv^*(z) = u(z)/k+iv(z)/k$ is a solution of Problem A for equation

$$w^*_{\overline{Z}}=g^*(Z), g^*(Z)=\frac{g(Z)}{kH(y)}=\frac{1}{H(y)}[A_1w^*+A_2\overline{w^*}+\frac{A_3}{k}] \text{ in } D_Z. \tag{2.26}$$

By the proof of the first estimate in (2.24), we can derive the estimate of the solution $w^*(z)$:

$$\hat{C}_\beta[w^*(z), \overline{D}] \le M_4 = M_4(\beta, k_0, H, D). \tag{2.27}$$

From the above estimate it follows that the second estimate of (2.24) holds with $k > 0$. If $k = 0$, we can choose any positive number ε to replace $k = 0$. By using the same proof as before, we have

$$\hat{C}_\beta[w(z), \overline{D}] \le M_4\varepsilon.$$

Let $\varepsilon \to 0$, it is obvious that the second estimate in (2.24) with $k = 0$ is derived.

2.3 Solvability of Riemann-Hilbert problem for degenerate elliptic complex equations

Theorem 2.5 *Suppose that equation* (2.2) *satisfies Condition* C. *Then Problem* A *for* (2.5) *has a unique solution in* D.

Proof We first verify the uniqueness of the solution of Problem A for system (2.2) or equation (2.5). Let $w_1(z), w_2(z)$ be any two solutions of

Problem A for (2.5). It is easy to see that $w(z) = w_1(z) - w_2(z)$ satisfy the homogeneous equation and boundary conditions

$$\begin{gathered} w_{\overline{Z}} = [A_1 w + A_2 \overline{w}]/H(y) \text{ in } D_Z, \\ \mathrm{Re}[\overline{\lambda((Z))} w(z(Z))] = 0 \text{ in } \partial D^*, \ \mathrm{Im}[\overline{\lambda(z_0)}\Phi(Z_0)] = 0. \end{gathered} \tag{2.28}$$

Due to the solution $w[z(Z)]$ possesses the expression (2.22), but $\psi(Z) = 0$ in D_Z, and the index $K = 0$ of $\lambda[z(Z)]$ on ∂D_Z, from Theorem 1.1, Chapter IV, [87]1), it is not difficult to derive that $\Phi(Z) = 0$ in D_Z, hence $w(z) = w_1(z) - w_2(z) = 0$ in D.

As for the existence of solutions of Problem A for equation (2.5), we can prove by using the method of parameter extension. In fact, the complex equation (2.5) can be rewritten as

$$\begin{gathered} w_{\overline{Z}} = F(Z, w), \\ F(Z, w) = \frac{1}{H(y)}\{A_1[z(Z)]w + A_2[z(Z)]\overline{w} + A_3[z(Z)]\} \text{ in } D_Z. \end{gathered} \tag{2.29}$$

In order to find a solution $w(z)$ of Problem A in D, we can express $w(z)$ in the form (2.22), and consider the equation with the parameter $t \in [0, 1]$:

$$w_{\overline{Z}} - tF(z, w) = S(z) \text{ in } \overline{D_Z}, \tag{2.30}$$

in which the function $S(z)$ satisfies the condition

$$H(y)X(Z)S(z) \in L_\infty(\overline{D_Z}), \tag{2.31}$$

where $X(Z)$ is as stated in (2.20). This problem is called Problem A_t.

When $t = 0$, the complex equation (2.30) becomes the equation

$$w_{\overline{Z}} = S(z) \text{ in } \overline{D_Z}. \tag{2.32}$$

It is clear that the unique solution of Problem A_0, i.e. Problem A for $w_{\overline{Z}} = S(z)$ can be found, namely $X(Z)w[z(Z)] = \Phi(Z) + TXS$. Suppose that when $t = t_0$ $(0 \le t_0 < 1)$, Problem A_{t_0} is solvable, i.e. Problem A_{t_0} for (2.30) has a solution $w_0(z)$ $(w_0(z) \in \hat{C}(\overline{D})$, i.e. $X[Z(z)]w_0(z) \in C(\overline{D}))$. We can find a neighborhood $T_\varepsilon = \{|t - t_0| \le \varepsilon, 0 \le t \le 1\}(0 < \varepsilon < 1)$ of t_0 such that for every $t \in T_\varepsilon$, Problem A_t is solvable. In fact, Problem A_t can be written in the form

$$w_{\overline{Z}} - t_0 F(z, w) = (t_0 - t)F(z, w) + S(z) \text{ in } \overline{D_Z}, \tag{2.33}$$

Replacing $w_0(z)$ into the right-hand side of (2.33) by a function $w_0(z) \in \hat{C}(\overline{D})$, especially, we select $w_0(z) = 0$ and substitute it into the right-hand side of (2.33), it is obvious that the boundary value problem for such equation in (2.33) then has a solution $w_1(z) \in \hat{C}(\overline{D})$. Using successive iteration, we obtain a sequence of solutions $w_n(z)\,(w_n(z) \in \hat{C}(\overline{D}),\, n = 1, 2, ...)$, which satisfy the equations

$$w_{n+1\overline{Z}} - t_0F(z, w_{n+1}) = (t-t_0)F(z, w_n) + S(z) \text{ in } \overline{D},$$
$$\mathrm{Re}[\overline{\lambda(z)}w_{n+1}(z)] = r(z) \text{ on } \partial D^*, \mathrm{Im}[\overline{\lambda(z_0)}w_{n+1}(z_0)] = b_0.$$

From the above formulas, it follows that

$$[w_{n+1} - w_n]_{\overline{Z}} - t_0[F(z, w_{n+1}) - F(z, w_n)]$$
$$= (t - t_0)[F(z, w_n) - F(z, w_{n-1})] \text{ in } D,$$
$$\mathrm{Re}[\overline{\lambda(z)}(w_{n+1}(z) - w_n(z))] = 0 \text{ on } \partial D^*,$$
$$\mathrm{Im}[\overline{\lambda(z_0)}(w_{n+1}(z_0) - w_n(z_0))] = 0.$$

Noting that

$$L_\infty[H(y)X(Z)(F(z, w_n) - F(z, w_{n-1})), \overline{D_Z}] \le 2k_0\hat{C}[w_n - w_{n-1}, \overline{D_Z}],$$

and then by Theorem 2.4, we can derive

$$\hat{C}[w_{n+1} - w_n, \overline{D_Z}] \le 2|t - t_0|M_4\hat{C}[w_n - w_{n-1}, \overline{D_Z}],$$

where the constant $M_4 = M_4(\beta, k_0, H, D)$ is as stated in (2.24). Choosing the constant ε so small such that $2\varepsilon M_4 \le 1/2$ and $|t - t_0| \le \varepsilon$, it follows that

$$\hat{C}[w_{n+1} - w_n, \overline{D_Z}] \le 2\varepsilon M_4\hat{C}[w_n - w_{n-1}, \overline{D_Z}] \le \frac{1}{2}\hat{C}[w_n - w_{n-1}, \overline{D_Z}],$$

and when $n, m \ge N_0 + 1\,(N_0$ is a positive integer),

$$\hat{C}[w_{n+1} - w_n, \overline{D_Z}] \le 2^{-N_0}\sum_{j=0}^{\infty}2^{-j}\hat{C}[w_1 - w_0, \overline{D_Z}] \le 2^{-N_0+1}\hat{C}[w_1 - w_0, \overline{D_Z}].$$

