COMPLEX WHITE NOISE AND THE INFINITE DIMENSIONAL UNITARY GROUP

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We discuss the infinite dimensional unitary group $U(E_c)$ in connection with complex white noise. Some important subgroups of $U(E_c)$ enjoy a beautiful structure expressed in terms of Lie algebras, and they play significant probabilistic roles.

Keywords: Complex white noise, Infinite-dimensional Unitary Group.

1. Introduction

What we are going to present are in line with white noise analysis. Hence, before we come to the main topic, it is better to remind the reader of the ideas and characteristics (in fact, advantages) of white noise theory. In this note we shall study a harmonic analysis of functionals of complex white noise by using the infinite dimensional unitary group.

White noise analysis has developed extensively during the past three decades, and it seems a good time to have the white noise variable complexified and to develop the theory of functionals of complex white noise.

We are pleased to describe three big advantages of white noise theory.

(1) Use of *generalized white noise functionals*. Our idea starts out with *reduction* of the given random system, so that a system of idealized elemental random variables would be given.[1] They are independent and are taken to be the variables of functionals that describes the random system in question. Once the variables are given, it is the standard way to start with defining basic functions,[2] namely polynomials in those variables. If the innovation is Gaussian white noise, the standard system of variables are $\dot{B}(t), t \in R$. Then, we need a clever trick to define polynomials, which are most basic functions, and more general nonlinear functions of the $\dot{B}(t)$. Namely, we use the *renormalization* technique, in addition to the orthogonalization that is done in the classical

stochastic calculus. So, our calculus is different from, and in fact beyond the classical stochastic analysis over the Wiener space. We give an *identity* to $\dot{B}(t)$ and is used without smearing it by smooth functions of t. We are, therefore led to introduce the class of generalized white noise functionals, so that our harmonic analysis becomes systematic and further enjoys much freedom to find directions of further development. For instance, a good domain of the Lévy Laplacian can be defined and we can discuss applications to path integrals (Feynman and Chern-Simons, for instance) which heavily use those generalized functionals; similarly infinite dimensional Dirichlet forms, and so forth.

(2) The infinite dimensional rotation group and the unitary group appear as the invariance of the white noise measure μ and complex white noise measure ν, respectively.

(3) Since the innovation is defined as the basic notion of the analysis, this idea of using the innovation can be extended to the study of random fields that are parametrized by higher dimensional (space-time) manifold. Needless to say, much fruitful results can be obtained. We may say that this advantage comes from the reduction theory.

2. Complex white noise

Let $(E^*, \mathbf{B}, \mu)$ be a white noise space. We take two copies of this measure space and form a direct product to define a complex white noise in the following manner. Let $(E_1^* \times E_2^*, \mathbf{B}_1 \times \mathbf{B}_2, \mu_1 \times \mu_2)$ be the product measure space, where $\mu_k, k = 1, 2$, are white noise measures with variance $\frac{1}{2}$. Set

$$z = x_1 + ix_2, (x_1, x_2) \in E_1^* \times E_2^*$$

and denote the vector space spanned by the z by E_c^*. Naturally we can form a *complex white noise* (E_c^*, ν), where $\nu = \mu_1 \times \mu_2$.

3. Infinite dimensional unitary group

The complex Hilbert space $L^2(E_c^*, \nu)$ is the basic space on which we shall carry out the analysis. In fact, a unitary group is defined in what follows, so that harmonic analysis can be done on $L^2(E_c^*, \nu)$.

Let g be a linear homeomorphism of $E_c = E_1 + iE_2$ such that

$$\|g\zeta\| = \|\zeta\| \qquad \text{for all } \zeta \in E_c.$$

The collection $U(E_c)$ of such g's forms a group under the usual product, and is called the *infinite dimensional unitary group*.

The group $U(E_c)$ is viewed as a complexification of the rotation group $O(E)$. On the Hilbert space $L^2(E_c^*, \nu)$ we can define unitary representation of the group $U(E_c)$, hence we can discuss the harmonic analysis.[3] Finer results on the analysis can be seen by observing significant subgroups of $U(E_c)$.

4. Subgroups of $U(E_c)$

1) Finite dimensional unitary group $U(n), n \geq 1$ and its limit.

As in the case of $G_n \subset O(E)$, we choose an orthonormal system $\zeta_k, k \geq 1$, in E_c. The first n members determine an n-dimensional subspace E_n and the unitary group $U(n)$ acting on E_n is defined. The group $U(n)$ can be embedded in the group $U(E_c)$ as its subgroup. Further, the projective limit $U(\infty)$ of the $U(n)$ is also introduced.

