Chapter 1

Introduction to harmonic maps

1.1 Dirichlet principle of harmonic maps

Harmonic maps are nonlinear extensions of harmonic functions. Just like harmonic functions, harmonic maps are critical points of a natural energy functional, called Dirichlet energy, of maps between two Riemannian manifolds.

Let (M, g) be a n-dimensional Riemannian manifold with or without boundary, endowed with a smooth Riemannian metric g. For any fixed point $p_0 \in M$, let $(x_1, \cdots, x_n)$ be a coordinate system near p_0 so that g can be represented by

$$g = \sum_{1 \le \alpha, \beta \le n} g_{\alpha\beta}\, dx_\alpha\, dx_\beta,$$

where $(g_{\alpha\beta})$ is a positive definitive symmetric $n \times n$ matrix. Let $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$ be the inverse matrix of $(g_{\alpha\beta})$ and $dv_g = \sqrt{g}\, dx = \sqrt{\det(g_{\alpha\beta})}\, dx$ be the volume element of (M, g). Let (N, h) be a l-dimensional compact Riemannian manifold without boundary which is endowed with a smooth Riemannian metric h.

Throughout this book we use the Einstein convention for summation. For any map $u \in C^2(M, N)$, we can define its Dirichlet energy as follows. For any fixed $p \in M$, there exist two normal coordinate charts $U_p \subset M$ of p and $V_q \subset N$ of $q = u(p)$ such that $u(U_p) \subset V_q$. The Dirichlet energy density function $e(u)$ is defined by

$$e(u)(x) \left(\equiv |\nabla u|_g^2\right) = \frac{1}{2} \sum_{\alpha,\beta} g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i}{\partial x_\alpha} \frac{\partial u^j}{\partial x_\beta}, \tag{1.1}$$

where (x_α) and (u^i) are the coordinate systems on U_p and V_q respectively. The Dirichlet energy functional is defined by

$$E(u) = \int_M e(u)\, dv_g. \tag{1.2}$$

Definition 1.1.1 A map $u \in C^2(M, N)$ is a harmonic map, if it is a critical point of the Dirichlet energy functional E.

We first have

Proposition 1.1.2 *A map $u \in C^2(M,N)$ is a harmonic map iff u satisfies*

$$\Delta_g u^i + g^{\alpha\beta}\Gamma^i_{jk}(u)\frac{\partial u^j}{\partial x_\alpha}\frac{\partial u^k}{\partial x_\beta} = 0 \quad \text{in} \quad M, \ (1 \le i \le l), \tag{1.3}$$

where Δ_g is Laplace-Beltrami operator on (M,g) given by

$$\Delta_g = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\alpha}\left(\sqrt{g}g^{\alpha\beta}\frac{\partial}{\partial x_\beta}\right)$$

and

$$\Gamma^i_{jk} = \frac{1}{2}h^{il}(h_{lj,k} + h_{kj,l} - h_{jk,l})$$

is the Christoffel symbol of the metric h on N.

Proof. Let $U \subset M$ be any coordinate chart and $\phi \in C^2_0(U, \mathbb{R}^l)$. Then we have

$$\begin{aligned} 0 &= \frac{d}{dt}|_{t=0}\left(\frac{1}{2}\int_M g^{\alpha\beta}h_{ij}(u+t\phi)(u^i_\alpha + t\phi^i_\alpha)(u^j_\beta + t\phi^j_\beta)\sqrt{g}\,dx\right) \\ &= \frac{1}{2}\int_M g^{\alpha\beta}h_{ij,k}(u)\phi^k u^i_\alpha u^j_\beta\sqrt{g}\,dx + \int_M g^{\alpha\beta}h_{ij}(u)u^i_\alpha\phi^j_\beta\sqrt{g}\,dx. \end{aligned}$$

This implies

$$\begin{aligned} \int_M \Delta_g u^i h_{ij}(u)\phi^j\,dv_g &= \frac{1}{2}\int_M g^{\alpha\beta}h_{ij,k}(u)u^i_\alpha u^j_\beta\phi^k\,dv_g \\ &\quad - \int_M g^{\alpha\beta}h_{ij,l}(u)u^i_\alpha u^l_\beta\phi^j\,dv_g. \end{aligned}$$