Hence $\{w_n(z)\}$ is a Cauchy sequence. According to the completeness of the Banach space $\hat{C}(\overline{D_Z})$, there exists a function $w_*(z) \in \hat{C}(\overline{D_Z})$, so that $\hat{C}[w_n - w_*, \overline{D_Z}] \to 0$ as $n \to \infty$, we can see that $w_*(z)$ is a solution of Problem A_t for every $t \in T_\varepsilon = \{|t - t_0| \le \varepsilon\}$. Because the constant ε is independent of $t_0\,(0 \le t_0 < 1)$, therefore from the solvability of Problem A_{t_0} when $t_0 = 0$, we can derive the solvability of Problem A_t when $t = \varepsilon, 2\varepsilon, ..., [1/\varepsilon]\varepsilon, 1$, where $[1/\varepsilon]$ means the integer part of $1/\varepsilon$. In particular, when $t = 1$ and $S(z) = 0$, Problem A_1, i.e. Problem A for (2.5) in D has a solution $w(z)$.

3 The Discontinuous Riemann-Hilbert Problem for Quasilinear Degenerate Elliptic Complex Equations of First Order

In this section we discuss the discontinuous Riemann-Hilbert Problem for quasilinear degenerate elliptic system of first order equations in a bounded simply connected domain. We first give the representation of solutions of the boundary value problem for the equations, and then prove the existence and uniqueness of solutions for the problem.

3.1 Formulation of discontinuous Riemann-Hilbert problem for degenerate elliptic complex equations

Let D be a simply connected bounded domain in the complex plane $\mathbf{C}$ with the boundary $\partial D = \Gamma \cup \gamma$, where $\Gamma(\subset \{y>0\}) \in C^1_\alpha(0<\alpha<1)$ with the end points $z=-1,1$ and $\gamma=(-1,1)$ on the x-axis. As stated in Section 2, there is no harm in assuming that the boundary $\Gamma(\in C^1_\alpha)$ is a curve with the form $x=-1+G(y)\,(-1\le x\le 0)$ and $x=1-G(y)\,(0\le x\le 1)$ near the points $z=-1,1$. We consider the quasilinear degenerate elliptic system of first order equations

$$\begin{cases} H(y)u_x - v_y = a_1u+b_1v+c_1 \\ H(y)v_x + u_y = a_2u+b_2v+c_2 \end{cases} \text{in } D, \tag{3.1}$$

in which $H(y)=\sqrt{K(y)}, Y=G(y)=\int_0^y H(t)dt, G'(y)=H(y)$, $K(y)$ is the same as stated in (2.1), and $a_j, b_j, c_j (j=1,2)$ are functions of $(x,y)(\in D), u, v(\in \mathbf{R})$. The following degenerate elliptic system is a special case of system (3.1) with $H(y)=y^{m/2}$:

$$\begin{cases} y^{m/2}u_x - v_y = a_1u+b_1v+c_1 \\ y^{m/2}v_x + u_y = a_2u+b_2v+c_2 \end{cases} \text{in } D, \tag{3.2}$$

where m is a positive constant. According to Section 2, the system (3.1) can be written in the complex form

$$\begin{aligned} &w_{\bar z} = F(z,w),\ F(z,w) \\ &= A_1(z,w)w + A_2(z,w)\overline{w} + A_3(z,w) = g(Z) \text{ in } D, \text{ i.e.} \\ &w_{\overline{Z}} = [A_1w + A_2\overline{w} + A_3]/H(y) = g(Z)/H(y) \text{ in } D_Z, \end{aligned} \tag{3.3}$$

where

$$A_1=\frac{1}{4}[a_1+ia_2-ib_1+b_2], A_2=\frac{1}{4}[a_1+ia_2+ib_1-b_2], A_3=\frac{1}{2}[c_1+ic_2],$$

in which $w=u+iv$, $Z=x+iG(y)$, D_Z is the image domain of D with respect to the mapping $Z=Z(z)$.

Suppose that equation (3.3) satisfies **Condition** C, namely

1) $A_j(z,w)\,(j=1,2,3)$ are measurable in D for all continuous functions $w(z)$ in $D^*=\bar{D}\backslash\{-1,1\}$, and satisfy

$$L_\infty[A_j,\overline{D}]\le k_0,\ j=1,2,\ L_\infty[A_3,\overline{D}]\le k_1 \text{ in } D. \tag{3.4}$$

2) For any continuously differentiable functions $w_1(z),w_2(z)$ in D^*, the equality

$$F(z,w_1)-F(z,w_2)=\tilde{A}_1(w_1-w_2)+\tilde{A}_2(\overline{w_1}-\overline{w_2}) \text{ in } D \tag{3.5}$$

holds, where $\tilde{A}_j=\tilde{A}_j(z,w_1,w_2)\,(j=1,2)$ satisfy the conditions

$$L_\infty[\tilde{A}_j,\overline{D}]\le k_0,\ j=1,2, \tag{3.6}$$

in (3.4), (3.6), k_0, k_1 are non-negative constants. In particular, when (3.3) is a linear equation, the condition (3.5) obviously holds.

Now we formulate the general discontinuous Riemann-Hilbert boundary value problem. Let $Z'=\{z_1=-1,...,z_n=1,...,z_m=z_0\}$ be m points on $\Gamma\cup\gamma$ arranged according to the positive direction successively. Denote by Γ_j the curve on Γ from z_{j-1} to z_j, and Γ_j does not include the end point $z_{j-1},z_j\,(j=1,2,...,m)$.