A member of $U(\infty)$ may be considered to be approximated by finite dimensional transformations.

2) Another generalization technique is as follows. Take members in E_c as many as n_k such that they form an orthonormal system. They span a subspace E_{n_k}. We take $E_{n_k}, k = 1, 2, \cdots, m$, which are mutually orthogonal. Thus a subspace $\bigoplus E_{n_k} \subset E_c$ is given. Hence, we can form a product $U(\mathbf{n}) = \bigotimes U(n_k)$ of the unitary groups $U(n_k)$ defined as above, where $\mathbf{n} = (n_1, n_2, \cdots, n_m)$. By letting $m \to \infty$ we are given an analogue of the windmill subgroup $\mathcal{W}$, which is *essentially infinite dimensional.*

We then come to subgroups which are coordinate-free. Namely, we take one-parameter subgroups that come from diffeomorphisms of the parameter space. We call such a one-parameter group a *whisker.*

3) Conformal group.

If, in the R^d-parameter case, the basic nuclear space E is taken to be D_0, which is isomorphic to $C^\infty(S^d)$, then we are given the conformal group $C(d)$ which is a subgroup of $O(E)$. Hence, the complex form of $C(d)$, denote it by $C_c(d)$, acting on the complexified space $D_{0,c}$, is a subgroup of $U(D_{0,c})$. We call it a *complex conformal group.* It is locally isomorphic to the (real) linear group $SO(d+1,1)$ and is generated by one-parameter groups including whiskers as many as $\frac{(d+1)(d+2)}{2}$. Their generators of shift, isotropic dilation , rotation $SO(d)$ and special conformal transformations are, respectively, as follows:

$$s = -\frac{d}{du_i}, \quad i = 1, 2, ..., d,$$

$$\tau = (u, \nabla) + \frac{d}{2}, \quad r = |u|$$

$$r_{j,k} = u_j \frac{\partial}{\partial u_k} - u_k \frac{\partial}{\partial u_j}, \quad 1 \le j \ne k \le d,$$

$$\kappa_j = u_j^2 \frac{\partial}{\partial u_j} + u_j, \; j = 1, 2, ..., d.$$

The one-parameter subgroup $\tau_t, t \in R$, with generator τ is an operator such that

$$\tau_t \zeta(u) = \zeta(ue^{-t})e^{-td/2}.$$

It is called an *isotropic dilation.*

Remark 4.1. In order to have a finite dimensional Lie algebra (group), we should have the isotropic dilation instead of general dilations.

4) Heisenberg group.

From now on, one can see the *effective use* of complex white noise with emphasis of the role of the Fourier transform, actually in many ways. The basic nuclear space E_c is now specified to be the *complex Schwartz space*

$$\mathcal{S}_c = \mathcal{S} + i\mathcal{S}.$$

4.1) The gauge transformation $I_t, t \in R$, is defined by

$$I_t : \zeta(u) \longrightarrow I_t\zeta(u) = e^{it}\zeta(u).$$

Obviously I_t is a member of $U(E_c)$, and $\{I_t\}$ forms a continuous one parameter subgroup. It is periodic with period 2π.

$$I_t I_s = I_{t+s}, \quad t, s \in R,$$

$$I_{t+2\pi} = I_t,$$

$$I_t \to I \text{ as } t \to 0.$$

The group $\{I_t, t \in R\}$ is called the *gauge group.* Actually, we have an abelian gauge group that is isomorphic to $U(1) \cong S^1$. Indeed. Let the unitary operator U_t be defined by U_{I_t}, which forms a one-parameter unitary group acting on $L^2(E_c^*, \nu)$. This group has only point spectrum on the subspace H_n involving complex multiple Wiener integrals of degree n. The eigenspace belonging to the eigenvalue $-n + 2k$ is $H_{(n-k,k)}$. Hence, the following proposition is proved:

Proposition 4.1. *The space $H_n, n > 1$, is classified, according to the action of I_t, into its subspaces $H_{(n-k,k)}$ for which eigenvalue $-n + 2k$ is associated.*

The infinitesimal generator of the gauge group is iI, where I is the identity operator.

4.2) The shift S_t^j. For the case $d = 1$, the shift S_t is the complex form of the shift belonging to $O(E)$.