Choosing $\phi^j = h^{ji}\eta_i$ for $\eta = (\eta_1, \cdots, \eta_l) \in C^2_0(U, \mathbb{R}^l)$, we obtain

$$\begin{aligned} &\int_M \Delta_g u^i\eta^i\,dv_g \\ &= \frac{1}{2}\int_M g^{\alpha\beta}h^{mk}(u)(h_{ij,k}(u) - h_{ik,j}(u) - h_{jk,i}(u))u^i_\alpha u^j_\beta\eta_m\,dv_g. \end{aligned}$$

This yields (1.3). □

1.2 Intrinsic view of harmonic maps

For $u \in C^2(M,N)$, let T^*M be the cotangent bundle of M and u^*TN be the pull-back of the tangent bundle of N by u. View $du = \frac{\partial u^i}{\partial x_\alpha}dx_\alpha \otimes \frac{\partial}{\partial u^i}$ as a section of the bundle $T^*M \otimes u^*TN$. Then $e(u)$ equals to

$$e(u) = \frac{1}{2}\langle du, du\rangle_{T^*M\otimes u^*TN} = \frac{1}{2}\mathrm{tr}_g(u^*h),$$

where $\langle,\rangle_{T^*M\otimes u^*TN}$ denotes the inner product on $T^*M \otimes u^*TN$ induced from T^*M and u^*TN, and u^*h is the pull back of the metric tensor h by u, i.e.,

$$(u^*h)\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) = h\left(\frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta}\right) = h_{ij}(u)u^i_\alpha u^j_\beta. \tag{1.4}$$

Let ∇ denote the covariant derivative on $T^*M \otimes u^*TN$ induced from T^*M and u^*TN. Then we have (cf. Eells-Lemaire [50, 51])

Proposition 1.2.1 *$u \in C^2(M,N)$ is a harmonic map iff u satisfies*

$$\tau(u) := tr_g(\nabla du) = 0 \quad \text{in } M. \tag{1.5}$$

Note that

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x_\beta}}(du) &= \nabla_{\frac{\partial}{\partial x_\beta}}\left(u^i_\alpha dx_\alpha \otimes \frac{\partial}{\partial u^i}\right)\\
&= \frac{\partial^2 u^i}{\partial x_\alpha \partial x_\beta} dx_\alpha \otimes \frac{\partial}{\partial u^i} + u^i_\alpha (\nabla^{T^*M}_{\frac{\partial}{\partial x_\beta}} dx_\alpha) \otimes \frac{\partial}{\partial u^i}\\
&+ u^i_\alpha u^j_\beta \left(\nabla^{TN}_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i}\right) \otimes dx_\alpha.
\end{aligned}$$

Also

$$\nabla^{TN}_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i} = (\Gamma^N)^k_{ij}(u) \frac{\partial}{\partial u^k} \qquad \text{and } \nabla^{T^*M}_{\frac{\partial}{\partial x_\beta}} dx_\alpha = -(\Gamma^M)^\alpha_{\beta\gamma}(x)\, dx_\gamma,$$

we conclude that (1.5) is equivalent to

$$\tau^k(u) = g^{\alpha\beta}\left(u^k_{\alpha\beta} - (\Gamma^M)^\gamma_{\alpha\beta} u^k_\gamma + (\Gamma^N)^k_{ij}(u) u^i_\alpha u^j_\beta\right) = 0 \quad \text{in } M,\ 1 \le k \le l. \tag{1.6}$$

1.3 Extrinsic view of harmonic maps

By the isometric embedding theorem by Nash [150], we can assume that (N,h) is isometrically embedded into an Euclidean space $\mathbb{R}^L$ for some $L \ge 1$. Then

$$C^2(M,N) = \left\{u = \left(u^1, \cdots, u^L\right) \in C^2(M,\mathbb{R}^L) \mid u(M) \subset N\right\}.$$

Hence for $u \in C^2(M,N)$ the Dirichlet energy density is

$$e(u) = \frac{1}{2} g^{\alpha\beta} u^i_\alpha u^i_\beta.$$

As $N \subset \mathbb{R}^L$ is a compact smooth submanifold, it is well-known that there exists $\delta = \delta(N) > 0$ such that the nearest point projection map $\Pi_N : N_\delta \to N$ is smooth, where

$$N_\delta = \left\{y \in \mathbb{R}^L \mid d(y,N) := \inf_{z\in N} |y - z| < \delta\right\},$$

and $\Pi_N(y) \in N$ is such that $|y - \Pi_N(y)| = d(y,N)$ for $y \in N_\delta$.