Problem B Find a continuous solution $w(z)$ of (3.3) in $D^*=\overline{D}\backslash Z'$, which satisfies the boundary conditions

$$\begin{gathered}\operatorname{Re}[\overline{\lambda(z)}w(z)]=r(z) \text{ on } \partial D^*=\{\Gamma\cup\gamma\}\backslash Z',\\ \operatorname{Im}[\overline{\lambda(z_j')}w(z_j')]=b_j,\ j=1,...,2K+1=J,\end{gathered} \tag{3.7}$$

in which $\lambda(z)=\operatorname{Re}\lambda(z)+i\operatorname{Im}\lambda(z)\ne 0$ on $\Gamma\cup\gamma$, $z_j'\,(\notin Z',\ j=1,...,J)$ are distinct points on $\Gamma\cup\gamma$, $b_j\,(j=1,...,J)$ are real constants and $\lambda(z),r(z),b_j\,(j=1,...,J)$ satisfy the conditions

$$\begin{gathered}C_\alpha[\lambda(z),\Gamma_j]\le k_0,\ C_\alpha[R_j(z)r(z),\Gamma_j]\le k_2,\ j=1,...,m,\\ C_\alpha[\lambda(z),\gamma]\le k_0,\ C_\alpha[r(z),\gamma]\le k_2,\ |b_j|\le k_2,\ j=1,...,J,\end{gathered} \tag{3.8}$$

where $R_j(z)=|z-z_{j-1}|^{\beta_{j-1}}|z-z_j|^{\beta_j}$, $\beta_j\,(j=1,...,m)$ are similar to those in (1.39) with $a_0=1/4$, $\alpha\,(0<\alpha<1), k_0, k_2$ are non-negative constants, and the number

$$K=\frac{1}{2}(K_1+\cdots+K_m) \tag{3.9}$$

is called the index of Problem B, where

$$\begin{gathered}K_j=\left[\frac{\phi_j}{\pi}\right]+J_j,\ J_j=0 \text{ or } 1,\\ e^{i\phi_j}=\frac{\lambda(z_j-0)}{\lambda(z_j+0)},\gamma_j=\frac{\phi_j}{\pi}-K_j, j=1,...,m.\end{gathered} \tag{3.10}$$

Here we choose the index $K\geq-1/2$. From Theorems 3.2 and 3.4 below, we shall see that Problem B for (3.3) is well-posed.

3.2 Representation and uniqueness of solutions of discontinuous Riemann-Hilbert problem for elliptic complex equations

Now we give the representation theorem of solutions for equation (3.3).

Theorem 3.1 *Suppose that equation* (3.3) *satisfies Condition C. Then any solution of Problem B for* (3.3) *can be expressed as*

$$w[z(Z)]=[\tilde\Phi(Z)+\tilde\psi(Z)]e^{\tilde\phi(Z)}=\Phi(Z)e^{\phi(Z)}+\psi(Z), \tag{3.11}$$

where

$$\tilde\psi(Z)=T\tilde f=-\frac{1}{\pi}\iint_{D_t}\frac{\tilde f(t)}{t-Z}d\sigma_t,\ \tilde f(Z)=\frac{A_3[z(Z)]}{H(y)}e^{-\tilde\phi(Z)},$$

$$\tilde\phi(Z)=T\tilde h=-\frac{1}{\pi}\iint_{D_t}\frac{\tilde h(t)}{t-Z}d\sigma_t \text{ in } D_Z,$$

$$\tilde h(Z)=\begin{cases}\dfrac{1}{H(y)}\{A_1[z(Z)]+A_2[z(Z)]\dfrac{\overline{w(z(Z))}}{w(z((Z))}\} \text{ if } w[z(Z)]\neq0, Z\in D_Z,\\ 0 \text{ if } w[z(Z)]=0,\ Z\in D_Z,\end{cases}$$

$$\psi(Z)=Tf=-\frac{1}{\pi}\iint_{D_t}\frac{f(t)}{t-Z}d\sigma_t, L_\infty[f(Z)H(y),D_Z]\leq k_3,$$

$$\phi(Z)=Th=-\frac{1}{\pi}\iint_{D_t}\frac{h(t)}{t-Z}d\sigma_t \text{ in } D_Z,$$

$$h(Z)=\begin{cases}\frac{1}{H(y)}\{A_1[z(Z)]+A_2[z(Z)]\frac{\overline{W(Z)}}{W(Z)}\} \text{ if } W(Z)\neq 0, Z\in D_Z,\\ 0 \text{ if } W(Z)=0,\ Z\in D_Z,\end{cases} \tag{3.12}$$

in which $W(Z) = w[z(Z)] - \psi(Z)$, $k_3 = k_3(k_0, k_1, k_2, H, D)$ *is a non-negative constant,* $Z = x + iY = x + iG(y)$, *and* $\Phi(Z)$ *is an analytic function in* D_Z *satisfying the boundary conditions*

$$\begin{aligned}&\text{Re}[\overline{\lambda(z(Z))}e^{\phi(Z)}\Phi(Z)]=r[z(Z)]-\text{Re}[\overline{\lambda(z(Z))}\psi(Z)] \text{ on } \partial D^*,\\ &\text{Im}[\overline{\lambda(z'_j)}e^{\phi(Z_j)}\Phi(Z_j)]=b_j-\text{Im}[\overline{\lambda(z'_j)}\psi(Z_j)], j=1,...,2K+1,\end{aligned} \tag{3.13}$$

where $Z_j=Z(z'_j)$, $j=1,...,2K+1$, *hence the function* $w[z(Z)]=\Phi(Z)e^{\phi(Z)}$ $+\psi(Z)$ *in* (3.11) *is just the solution of Problem* B *in* D_Z *for equation* (3.3).

Proof Let $w(z)$ be a solution of Problem B for equation (3.3), and be substituted in the positions of w in (3.3), thus the coefficients A_j $(j = 1, 2, 3)$ be determined. Moreover according to the method in the proof of Theorem 2.5, we can find the solution $\psi(Z)$ of the linear complex equation

$$w_{\overline{Z}} = [A_1 w + A_2 \overline{w} + A_3]/H(y) \text{ in } D_Z, \tag{3.14}$$

and the function $\psi(Z) = \tilde{\psi}(Z)e^{\tilde{\phi}(Z)}$, herein $\tilde{\phi}(Z)$, $\tilde{\psi}(Z)$ are two double integrals as stated in the proof of Theorem 2.3 and satisfy the similar estimate in (2.13). Moreover the function $\phi(Z)$ is determined as stated in (3.12), and $\Phi(Z)$ is an analytic function in D_Z satisfying the boundary condition (3.13). It is clear that $w[z(Z)]$ possesses the representation (3.11).

Theorem 3.2 *Suppose that equation* (3.3) *satisfies Condition* C. *Then Problem* B *for* (3.3) *has at most one solution in* D.

Proof Let $w_1(z), w_2(z)$ be any two solutions of Problem B for (3.3). It is easy to see that $w(z) = w_1(z) - w_2(z)$ satisfy the homogeneous equation

$$w_{\overline{Z}} = [\tilde{A}_1 w + \tilde{A}_2 \overline{w}]/H(y) \text{ in } D_Z, \tag{3.15}$$

and homogeneous boundary condition of (3.13), i.e.

$$\begin{aligned}&\text{Re}[\overline{\lambda(z(Z))}e^{\phi(Z)}\Phi(Z)] = 0 \text{ in } \partial D_Z,\\ &\text{Im}[\overline{\lambda(z'_j)}e^{\phi(Z'_j)}\Phi(Z'_j)]=0,\ j=1,...,2K+1,\end{aligned} \tag{3.16}$$

where $Z_j' = Z(z_j')\,(j = 1, ..., m)$. According to the proof of Theorem 2.5, we can prove $\Phi(Z) = 0$ in D_Z, thus $w(z) = w_1(z) - w_2(z) = 0$ in D.