We can easily extend to the R^d-parameter case. The generators are

$$-\frac{\partial}{\partial u_j}, \quad j = 1, 2, \cdots, d.$$

4.3) Multiplication $\pi_t^j, j = 1, 2, \cdots, d$. Let them be defined to be the conjugate to the shifts via the Fourier transform $\mathcal{F}$:

$$\pi_t^j = \mathcal{F} S_t^j \mathcal{F}^{-1}.$$

Actual expressions are

$$(\pi_t^j \zeta)(u) = e^{itu_j} \zeta(u), \quad u \in R^d.$$

Its infinitesimal generator is denoted by π_j and it is expressed as

$$\pi_j = iu_j.$$

Definition 4.1. The subgroup of the $U(\mathcal{S}_c)$ generated by the gauge group, the shifts and the multiplication is called the *Heisenberg group.*

It should be noted that we have the commutation relation

$$\pi_t S_s = I_{st} S_s \pi_t.$$

In terms of the generators

$$[\pi, s] = iI,$$

which is a most significant relation (actually, it is nothing but the uncertainty principle).

Gauge transformations (continued)

We can extend the Heisenberg group which shows an invariance of complex white noise.

The Heisenberg group introduced above involves three kinds of one-parameter groups. In addition to them, we have a new class of continuous linear operators acting on E_c as a generalization of the $\{I_t\}$.

noindent 4.4) Define I_α by

$$I_\alpha : \ \zeta(u) \ \rightarrow \ (I_\alpha \zeta)(u) = e^{i\alpha(u)}\zeta(u),$$

where α is a member of the (real) Schwartz space S. The operator I_α is a unitary operator on $L^2(R)$ and it is called *S-gauge transformation.*

The infinitesimal generator of $I_{\alpha t}$ is $i\alpha$. The collection $\{I_\alpha, \alpha \in S\}$ forms a group under the usual product and is a subgroup of $U(S_c)$. The group is called *S-gauge transformation* group. Obviously it is an *abelian* subgroup of $U(S_c)$.

We recall the infinitesimal generators of the whiskers obtained so far.

$$iI; \ s = -\frac{d}{du}, \ \pi = iu\cdot, \ i\alpha\cdot,$$

where $\alpha \in S$.

Non-trivial commutation relations are

$$[\pi, \alpha] = 0, \ [s, \alpha] = \alpha' \ (\in S).$$

The adjoint I_α^* acts on $z \in S_c$ in such a way that $e^{-i\alpha(u)}z(u)$, and the

$$I_\alpha^* = e^{-i\alpha(u)}, \quad \alpha \in S,$$

is an S^*-*gauge transformation.*

The collection

$$\{I_\alpha^* = e^{-i\alpha(u)} , \quad \alpha \in S\}$$

is called S^*-*gauge transformation.*

The following theorem can be proved without any difficulty.

Theorem 4.1. *i) The generators listed above form a Lie algebra under the Lie product* [,]. *It is also an algebra (in the ordinary sense).*

ii) The S^*-*gauge transformation acts on* $z = (x, y)$-*space, and* ν *measure is invariant under the action of the adjoint* I_α^*.

5) The Fourier-Mehler transforms $\mathcal{F}_\theta$.

Since particularly important roles of the Fourier transform can be seen in the study of complex white noise, we shall further proceed to the fractional powers of the ordinary Fourier transform. Actually, we define a one-parameter system of unitary operators $\mathcal{F}_\theta, \theta \in [0, 2\pi]$ such that $\mathcal{F}_\theta$ is viewed as the $\frac{2\theta}{\pi}$-th (fractional) power of the ordinary Fourier transform, where θ is considered mod 2π.

The operator in question is defined by the integral kernel $K_\theta(u, v)$:

$$K_\theta(u,v) = (\pi(1 - \exp[2i\theta]))^{-1/2} \exp\left[-\frac{i(u^2+v^2)}{2\tan\theta} + \frac{iuv}{\sin\theta}\right].$$

It defines an operator $\mathcal{F}_\theta$ by the formula

$$(\mathcal{F}_\theta \zeta)(u) = \int_{-\infty}^{\infty} K_\theta(u,v)\zeta(v)dv,$$

where $\theta \neq \frac{1}{2}k\pi, \ k \in Z$.