Note that $P(y) = \nabla \Pi_N(y) : \mathbb{R}^L \to T_yN,\ y \in N$, is an orthogonal projection map, and

$$A(y) = \nabla P(y) : T_yN \otimes T_yN \to (T_yN)^\perp,\ y \in N,$$

is the second fundamental form of $N \subset \mathbb{R}^L$.

Now we have

Proposition 1.3.1 *$u \in C^2(M,N)$ is a harmonic map iff u satisfies*

$$\Delta_g u \perp T_u N. \tag{1.7}$$

Proof. For $\phi \in C_0^2(M, \mathbb{R}^L)$, one has

$$\begin{aligned} 0 &= \frac{d}{dt}|_{t=0} \int_M |\nabla (\Pi(u+t\phi))|^2 \, dv_g \\ &= 2\int_M \langle \nabla u, \nabla (P(u)(\phi))\rangle_g \, dv_g \\ &= -2\int_M \langle \Delta_g u, P(u)(\phi)\rangle \, dv_g \\ &= -2\int_M \langle P(u)(\Delta_g u), \phi\rangle \, dv_g. \end{aligned}$$

This clearly implies (1.7). □

Let $\{\nu_{l+1}(u), \cdots, \nu_L(u)\}$ be a local orthonormal frame of the normal bundle $(T_uN)^\perp$. Then (1.7) implies

$$\Delta_g u = \sum_{l+1\le i\le L} \lambda_i(x)\nu_i(u)$$

for some functions $(\lambda_{l+1}, \cdots, \lambda_L)$ on M. Moreover, for $l+1 \le i \le L$,

$$\begin{aligned} \lambda_i &= \Delta_g u \cdot \nu_i(u) \\ &= \mathrm{div}_g (\nabla u \cdot \nu_i(u)) - \nabla u \cdot \nabla (\nu_i(u)) \\ &= -(\nabla \nu_i)(u)(\nabla u, \nabla u) \end{aligned}$$

where we have used $\nabla u \cdot \nu_i(u) = 0$, and div_g is the divergence operator on (M, g) given by

$$\mathrm{div}_g = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_\alpha} \left(\sqrt{g} g^{\alpha\beta}\right).$$

Therefore we obtain the analytic version of (1.7):

$$\Delta_g u + A(u)(\nabla u, \nabla u) = 0 \ \text{ in } M, \tag{1.8}$$

where

$$A(u)(\nabla u, \nabla u) = \sum_{l+1\le i\le L} g^{\alpha\beta} A^i(u)(u_\alpha, u_\beta)\,\nu_i(u),$$

and $A^i = \nabla \nu_i$ is the second fundamental form of N in the normal direction ν_i.

Example 1.3.2 Let $M = T^n$ be the n-dimensional torus, and $N = S^k \subset \mathbb{R}^{k+1}$ be the unit sphere. Then $u \in C^2(T^n, S^k)$ is a harmonic map iff

$$\Delta u + |\nabla u|^2 u = 0 \ \text{ in } T^n. \tag{1.9}$$

1.4 A few facts about harmonic maps

Proposition 1.4.1 *If $\Phi : M \to M$ is a C^2-diffeomorphism and $u \in C^2(M, N)$ is a harmonic map with respect to (M, g), then $u \circ \Phi \in C^2(M, N)$ is a harmonic map with respect to $(M, \Phi^* g)$.*

Proof. This is an easy consequence of the identity:

$$\int_M |\nabla v|_g^2 \, dv_g = \int_M |\nabla (v \circ \Phi)|_{\Phi^* g}^2 \, dv_{\Phi^* g}$$

for all $v \in C^2(M, N)$. □

Proposition 1.4.2 *Let (M, g_1) be a Riemannian surface, $\Phi : (M, g_1) \to (M, g_2)$ be a conformal map. If $u \in C^2(M, N)$ is a harmonic map with respect to (M, g_2), then $u \circ \Phi \in C^2(M, N)$ is a harmonic map with respect to (M, g_1).*