3.3 Estimates and existence of solutions of Riemann-Hilbert problem for degenerate elliptic complex equations

Now we shall give the estimates of the solutions of Problem B for (3.3) in $\overline{D}$. We rewrite equation (3.3) in the form

$$w_{\overline{z}} = F(z, w),\ F(z, w) = A_1 w + A_2 \overline{w} + A_3, \tag{3.17}$$

in which $A_j\,(j = 1, 2, 3)$ are as stated in (3.3).

Theorem 3.3 *Let equation* (3.3) *satisfy Condition C. Then any solution $w(z)$ of Problem B satisfies the estimates*

$$\begin{gathered}\hat{C}_\delta[w(z), \overline{D}] = C_\delta[X(Z)w(z(Z)), \overline{D_Z}] \le M_1,\\ \hat{C}_\delta[w(z), \overline{D}] \le M_2(k_1 + k_2),\end{gathered} \tag{3.18}$$

where

$$X(Z) = \prod_{j=0}^{m} |Z(z) - Z(t_j)|^{\eta_j}, \eta_j = \begin{cases} \max(-4\gamma_j, \beta_j) + 8\delta, j = 1, n,\\ \max(-\gamma_j, \beta_j) + 2\delta, j = 2, ..., m, j \ne n\end{cases} \tag{3.19}$$

herein $\gamma_j\,(j = 1, ..., m)$ are as stated in (3.10), *and $t_1 = z_1 = -1, ..., t_n = z_n = 1, ..., t_m = z_m$, $k = (k_0, k_1, k_2)$, and δ is a sufficiently small positive constant, and $M_1 = M_1(\delta, k, H, D)$, $M_2 = M_2(\delta, k_0, H, D)$ are non-negative constants.*

Proof Taking into account $A_j[z, w(z)] \in L_\infty(D_Z)$, $j = 1, 2, 3$, and applying (2.25), we get

$$C_\beta[\psi(Z), \overline{D_Z}] \le M_3,\ C_\beta[\phi(Z), \overline{D_Z}] \le M_3, \tag{3.20}$$

where $\phi(z)$, $\psi(z)$ are the functions as in (3.11), β is as stated in (2.25), and $M_3 = M_3(\beta, k, H, D)$ is a non-negative constant. Moreover due to the analytic function $\Phi(z)$ satisfies the boundary condition (3.13), similarly to (2.20), we can obtain the estimate

$$\hat{C}_\delta[\Phi(z), \overline{D}] \le M_4 = M_4(\delta, k, H, D). \tag{3.21}$$

Combining (3.20), (3.21), the first estimate in (3.18) is derived.

As for the second estimate in (3.18), which can be verified according to the proof of Theorem 2.4.

Now we prove the existence of solutions of Problem B for equation (3.3) by the method of continuity.

Theorem 3.4 *Suppose that equation* (3.3) *satisfies Condition C. Then the discontinuous Riemann-Hilbert problem* (*Problem* B) *for* (3.3) *has a solution.*

Proof We discuss the complex equation (3.17), i.e.

$$w_{\overline{Z}}=F(z,w),\ F(z,w)=[A_1w+A_2\overline{w}+A_3]/H(y) \text{ in } D_Z. \tag{3.22}$$

In order to find a solution $w(z)$ of Problem B in D by the method of continuity, we consider Problem B for the complex equation with the parameter $t\in[0,1]$:

$$w_{\overline{Z}}-tF(z,w)=S(z) \text{ in } \overline{D_Z}, \tag{3.23}$$

in which the function $S(z)$ satisfies the condition

$$H(y)X(Z)S(z)\in L_\infty(\overline{D_Z}). \tag{3.24}$$

This problem is called Problem B_t.

Let T be a point set in the interval [0,1], such that for every $t\in T$, Problem B_t for equation (3.23) has a solution $w(Z)\in\hat{C}_\delta(\overline{D_Z})$ for every function $S(Z)$ satisfying the condition (3.24). It is clear that when $t=0$, Problem B_0 for $w_{\overline{Z}}=S(Z)$ has a solution

$$X(Z)w(Z)=\Phi(Z)+TXS, \tag{3.25}$$

where $\Phi(Z)$ is an analytic function in D_Z. Hence T is non-empty. If we can prove that T is both open and closed in [0,1], then we can derive that $T=[0,1]$. In particular, when $t=1$ and $S(Z)=0$, Problem B_1 possesses a solution, i.e. Problem B for equation (3.22) is solvable.

In order to prove that T is a open set in [0,1], let $t_0\in T$. We rewrite (3.23) in the form

$$w_{\overline{Z}}-t_0F(z,w)=(t-t_0)F(z,w)+S(z) \text{ in } \overline{D_Z}, \tag{3.26}$$

Replacing $w_0(z)(\in\hat{C}(\overline{D}))$ into the right-hand side of (3.26) by a function $w_0(z)$, especially, we select $w_0(z)=0$ and substitute it into the right-hand side of (3.26), it is obvious that Problem B_{t_0} for such equation in (3.26) then has a solution $w_1(z)\,(w_1(z)\in\hat C(\overline{D}))$. Using successive iteration, we obtain a sequence of solutions $w_n(z)\,(w_n(z)\in\hat C(\overline{D}),\ n=1,2,...)$, which satisfy the equations and the boundary conditions

$$w_{n+1\overline{Z}}-t_0F(z,w_{n+1})=(t-t_0)F(z,w_n)+S(z)\ \text{ in }\ \overline{D},\tag{3.27}$$

$$\mathrm{Re}[\overline{\lambda(z)}w_{n+1}(z)]=r(z)\ \text{on}\ \partial D^*,\mathrm{Im}[\overline{\lambda(z_j')}w_{n+1}(z_j')]=b_j,j=1,...,2K+1.\tag{3.28}$$

From the above formulas, it follows that

$$\begin{aligned}&[w_{n+1}-w_n]_{\overline{Z}}-t_0[F(z,w_{n+1})-F(z,w_n)]\\&=(t-t_0)[F(z,w_n)-F(z,w_{n-1})]\ \text{ in }\ D,\\&\mathrm{Re}[\overline{\lambda(z)}(w_{n+1}(z)-w_n(z))]=0\ \text{ on }\ \partial D^*,\\&\mathrm{Im}[\overline{\lambda(z_j')}(w_{n+1}(z_j')-w_n(z_j'))]=0,j=1,...,2K+1.\end{aligned}\tag{3.29}$$