We now have some observations about the actions of $\mathcal{F}_\theta$. Set

$$\xi_n(u) = (2^n n! \sqrt{\pi})^{-1/2} H_n(u) \exp\left[-\frac{u^2}{2}\right].$$

Then, it is proved that

$$\mathcal{F}_\theta \xi_n(u) = e^{in\theta}\xi_n(u), \quad n \geq 0.$$

With this relationship we can prove that $\mathcal{F}_\theta$ is well defined for every θ (by interpolation), and further

$$\mathcal{F}_\theta \mathcal{F}_{\theta'} = \mathcal{F}_{\theta+\theta'} = \mathcal{F}_{\theta''}, \quad \theta + \theta' = \theta'' \ (\mathrm{mod}\, 2\pi).$$

$$\mathcal{F}_\theta \to I, \text{as } \theta \to 0.$$

Particular choices of θ give

$$\mathcal{F}_{\pi/2} = \mathcal{F}, \quad \mathcal{F}_{(3/2)\pi} = \mathcal{F}^{-1}.$$

Thus, we have obtained a periodic one-parameter unitary group including the Fourier transform and its inverse.

The kernel function K_θ illustrates this fact. Moreover one can see that the $\mathcal{F}_\theta$ defines a fractional power of the Fourier transform as is shown in what follows.

First we note an identity

$$vK_\theta(u,v) = (u\cos\theta - i\frac{\partial}{\partial u}\sin\theta)K_\theta(u,v).$$

Namely, the multiplication operator "$u\cdot$ is transformed by $\mathcal{F}_\theta$ by the following formula

$$u \longrightarrow u\cos\theta + \frac{1}{i}\frac{\partial}{\partial u}\sin\theta.$$

This comes from the above identity.

Similar computation proves that we are given, under $\mathcal{F}_\theta$,

$$\frac{1}{i}\frac{\partial}{\partial u} \mapsto -u\sin\theta + \frac{1}{i}\frac{\partial}{\partial u}\cos\theta.$$

Symbolically writing, we have established

$$\mathcal{F}_\theta : \begin{pmatrix} u \\ \frac{1}{i}\frac{\partial}{\partial u} \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ \frac{1}{i}\frac{\partial}{\partial u} \end{pmatrix}$$

We are happy to see that there appeared the group $SO(2)$ again.

The infinitesimal generator of $\mathcal{F}_\theta$ is denoted by if and is expressed in the form

$$if = -\frac{1}{2}i\left(\frac{d^2}{du^2} - u^2 + I\right).$$

Observing the commutation relations of the generators, so as to have a finite dimensional Lie algebra, either real form or complex form, we are naturally given a new generator σ' which is expressed in the form

$$\sigma' = \frac{1}{2}\left(\frac{d^2}{du^2} + u^2\right).$$

We are particularly interested in the probabilistic roles of this operator (generator) in quantum dynamics.

An easy and formal interpretation of σ' is that

$$\frac{1}{i}\partial_t\Psi = \sigma'\Psi$$

is the Schrödinger equation for the *repulsive oscillator.*

For our purpose, it is convenient to take $\sigma = \sigma' + \frac{i}{2}I$, namely we introduce

$$\sigma = \frac{1}{2}\left(\frac{d^2}{du^2} + u^2 + iI\right).$$

A one-parameter group with the generator σ can be defined locally in space-time only. It is, however, interesting to discuss the operator σ in connection with the dynamics having a potential of repulsive force.

Lie algebras of infinitesimal generators.

We have so far various infinitesimal generators. For simplicity we first consider the case $d = 1$.

The Lie algebra of the conformal group is

$$\begin{aligned} s &= -\frac{d}{du}, \\ \tau &= u\frac{d}{du} + \frac{1}{2}, \\ \kappa &= u^2\frac{d}{du} + u, \end{aligned}$$

Then, we come to the complex world, where one can find not only complexification of the real Lie algebras that have already been given, but also new members having close connection with the Fourier transform.

The Heisenberg group defines the Lie algebra generated by

$$\begin{aligned} &iI, \\ s &= -\frac{d}{du}, \\ i\pi &= iu, \end{aligned}$$

Hereafter, the one-parameter subgroups $I_{\alpha t}$ will be treated separately. One reason lies in the fact that we wish to remain within finite dimensional Lie algebras (Lie subgroups).

There are two interesting generators related to the Fourier transform. They are

$$\begin{aligned} if &= -\frac{i}{2}\left(\frac{d^2}{du^2} - u^2 + I\right), \\ \sigma &= \frac{1}{2}\left(\frac{d^2}{du^2} + u^2 + iI\right). \end{aligned}$$

We are now ready to consider the structure of the Lie algebras.