Proof. This follows from the conformal invariance of the Dirichlet energy functional E in dimension two. In fact, let $\phi \in C^2(M)$ be such that $\Phi^* g_2 = e^{2\phi} g_1$. Then we have, for any $v \in C^2(M, N)$,

$$\begin{aligned} E(v \circ \Phi, g_1) &= \frac{1}{2} \int_M \mathrm{tr}_{g_1} \left((v \circ \Phi)^* h\right) dv_{g_1} \\ &= \frac{1}{2} \int_M \mathrm{tr}_{e^{-2\phi} \Phi^* g_2} \left(\Phi^* \circ v^* h\right) e^{-2\phi} \, dv_{\Phi^* g_2} \\ &= \frac{1}{2} \int_M \mathrm{tr}_{\Phi^* g_2} \left(\Phi^* \circ (v^* h)\right) dv_{\Phi^* g_2} \\ &= \frac{1}{2} \int_M \mathrm{tr}_{g_2} (v^* h) \, dv_{g_2} = E(v, g_2). \end{aligned}$$

This completes the proof of Proposition 1.4.2. □

Remark 1.4.3 (a) Harmonic maps from S^1 to N correspond to closed geodesics in N.
(b) The set of harmonic maps from a Riemannian surface M depends only on the conformal structures of M.
(c) Let $\mathrm{Id} : (M, g) \to (M, g)$ be the identity map. Then Id is a harmonic map.
(d) For $n = \dim(M) = 2$, any conformal map $\phi : (M, g_1) \to (M, g_2)$ is a harmonic map.

Proof. We only indicate the proof of (c). Denote $u(x) = \mathrm{Id}(x) = x$. Then we have

$$\begin{aligned} \tau^k(u) &= g^{\alpha\beta} \left(u^k_{\alpha\beta} - (\Gamma^M)^\gamma_{\alpha\beta} u^k_\gamma + (\Gamma^N)^k_{ij}(u) u^i_\alpha u^j_\beta \right) \\ &= g^{\alpha\beta} \left(0 - (\Gamma^M)^\gamma_{\alpha\beta} \delta_{k\gamma} + (\Gamma^M)^k_{ij}(u) \delta_{i\alpha} \delta_{i\beta} \right) = 0 \end{aligned}$$

for $1 \le k \le n$. □

1.5 Bochner identity for harmonic maps

One of the most important formulas for a harmonic map $u : M \to N$ is the differential equation satisfied by the energy density $e(u)$.

Denote by R^M and R^N the Riemannian curvature tensors of M and N respectively, Ric^M be the Ricci curvature of M. For $x_0 \in M$, in a local coordinate system centered at x_0, write

$$R^M = (R_{\alpha\beta\gamma\delta}),\ \mathrm{Ric}^M = (R_{\alpha\beta}),\ \text{and } R^N = (\hat{R}_{ijkl}),$$

and K^N denotes the sectional curvature of N

Theorem 1.5.1 *If $u \in C^2(M, N)$ is a harmonic map, then in a local coordinate system it holds*

$$\Delta_g e(u) = |\nabla du|^2 + R_{\alpha\beta}\left(u_\alpha, u_\beta\right) - \hat{R}_{ijkl}(u)\left(u^i_\alpha, u^j_\beta, u^k_\alpha, u^l_\beta\right) \tag{1.10}$$

where ∇ denotes the covariant derivative on $T^*M \otimes u^*TN$.

Proof. For $x_0 \in M$, let (x_α) be the normal coordinate system centered at x_0. Assume that (N, h) is isometrically embedded in $\mathbb{R}^L$. Then we have

$$\begin{aligned} \Delta_g e(u) &= |u_{\alpha\beta}|^2 + \langle u_\alpha, u_{\beta\alpha,\beta}\rangle \\ &= |u_{\alpha\beta}|^2 + \langle u_\alpha, u_{\beta\beta,\alpha}\rangle + R_{\alpha\beta}\left(u_\alpha, u_\beta\right) \\ &= |u_{\alpha\beta}|^2 + \langle u_\alpha, \left(\Delta_g u\right)_\alpha\rangle + R_{\alpha\beta}\left(u_\alpha, u_\beta\right) \end{aligned}$$