Noting that

$$L_\infty[H(y)X(Z)(F(z,w_n)-F(z,w_{n-1})),\overline{D_Z}]\le 2\hat C[w_n-w_{n-1},\overline{D_Z}],\tag{3.30}$$

and according to the proof of Theorem 2.5, we can derive

$$\hat C[w_{n+1}-w_n,\overline{D}]\le 2|t-t_0|M_2\hat C[w_n-w_{n-1},\overline{D}],\tag{3.31}$$

where the constant $M_2=M_2(\delta,k_0,H,D)$ is as stated in (3.18). Choosing the constant ε so small such that $2\varepsilon M_2\le 1/2$ and $|t-t_0|<\varepsilon$, it follows that

$$\hat C[w_{n+1}-w_n,\overline{D}]\le 2\varepsilon M_2\hat C[w_n-w_{n-1},\overline{D}]\le\frac{1}{2}\hat C[w_n-w_{n-1},\overline{D}],\tag{3.32}$$

and when $n,m\ge N_0+1\,(N_0$ is a positive integer),

$$\hat C[w_{n+1}-w_n,\overline{D}]\le 2^{-N_0}\sum_{j=0}^{\infty}2^{-j}\hat C[w_1-w_0,\overline{D}]\le 2^{-N_0+1}\hat C[w_1-w_0,\overline{D}].$$

Hence $\{w_n(z)\}$ is a Cauchy sequence. According to the completeness of the Banach space $\hat C(\overline{D})$, there exists a function $w_*(z)\in\hat C(\overline{D})$, so that $\hat C[w_n-w_*,\overline{D}]\to 0$ as $n\to\infty$. Obviously $w_*(z)$ is a solution of Problem B_t for every $t\in T_\varepsilon=\{|t-t_0|<\varepsilon\}$. Because the constant ε is independent of $t_0\,(0\le t_0<1)$, therefore from the solvability of Problem B_{t_0} when $t_0=0$, we can derive the solvability of Problem B_t for equation (3.23) when $t \in T_\varepsilon$. This shows that the set T in [0,1] is open.

Finally we verify that T is closed in [0,1]. Let $t_n \in T\,(n = 1, 2, ...)$, and $t_n \to t_0$ as $n \to \infty$. We shall prove that Problem B_{t_0} for equation (3.23) is solvable. Denote by $w_n(z)\,(n = 1, 2, ...)$ the solutions of Problems $B_{t_n}\,(t_n \in T,\, n = 1, 2, ...)$ for the corresponding equations (3.23), which can be expressed by

$$X(Z)w_n[z(Z)] = \Phi_n(Z)e^{\phi_n(Z)} + \psi_n(Z),\, n = 1, 2, ...$$

and satisfy the estimate (3.18). Hence from $\{w_n(z)\}$, we can choose a subsequence $\{w_{n_k}(z)\}$, such that $X(Z)w_{n_k}[z(Z)]$ uniformly converges a function $X(Z)w_0[z(Z)]$ in $\overline{D_Z}$, it is clear that the function $w_0(z)$ is just the solution of Problem B_{t_0} for equation (3.23) with $t = t_0$. This completes the proof.

4 The Riemann-Hilbert Problem for Degenerate Elliptic Complex Equations of First Order in Multiply Connected Domains

This section deals with the Riemann-Hilbert problem for degenerate elliptic complex equations of first order in multiply connected domains. We first give the representation of solutions of the boundary value problem for the equations, and then prove the uniqueness and existence of solutions for the problem.

4.1 Formulation of Riemann-Hilbert problem for degenerate elliptic complex equations in multiply connected domains

Let D be an $(N + 1)$-connected bounded domain in the upper half-plane with the boundary $\Gamma = \cup_{j=0}^{N}\Gamma_j \in C_\alpha(0 < \alpha < 1)$, where $\Gamma_j(j = 1, ..., N)$ are located in the domain D_0 bounded by $\Gamma_0 = \Gamma_{N+1}$, there is no harm in assuming that $\Gamma_0 = \Gamma' \cup \gamma$, herein $\gamma = \{-1 < x < 1, y = 0\}$ and $\Gamma'(\in \{y > 0\})$ is a curve with the end points $z = \pm 1$, and the inner angles of Γ' and γ at $z = \pm 1$ are equal to π, because otherwise through a conformal mapping the above requirement can be realized. We consider the quasilinear degenerate elliptic equation of first order: (3.1) with **Condition**

C, its complex form is as follows

$$\begin{gathered} w_{\overline{z}}=F(z,w), F=A_1(z,w)w+A_2(z,w)\overline{w}+A_3(z,w), \text{ i.e.} \\ H(y)w_{\overline{Z}} = g(Z) \text{ in } D,\ A_3 = \frac{1}{2}[c_1+ic_2], \\ A_1=\frac{1}{4}[a_1+ia_2-ib_1+b_2], A_2=\frac{1}{4}[a_1+ia_2+ib_1-b_2], \end{gathered} \tag{4.1}$$

where the coefficients A_j $(j=1,2,3)$ in (3.1) satisfy

$$L_\infty[A_j,\overline{D}], L_\infty[\tilde{A}_j,\overline{D}] \le k_0, j=1,2,\ L_\infty[A_3,\overline{D}] \le k_1 \text{ in } D, \tag{4.2}$$

besides $\tilde{A}_j(j=1,2)$ are as stated in (3.6), and k_0, k_1 are non-negative constants. We mention that under Condition C, the above solution of equation (4.1) in D is a generalized solution, and if $A_j \in C_\alpha(D)$, then the solution of (4.1) is a classical solution.

The Riemann-Hilbert boundary value problem for equation (4.1) may be formulated as follows:

Problem A Find a continuous solution $w(z)$ of (4.1) in $\overline{D}$ satisfying the boundary condition

$$\mathrm{Re}[\overline{\lambda(z)}w(z)] = r(z),\ z \in \Gamma, \tag{4.3}$$

where $\lambda(z) \ne 0$, $z \in \Gamma$, and $\lambda(z)$, $r(z)$ satisfy the conditions

$$C_\alpha[\lambda(z),\Gamma] \le k_0,\ C_\alpha[r(z),\Gamma] \le k_2, \tag{4.4}$$

in which $\alpha\,(0<\alpha<1), k_2$ are non-negative constants.

The integer

$$K = \frac{1}{2\pi}\Delta_\Gamma \arg\lambda(z)$$

is the index of Problem A. When the index $K<0$, Problem A may not be solvable, when $K \ge 0$, the solution of Problem A is not necessarily unique. Hence we consider the well posedness of Problem A with the modified boundary conditions for the complex equation (4.1) as follows.

