Preface

Harmonic maps between Riemannian manifolds are canonical objects from the points of view of topology and calculus of variations. These maps provide a rich display of both differential geometric and analytic phenomena. Much of the study of these maps serves as a model for many other challenging problems in geometric analysis and has been the source of inspiration and undiminishing fascination.

The study of harmonic maps in one dimension is equivalent to the study of the shortest paths—geodesics in Riemannian manifolds. The classical Morse theory (see Milnor [143]) and the Lusternik-Schnirelmann theory [130] were created from the study of such objects.

Harmonic maps with two-dimensional domains present special features that are crucial for applications to minimal surfaces (i.e., conformal harmonic maps) and to the deformation theory of Riemann surfaces—Teichmüller theory (see Wolf [213]). We refer to books by Jost [103] and Helein [93] and surveys by Schoen [167, 168] for detailed and systematic presentations of the various aspects of the theory. Many of the geometric and analytic methods used in the study of two-dimensional harmonic mapping problems can be adapted and generalized to the study of other geometrical objects such as constant mean curvature surfaces, Willmore surfaces, and pseudo-holomorphic curves. This last topic has played a fundamental role in the study of four-dimensional topology as well as symplectic and Kähler manifolds; see for example Gromov [73] and McDuff and Salamon [144].

We would also point out that the study of harmonic diffeomorphisms between two-dimensional domains (see Jost [103], Hélein [89], Wan [204], Tam-Wan [199], Han-Tam-Treibergs-Wan [78], and Li-Tam-Wang [119]) is closely related to the study of another important classical geometry problem: isometric embedding. Indeed, Lewy [129] reduced the study of Monge-Ampère equations for isometric embeddings to the study of the Darboux system which describes a special class of generalized harmonic maps that led him to develop solutions of the Weyl isometric embedding and Minkowski problems in two-dimensions for analytic metrics. Heinz [87, 88] further generalized Lewy's approach, and the work of F. Labourie [113] can be viewed as an interpretation in the language of Gromov, see for example Lin [121].

The study of harmonic maps from a compact Riemannian manifold M into another compact Riemannian manifold N in higher dimensions probably began with the ground-breaking work of Eells and Sampson [49]. They proved, in particular, that any homotopy class of maps from M into N contains a smooth harmonic map whenever the target manifold N is nonpositively curved. The result has shown to be extremely useful for establishing certain rigidity and vanishing theorems which can be seen in Siu [190], Corlette [37], Jost and Yau [106, 107], and Gromov and Schoen [74]. There have been very important studies on harmonic maps from suitable metric spaces into Alexandrov spaces of nonpositively curvature by Korevaar and Schoen [111, 112], and Jost [104, 105]. There have been important works on both harmonic maps and their heat flows on complete, noncompact Riemannian manifolds into compact Riemannian manifolds with nonpositive curvature by Li and Tam [116, 117, 118].

The theory of harmonic maps is remarkably rich. It took "A report on harmonic maps" [50] in 1978 and "Another report on harmonic maps" [51] in 1988 by Eells and Lemaire to give a brief survey on the subject. These reports contain nearly one thousand relevant references. Since then there have been many more developments, especially during the past two decades. In addition to books by Jost [102, 103] and Helein [93] that we mentioned earlier, there are books by Giaquinta, Modica, and Souček [70, 71], Schoen and Yau [176], lecture notes by Struwe [196] and Simon [187, 188], and surveys by Schoen [167, 168, 169], Hardt [80], Brezis [16, 17] and Helein [94]. Therefore, it is almost impossible to write a book on harmonic maps that will be a comprehensive and complete account of the entire theory.

Our goal in this book is to present a significant portion of the analytic aspects of the theory of harmonic maps and the associated heat flows. These ideas and techniques are central to the development of many other related studies on the general Gauge theory, the theory of liquid crystals, and the theory of Ginzburg-Landau equations. We shall not discuss these theories or their applications. We shall also omit entirely a discussion of the geometric aspects of harmonic maps and various beautiful and important applications. Interested readers may find some of the references mentioned herein to be quite informative on these topics.