where we have used the Ricci identity

$$u_{\beta\alpha,\beta} = u_{\beta\beta,\alpha} + R_{\alpha\beta}u_\beta.$$

On the other hand, since u is harmonic map, (1.8) implies

$$\begin{aligned} \langle u_\alpha, \left(\Delta_g u\right)_\alpha\rangle &= -\langle u_\alpha, \left(A(u)(\nabla u, \nabla u)\right)_\alpha\rangle \\ &= \langle \Delta_g u, A(u)\left(\nabla u, \nabla u\right)\rangle \\ &= -\langle A(u)\left(\nabla u, \nabla u\right), A(u)\left(\nabla u, \nabla u\right)\rangle \\ &= -\langle A(u)\left(u_\alpha, u_\alpha\right), A(u)\left(u_\beta, u_\beta\right)\rangle \end{aligned}$$

where we have used the fact that

$$\langle u_\alpha, A(u)\left(\nabla u, \nabla u\right)\rangle = 0.$$

For $u_{\alpha\beta}$, it is easy to see that

$$|u_{\alpha\beta}|^2 = |P(u)(u_{\alpha\beta})|^2 + |A(u)\left(u_\alpha, u_\beta\right)|^2 = |\nabla du|^2 + |A(u)\left(u_\alpha, u_\beta\right)|^2.$$

Putting all these identities together, we obtain

$$\begin{aligned} \Delta_g e(u) &= |\nabla du|^2 + \mathrm{Ric}^M\left(\nabla u, \nabla u\right) \\ &- \left\{\langle A(u)\left(u_\alpha, u_\alpha\right), A(u)\left(u_\beta, u_\beta\right)\rangle - |A(u)\left(u_\alpha, u_\beta\right)|^2\right\} \end{aligned}$$

This, with the help of Gauss-Kodazi equation (see [63, 175]):

$$\langle R^N(u)(X,Y)X, Y\rangle = \langle A(u)(X,X), A(u)(Y,Y)\rangle - |A(u)(X,Y)|^2,\ \forall X, Y \in T_u N$$

yields (1.10). □

Proposition 1.5.2 *If (M,g) is compact without boundary with $Ric^M \geq 0$ and the sectional curvature of N, K^N, is non-positive. Then any harmonic map $u \in C^2(M,N)$ is totally geodesic. If $Ric^M > 0$ at a point in M, then u is constant. If $K^N < 0$, then either u is constant or $u(M)$ is contained in a closed geodesic.*

Proof. It follows from (1.10) that $e(u)$ is a subharmonic function on M. Hence the maximum principle implies $e(u) =$ constant and hence $|\nabla du| = 0$. This says that u is totally geodesic.
If $Ric^M(p_0) > 0$, then $\nabla u(p_0) = 0$ and hence $e(u) \equiv 0$ and u is constant map.
If $K^N < 0$, then the linear space span $\{u_1, \cdots, u_n\}$ is at most dimension one. Hence either u is constant or the image of u lies inside a geodesic. □

1.6 Second variational formula of harmonic maps

In this section, we derive the second variational formula for harmonic maps into spheres and general target manifolds.

Proposition 1.6.1 *If $u \in C^2(M, S^k)$ is a harmonic map and $\phi \in C_0^2(M, \mathbb{R}^{k+1})$, then*

$$\frac{d^2}{dt^2}|_{t=0}\left(\frac{1}{2}\int_M |\nabla(\frac{u+t\phi}{|u+t\phi|}|^2\,dv_g\right) = \int_M \left(|\nabla\hat{\phi}|^2 - |\nabla u|^2|\hat{\phi}|^2\right)dv_g, \tag{1.11}$$

where $\hat{\phi}(\equiv \phi - \langle u,\phi\rangle u)$ is the tangential component of ϕ.

Proof. For $\phi \in C_0^\infty(M,\mathbb{R}^{k+1})$ and small $t \in \mathbb{R}$, denote $u_t = \frac{u+t\phi}{|u+t\phi|}$. Then direct calculations give

$$\frac{du_t}{dt}|_{t=0} = \phi - \langle u,\phi\rangle u = \hat{\phi},$$

and

$$\frac{d^2u_t}{dt^2}|_{t=0} = 3\langle u,\phi\rangle^2 u - |\phi|^2 u - 2\langle u,\phi\rangle\phi.$$