Problem B Find a continuous solution $w(z)$ of equation (4.1) satisfying the boundary condition

$$\mathrm{Re}[\overline{\lambda(z)}w(z)] = r(z)+h(z),\ z \in \Gamma, \tag{4.5}$$

where

$$h(z)=\begin{cases} 0,\ z\in\Gamma, & \text{if } K\geq N,\\ \left.\begin{array}{l} h_j,\ z\in\Gamma_j,\ j=1,...,N-K,\\ 0,\ z\in\Gamma_j, j=N-K+1,...,N+1 \end{array}\right\} & \text{if } 0\leq K<N,\\ \left.\begin{array}{l} h_j,\ z\in\Gamma_j,\ j=1,...,N,\\ h_0+\text{Re}\displaystyle\sum_{m=1}^{-K-1}(h_m^+ +ih_m^-)[\zeta(z)]^m, z\in\Gamma_0 \end{array}\right\} & \text{if } K<0, \end{cases} \tag{4.6}$$

in which $h_j\,(j=0,1,...,N)$, $h_m^\pm\,(m=1,...,-K-1,K<0)$ are unknown real constants to be determined appropriately, $\zeta=\zeta(z)$ is a conformal mapping from the bounded domain with the boundary Γ_0 onto $|\zeta|<1$. In addition, for $K\geq 0$ the solution $w(z)$ is assumed to satisfy the point conditions

$$\text{Im}[\overline{\lambda(a_j)}w(a_j)]=b_j, j\in J=\begin{cases} 1,...,2K-N+1, & \text{if } K\geq N,\\ N-K+1,...,N+1, & \text{if } 0\leq K<N, \end{cases} \tag{4.7}$$

where $a_j\in\Gamma_j\,(j=1,...,N)$, $a_j\in\Gamma_0\,(j=N+1,...,2K-N+1,K\geq N)$ are distinct points, and $b_j\,(j\in J)$ are all real constants satisfying the conditions

$$|b_j|\leq k_2,\ j\in J, \tag{4.8}$$

herein k_2 is a nonnegative constant.

4.2 Representation and uniqueness of solutions of Riemann-Hilbert problem for degenerate elliptic complex equations

It is easy to see that the complex equation

$$w_{\bar{\tilde{z}}}=0 \text{ in } \overline{D},\ \text{i.e. } w_{\overline{Z}}=0 \text{ in } D_Z \tag{4.9}$$

is a special case of equation (4.1). On the basis of the result in [86]9), we can find a unique solution of Problem B for equation (4.9) in $\overline{D_Z}$. Now we give the representation theorem of solutions for equation (4.1).

Theorem 4.1 *Suppose that the equation* (4.1) *satisfies Condition C. Then any solution of Problem B for* (4.1) *can be expressed as*

$$w[z(Z)] = W(Z) + \psi(Z) = \Phi(Z)e^{\phi(Z)}+\psi(Z), \tag{4.10}$$

where $\Phi(Z), \phi(Z), \psi(Z)$ *are as stated in* (2.22), $W(z)$ *is a solution of equation*

$$W_{\overline{Z}} = [A_1W + A_2\overline{W}]/H(y) \text{ in } D_Z, \tag{4.11}$$

and $\psi(Z)$ *is a solution of equation* (4.1) *in* D_Z *and possesses the form*

$$\psi(Z)=Tf=-\frac{1}{\pi}\iint_{D_t}\frac{f(t)}{t-Z}d\sigma_t \text{ in } D_Z, \tag{4.12}$$

$$f(Z) = [A_1\psi + A_2\overline{\psi}+A_3]/H(y) \text{ in } D_Z, \tag{4.13}$$

in which $Z = x + iY(y) = x + iG(y)$, *and* $W[Z(z)]$ *in* D *satisfies the boundary conditions*

$$\begin{aligned}&\text{Re}[\overline{\lambda(z)}W(z(Z))]=r(z)+h(z)-\text{Re}[\overline{\lambda(z)}\psi(Z(z))], z\in\Gamma,\\ &\text{Im}[\overline{\lambda(a_j)}W(a_j)] = b_j - \text{Im}[\overline{\lambda(a_j)}\psi(Z(a_j))],\ j \in J.\end{aligned} \tag{4.14}$$

Proof Let $w(z)$ be a solution of Problem B for equation (4.1), and be substituted in the coefficients of equation (4.1). By using the method in the proof of Theorem 3.4, we can find a solution $\psi(z)$ of such equation (4.1), and $\psi(z)$ possesses the form (4.12), (4.13). Moreover we can find the solution $W(z)$ in $\overline{D}$ of (4.11) with the boundary condition (4.14), thus

$$w[z(Z)] = W(Z) + \psi(Z) \text{ in } D \tag{4.15}$$

is the solution of Problem B in D_Z for equation (4.1), where $W(z) = \Phi(z)e^{\phi(z)}$ is as stated in (4.10).

Theorem 4.2 *Suppose that equation* (4.1) *satisfies Condition C. Then Problem B for* (4.1) *has at most one solution in D.*

Proof Let $w_1(z), w_2(z)$ be any two solutions of Problem B for (4.1). It is easy to see that $w(z) = w_1(z) - w_2(z)$ satisfies the homogeneous equation and boundary conditions

$$w_{\overline{z}} = \tilde{A}_1w + \tilde{A}_2\overline{w} \text{ in } D, \tag{4.16}$$

$$\text{Re}[\overline{\lambda(z)}w(z)] = 0 \text{ on } \Gamma, \text{Im}[\overline{\lambda(a_j)}w(a_j)] = 0,\ j \in J. \tag{4.17}$$

Noting the function $g(Z)$ in (4.1) with the condition $g(Z) \in L_\infty(D_Z)$, similarly to Theorem 2.5, by using the way as in Theorem 1.2, Chapter I and Theorem 4.1, Chapter II, [87]1), if the function $w(z) \not\equiv 0$ in $\overline{D}$, we can derive the absurd inequalities

$$\begin{gathered} 2K+1 \le 2N_D + N_\Gamma \le 2K, \ \text{ when } \ K \ge 0, \\ 2K-2N \le 2N_D + N_\Gamma \le 2N-2K-2, \ \text{ when } K<0, \end{gathered} \tag{4.18}$$

where N_D, N_Γ are denoted the totals of zero points of the solution $w(z)$ in D and Γ respectively. Hence $w(z) = w_1(z) - w_2(z) = 0$ in D. This proves the uniqueness of solutions of Problem B for (4.1).