Recall the Lie algebra $\mathbf{c}(d)$ of the conformal group. The $\mathbf{c}(1)$ is

$$\mathbf{c}(1) = \{s, \tau, \kappa\}.$$

Note. In the multi-dimensional case, say d-dimensional, we add the rotations $\gamma_{j,k}$, $1 \le j, k \le d$, to have $c(d)$:

$$\gamma_{j,k} = u_j \frac{\partial}{\partial u_k} - u_k \frac{\partial}{\partial u_j}, \quad 1 \le j \ne k \le d.$$

Behind the construction of the algebra $\mathbf{c}(d)$, the reflection R is involved. Thus, in order to define the conformal group it is recommended to take the basic nuclear space to be D_0, not to be an arbitrary nuclear space E.

The Lie algebra of the d-dimensional Heisenberg group is denoted by $\mathbf{h}(d)$. For the present case $d = 1$ we have

$$\mathbf{h}(1) = \{iI, s, \pi = iu\}.$$

The group and the algebra are based on complex white noise.

Keep the following two concepts in mind.

1. There the ordinary Fourier transform $\mathcal{F}$ plays a key role.
2. It is a member of the unitary group $U(S_c)$.

Then we are naturally led to the Fourier-Mehler transform $\mathcal{F}_\theta$, the fractional power of the Fourier transform.[4] The generator if of $\mathcal{F}_\theta$ is $-\frac{i}{2}(\frac{d^2}{du^2} - u^2 + I)$. Remind that we have introduced the operators, first σ', then σ: in fact, by hand for a moment, and later some interpretation is given so that the commutation relations appear in good shape. We have therefore had

$$\sigma = \frac{1}{2}\left(\frac{d^2}{du^2} + u^2 + iI\right).$$

Summing up

Proposition 4.2. *Based on the set of operators*

$$\{iI, s, \pi, \tau, f, \sigma\}$$

we have 6-dimensional complex Lie algebra $\mathbf{g}$.

This algebra has the real form as is easily seen.

Table of commutation relations. For $\mathbf{c}(1)$,

$$[\tau, s] = -s,$$

$$[\tau, \kappa] = \kappa,$$

$$[s, \kappa] = -2\tau.$$

For $\mathbf{h}(1)$,

$$[\pi, s] = I.$$

For the algebra $\mathbf{g}$,

$$[f, s] = \pi, \quad [\sigma, s] = \pi,$$

$$[f, \pi] = s, \quad [\sigma, \pi] = -s,$$

$$[f, \tau] = -2\sigma + iI, \quad [\sigma, \tau] = 2f - I.$$

The algebra $\mathbf{h}(1)$ is an ideal of $\mathbf{g}$ and is the maximum solvable Lie subalgebra. In short we may state

Proposition 4.3. *The ideal* $\mathbf{h}(1)$ *is the radical of* $\mathbf{g}$.

Proof is given by the actual and rather easy computations.

It seems necessary to give some interpretation, from various viewpoints, to the generator κ, a member of the special conformal transformation.

1. The reason why the κ has been taken.[1]
   (i) Obviously the κ is a good candidate to be invited to a class of possible generators expressed in the the form $a(u)\frac{d}{du} + \frac{1}{2}a'(u)$. We see that when the basic nuclear space E is taken to be D_0, the κ is well acceptable. Having had the κ in our class, we have formed the algebra generated by those admissible generators and we see that the algebra is isomorphic to $sl(2, R)$. This is a beautiful result.
   (ii) Similar to s, the κ is transversal to τ. This is a significant property, and in fact, it defines a flow of the Ornstein-Uhlenbeck process.
   (iii) The κ is related to the reflection with respect to the unit sphere. Hence, the parameter space must be $R^d \bigcup \{\infty\}$. It is, however, useful when variational calculus is applied for random fields parameterized by a smooth simple surface.
2. On the other hand, there are good reasons why κ should not be involved in the algebra $\mathbf{g}$.

(i) From our viewpoint that the Fourier transform is particularly emphasized. So the complex Schwartz space is fitting for the complex analysis. Namely, the Schwartz space S_c, which is invariant under the Fourier transform, is more significant. While, in order to introduce the κ we need another space like D_0, instead of S_c.
(ii) Under the Fourier transform we have a formal adjoint which is not suitable for our purpose.

So far we have seen the beautiful structure of generators in terms of the Lie algebra. Important note is that their probabilistic roles are in cooperation with the beauty of the algebra.

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