Hence we have

$$\begin{aligned}
&\frac{d^2}{dt^2}|_{t=0}\left(\frac{1}{2}\int_M |\nabla(\frac{u+t\phi}{|u+t\phi|})|^2\,dv_g\right)\\
= &\int_M \left(|\nabla(\frac{du_t}{dt}|_{t=0})|^2 + \langle\nabla u, \nabla(\frac{d^2u_t}{dt^2}|_{t=0})\rangle\right)dv_g\\
= &\int_M \left(|\nabla\hat{\phi}|^2 - \langle\Delta_g u, 3\langle u,\phi\rangle^2 u - |\phi|^2 u - 2\langle u,\phi\rangle\phi\rangle\right)dv_g\\
= &\int_M \left(|\nabla\hat{\phi}|^2 + \langle|\nabla u|^2 u, 3\langle u,\phi\rangle^2 u - |\phi|^2 u - 2\langle u,\phi\rangle\phi\rangle\right)dv_g\\
= &\int_M \left(|\nabla\hat{\phi}|^2 - |\nabla u|^2(|\phi|^2 - |\langle u,\phi\rangle|^2)\right)dv_g\\
= &\int_M \left(|\nabla\hat{\phi}|^2 - |\nabla u|^2|\hat{\phi}|^2\right)dv_g.
\end{aligned}$$

This completes the proof of (1.11). □

Next we derive a general second variational formula for the Dirichlet energy functional.

Proposition 1.6.2 *Let $u \in C^2(M,N)$ be a harmonic map, and $u_t \in C^2([0,1] \times M, N)$ be a family of smooth variations of u, i.e., $u_0 = u$. Let $v = \frac{du_t}{dt}|_{t=0} \in C^2(M, u^*TN)$. Then*

$$\frac{d^2}{dt^2}|_{t=0}\left(\frac{1}{2}\int_M |\nabla u_t|_g^2\, dv_g\right) \tag{1.12}$$

$$= \int_M \left(|\nabla v|_g^2 - \mathrm{tr}_g\langle R^N(v, \nabla u)\, v, \nabla u\rangle\right) dv_g. \tag{1.13}$$

In particular, if $K^N \le 0$, then u is stable, i.e.,

$$\frac{d^2}{dt^2}|_{t=0}\left(\frac{1}{2}\int_M |\nabla u_t|_g^2\, dv_g\right) \ge 0.$$

Proof. Let $(\frac{\partial}{\partial x_\alpha})$ be a local coordinate frame on M. Then we have

$$\frac{d}{dt}|_{t=0}\frac{\partial u_t}{\partial x_\alpha} = \nabla^{u^*TN}_{\frac{\partial}{\partial t}}\left(\frac{\partial u_t}{\partial x_\alpha}\right) = \nabla^{u^*TN}_{\frac{\partial}{\partial x_\alpha}} v,$$

as $[\frac{\partial u_t}{\partial t}, \frac{\partial u_t}{\partial x_\alpha}] = 0$. Hence, we have

$$\begin{aligned}\frac{d^2}{dt^2}|_{t=0}\frac{\partial u_t}{\partial x_\alpha} &= \nabla^{u^*TN}_{\frac{\partial}{\partial t}}\nabla^{u^*TN}_{\frac{\partial}{\partial x_\alpha}} v \\ &= \nabla^{u^*TN}_{\frac{\partial}{\partial x_\alpha}}\nabla^{u^*TN}_{\frac{\partial}{\partial t}} v + R^N\left(v, \frac{\partial u}{\partial x_\alpha}\right)v.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\frac{d^2}{dt^2}\left(\frac{1}{2}\int_M |\nabla u_t|_g^2\, dv_g\right) \\ =& \int_M \left(|\nabla v|_g^2 + \langle \nabla u, \nabla\left(\frac{d^2 u_t}{dt^2}|_{t=0}\right)\rangle_g\right) dv_g \\ =& \int_M \left(|\nabla v|_g^2 + \langle \frac{\partial u}{\partial x_\alpha}, \nabla^{u^*TN}_{\frac{\partial}{\partial x_\alpha}}(\nabla^{u^*TN}_v v)\rangle - \mathrm{tr}_g(R^N(v, \nabla u)v, \nabla u)\right) dv_g \\ =& \int_M \left(|\nabla v|_g^2 - \langle \tau(u), \nabla^{u^*TN}_v v\rangle - \mathrm{tr}_g\left(R^N(v, \nabla u)v, \nabla u\right)\right) dv_g.\end{aligned}$$

Since $\tau(u) = 0$, this implies (1.13). If $K^N \le 0$, then we can easily conclude that u is stable. □