4.3 Estimates of solutions of Riemann-Hilbert problem for degenerate elliptic equations

Now we shall give the estimates of the solutions of Problem B for (4.1) in $\overline{D}$, namely

Theorem 4.3 *If equation* (4.1) *satisfies Condition C, then any solution $w(z)$ of Problem B satisfies the estimates*

$$C_\delta[w(z(Z)), \overline{D_Z}] \le M_1, \ C_\delta[w(z(Z)), \overline{D_Z}] \le M_2(k_1+k_2), \tag{4.19}$$

here δ is a sufficiently small positive constant, and $M_1 = M_1(\delta, k, H, D)$, $M_2 = M_2(\delta, k_0, H, D)$ are non-negative constants.

Proof We first prove that if the solution $w(z)$ of Problem B satisfies the estimate of boundedness, i.e.

$$C[w(z(Z)), \overline{D_Z}] \le M_3, \tag{4.20}$$

where $M_3 = M_3(\delta, k, H, D)$ is a positive constant, then the first estimate of (4.19) will be derived, because from Lemma 2.1, it follows that $F(z, w) \in L_\infty(D_Z)$, hence $C_\beta[\psi(Z), \overline{D_Z})] \le M_4 = M_4(\beta, k, H, D_Z, M_3) < \infty$, β is as stated in (2.13). On basis of the representation (4.10), the function $W(Z) = w[z(Z)] - \psi(Z) = \Phi(Z)e^{\phi(Z)}$ in D_Z satisfies the boundary conditions

$$\begin{gathered} \operatorname{Re}[\overline{\lambda[z(Z)]}W(Z)] = r[z(Z)] - \operatorname{Re}[\overline{\lambda[z(Z)]}\psi(Z)] + h[z(Z)], Z \in \partial D_Z, \\ \operatorname{Im}[\overline{\lambda(a_j)}W[Z(a_j)] = b_j - \operatorname{Im}[\overline{\lambda(a_j)}\psi[Z(a_j)]], \ j \in J, \end{gathered} \tag{4.21}$$

where $\partial D_Z = Z(\Gamma)$, hence the analytic function $\Phi(Z) = W(Z)e^{-\phi(Z)}$ in D_Z satisfies the estimate $C_\delta[\Phi(Z), \overline{D_Z}] \le M_5 = M_5(\delta, k, H, D, M_3)$. Now we use the reduction to absurdity. Suppose that (4.20) is not true, then there exist sequences of coefficients $\{A_l^{(m)}\}$ $(l = 1, 2, 3)$, $\{\lambda^{(m)}(z)\}$, $\{r^{(m)}(z)\}$ and $\{b_j^{(m)}\}$, which satisfy the same conditions of coefficients as stated in (4.2), (4.4) and (4.8), such that $\{A_l^{(m)}\}$ weakly converge to $A_l^{(0)}$ $(l = 1, 2, 3)$ in D and $\{\lambda^{(m)}(z)\}$, $\{r^{(m)}(z)\}$, $\{b_j^{(m)}\}$ on Γ uniformly converge to $\lambda^{(0)}(z)$, $r^{(0)}(z)$, $b_j^{(0)}$ $(j \in J)$, and the solutions of the corresponding boundary value problems

$$w_{\bar{z}}^{(m)} = F^{(m)}(z, w^{(m)}), F^{(m)}(z, w^{(m)}) = A_1^{(m)} w^{(m)} + A_2^{(m)} \overline{w^{(m)}} + A_3^{(m)} \text{ in } \overline{D}, \tag{4.22}$$

$$\mathrm{Re}[\overline{\lambda^{(m)}(z)} w^{(m)}(z)] = r^{(m)}(z) \text{ on } \Gamma, \mathrm{Im}[\overline{\lambda^{(m)}(a_j)} w^{(m)}(a_j)] = b_j^{(m)}, j \in J, \tag{4.23}$$

have the solutions $w^{(m)}(z)$, but $C[w^{(m)}(z), \overline{D}]$ $(m = 1, 2, ...)$ are unbounded, hence we can choose a subsequence of $\{w^{(m)}(z)\}$ denote by $\{w^{(m)}(z)\}$ again, such that $H_m = C[w^{(m)}(z), \overline{D}] \to \infty$ as $m \to \infty$, we can assume $H_m \ge \max[k_1, k_2, 1]$. It is obvious that $\hat{w}^{(m)}(z) = w^{(m)}(z)/H_m$ are solutions of the boundary value problems

$$\hat{w}_{\bar{z}}^{(m)} = F^{(m)}(z, \hat{w}^{(m)}), F^{(m)}(z, \hat{w}^{(m)}) = A_1^{(m)} \hat{w}^{(m)} + A_2^{(m)} \overline{\hat{w}^{(m)}} + \frac{A_3^{(m)}}{H_m} \text{ in } \overline{D}, \tag{4.24}$$

$$\mathrm{Re}[\overline{\lambda^{(m)}(z)} \hat{w}^{(m)}(z)] = \frac{r^{(m)}(z)}{H_m} \text{ on } \Gamma, \mathrm{Im}[\overline{\lambda^{(m)}(a_j)} \hat{w}^{(m)}(a_j)] = \frac{b_j^{(m)}}{H_m}, j \in J. \tag{4.25}$$

It is easy to see that the functions in above boundary value problems satisfy the conditions

$$\begin{gathered} L_\infty[A_l, \overline{D}] \le k_0, l = 1, 2, L_\infty[A_3/H_m, \overline{D}] \le 1, C_\alpha[\lambda^{(m)}(z), \Gamma] \le k_0, \\ C_\alpha[r^{(m)}(z)/H_m, \Gamma] \le 1, \ |b_j^{(m)}/H_m| \le 1, \ j \in J. \end{gathered} \tag{4.26}$$

From the representation (4.10), the above solutions can be expressed as

$$\begin{gathered} \hat{w}^{(m)}(z) = \hat{W}^{(m)}(Z) + \hat{\psi}^{(m)}(Z), \\ \hat{\psi}^{(m)}(Z) = -\frac{1}{\pi} \iint_{D_t} \frac{\hat{f}^{(m)}(t)}{t - Z} d\sigma_t \text{ in } \overline{D_Z}, \end{gathered} \tag{4.27}$$

noting that $L_\infty[H(y)\hat{f}^{(m)}(Z), \overline{D_Z}] \le M_6 = M_6(k_0, H, D)$, we can derive that

$$C_\delta[\hat{\psi}^{(m)}(Z), D_Z] \le M_7 = M_7(\delta, k, H, D). \tag{4.28}$$

Due to the functions $\hat{W}^{(m)}(Z)$ are the solutions of the equation corresponding to (4.11) in $\overline{D_Z}$ and $\hat{w}^{(m)}(z)=\hat{W}^{(m)}(Z)+\hat{\psi}^{(m)}(Z)$ satisfy the boundary conditions as in (4.25), we can obtain the estimate

$$C_\delta[\hat{W}^{(m)}(Z),\overline{D_Z}]\le M_8=M_8(\delta,k,H,D). \tag{4.29}$$

Thus from $\{\hat{w}^{(m)}(z)\}=\{\hat{W}^{(m)}(Z)+\hat{\psi}^{(m)}(Z)\}$, we can choose a subsequence denoted by $\{\hat{w}^{(m)}(z)\}$ again, and $\{\hat{w}^{(m)}(z)\}=\{\hat{W}^{(m)}(Z)+\hat{\psi}^{(m)}(Z)\}$ uniformly converge to $\hat{w}^{(0)}(z)$, it is clear that $\hat{w}^{(0)}(z)$ is a solution of the homogeneous problem of Problem B, on the basis of Theorem 4.2, the solution $\tilde{w}^{(0)}(z)=0$ in $\overline{D}$, however, from $C[\hat{w}^{(m)}(z),\overline{D}]=1$, we can derive that there exists a point $z^*\in\overline{D}$, such that $\hat{w}^{(0)}(z^*)\ne 0$, it is impossible. This shows that (4.20) is true. By using the method from (4.20) to (4.28), (4.29), we can obtain the first estimate in (4.19). Moreover we can verify the second estimate in (4.19).

4.4 Existence of solutions of Riemann-Hilbert problem for degenerate elliptic equations

In this section, we prove the existence of solutions of Problem B for equation (4.1).

Theorem 4.4 *Let equation* (4.1) *satisfy Condition C. Then the Riemann-Hilbert problem* (*Problem B*) *for* (4.1) *in the multiply connected domain D has a unique solution.*

Proof In order to find a solution $w(z)$ of Problem B for equation (4.1) in D by the Leray-Schauder theorem, we consider the equation (4.1) with the parameter $t\in[0,1]$:

$$w_{\overline{\overline{z}}}=tF(z,w),\ F(z,w)=G(Z)=A_1w+A_2\overline{w}+A_3\ \text{ in } D_Z, \tag{4.30}$$

and introduce a bounded open set B_M of the Banach space $B=C_\delta(\overline{D_Z})$, whose elements are functions $w(z)$ satisfying the condition

$$w(z)\in C_\delta(\overline{D}),\ \ C_\delta[w(z(Z)),\overline{D_Z}]<M_9=1+M_1, \tag{4.31}$$

where δ, M_1 are constants as stated in (4.19). We choose an arbitrary function $W(z)\in B_M$ and substitute it in the position of w in $F(z,w)$. By Theorem 4.1, a solution $w(z)=\Phi(Z)+\Psi(Z)=W(Z)+T(tF)$ of Problem B for the complex equation

$$w_{\overline{\overline{z}}}=tF(z,W) \tag{4.32}$$

can be found. Noting that $tF[z(Z), W(z(Z))] \in L_\infty(\overline{D_Z})$, the above solution of Problem B for (4.32) is unique. Denote by $w(z) = T[W, t]\,(0 \le t \le 1)$ the mapping from $W(z)$ to $w(z)$. From Theorem 4.3, we know that if $w(z)$ is a solution $w(z)$ of Problem B for the equation

$$w_{\overline{z}} = tF(z, w) \text{ in } D_Z, \tag{4.33}$$

then the function $w(z)$ satisfies the estimate

$$C_\delta[w, \overline{D_Z})] < M_9. \tag{4.34}$$

Set $B_0 = B_M \times [0, 1]$. Now we verify the three conditions of the Leray-Schauder theorem:

1. For every $t \in [0, 1]$, $T[W, t]$ continuously maps the Banach space B into itself, and is completely continuous on B_M. In fact, arbitrarily select a sequence $W_n(z)$ in B_M, $n = 0, 1, 2, ...$, such that $C_\delta[W_n - W_0, \overline{D_Z}] \to 0$ as $n \to \infty$. By Condition C, we see that $L_\infty[F(z, W_n) - F(z, W_0), \overline{D}] \to 0$ as $n \to \infty$. Moreover, from $w_n = T[W_n, t]$, $w_0 = T[W_0, t]$, it is easy to see that $w_n - w_0$ is a solution of Problem B for the following complex equation

$$(w_n - w_0)_{\overline{z}} = t[F(z, W_n) - F(z, W_0)] \text{ in } D, \tag{4.35}$$

and then we can obtain the estimate

$$C_\delta[w_n - w_0, \overline{D}] \le 2k_0 C[W_n(z) - W_0(z), \overline{D}]. \tag{4.36}$$

Hence $C_\delta[w_n - w_0, \overline{D}] \to 0$ as $n \to \infty$. In addition for $W_n(z) \in B_M$, $n = 1, 2, ...$, we have $w_n = T[W_n, t]$, $w_m = T[W_m, t]$, $W_n, W_m \in B_M$, and then

$$(w_n - w_m)_{\overline{z}} = t[F(z, W_n) - F(z, W_m)] \text{ in } D, \tag{4.37}$$

where $L_\infty[F(z, W_n) - F(z, W_m), D_Z] \le 2k_0 M_9$, hence from (4.19), we can obtain the estimate

$$C_\delta[w_n - w_m, \overline{D_Z}] \le 2M_2 k_0 M_9. \tag{4.38}$$

Thus there exists a function $w_0(z) \in B_M$, from $\{w_n(z)\}$ we can choose a subsequence $\{w_{n_k}(z)\}$ such that $C_\delta[w_{n_k} - w_0, \overline{D_Z}] \to 0$ as $k \to \infty$. This shows that $w = T[W, t]$ is completely continuous in B_M. Similarly we can also prove that for $W(z) \in B_M$, $T[W, t)$ is uniformly continuous with respect to $t \in [0, 1]$.

2. For $t = 0$, it is evident that $w = T[W, 0] = \Phi(Z) \in B_M$.

3. From the estimate (4.19), we see that $w = T[W, t]\,(0 \le t \le 1)$ does not have a solution $w(z)$ on the boundary $\partial B_M = \overline{B_M} \backslash B_M$.

Hence by the Leray-Schauder theorem, there exists a function $w(z) \in B_M$ such that $w(z) = T[w(z), t]$, and the function $w(z) \in C_\delta(\overline{D_Z})$ is just a solution of Problem B for the complex equation (4.1).

By a similar way as in the proof of Theorem 4.8, Chapter II, [87]1), from Theorem 4.4 the following result can be derived.

Theorem 4.5 *Under the same conditions as in Theorem* 4.4*, the following statements hold.*

(1) *If the index $K \ge N$, then Problem A for* (4.1) *is solvable.*

(2) *If $0 \le K < N$, then the total number of solvability conditions for Problem A does not exceed $N - K$.*

(3) *If $K < 0$, then Problem A has $N - 2K - 1$ solvability conditions.*

In latter chapters the notations $M_j = M_j(p_0, \delta, k, D)$, $M'_j = M'_j(p_0, \delta, k, D)$ (j is a positive integer) mean all non-negative constants dependent on p_0, δ, k, D.