1

Introduction

1.1 Motivation and overview

The reasons for studying accelerator physics fall into two basic categories: first, to design, build, and improve real accelerators for use as tools, and second, to study the dynamics of particle motion in a non-linear system.

Some of the tools provide energy to process matter such as electron beam welding, x-ray lithography, cancer therapy, preparation of radioisotopes for medicine, food sterilization and Star Wars. Other accelerators are used as microscopes to study matter and fundamental interactions of nature. Examples of these are cathode ray tubes, electron microscopes, accelerators for studying nuclear and high energy physics.

A particle accelerator can also be used as an analog computer to test models of non-linear dynamics. The two mechanical systems in the physical universe, that are closest to being linear systems, are accelerators, and the solar system. Of the two, accelerators provide the more linear case. They can be observed over many more cycles, since the particles in an accelerator usually travel much faster than planets, and their orbits are much smaller than planets.

In this book we only consider accelerators that accelerate microscopic particles such as electrons, protons, and ions under the influence of electromagnetic fields. This influence can be divided into two effects: longitudinal acceleration due to electric fields along the direction of motion of the particle, and transverse bending of the trajectory due to transverse electric and magnetic fields. The motion of charged particles is determined by the relativistic extensions to Newton's laws and the Lorentz force,

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}). \tag{1.1}$$

For particles with spin, there is also the the Stern-Gerlach force which in the particle's rest frame is

$$\vec{F} = \nabla(\vec{\mu} \cdot \vec{B}), \tag{1.2}$$

where $\vec{\mu}$ is the magnetic moment of the particle. This spin force is much smaller (of order $\hbar$) than the Lorentz force for a charged particle, but for a neutral particle of

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low momentum, it can be a useful effect.

Particle accelerators for the most part are divided into linear accelerators that accelerate the beam in a single pass, and circular accelerators that recirculate the beam through the accelerating voltage many times. Some examples of the linear type are the Van de Graaff generator, the Cockcroft-Walton cascade generator, the radio frequency quadrupole (RFQ), and the drift tube linac (DTL). The circular accelerators include cyclotrons, synchrocyclotrons, betatrons, microtrons, and synchrotrons.

Some simple parameters of importance to the user are: input power, the type of particle accelerated, the average beam energy, the distribution of energy of the particles in the beam, the angular divergence, intensity, duty factor, repetition rate, background rates, the polarization of the beam and target, and the cost. These are fairly obvious concepts; however, some of the definitions used for intensity deserve a little clarification.

For example let us consider the requirements for a high energy physics experiment. The experiment is trying to measure some esoteric process like[1]

$$e^+ + e^- \rightarrow \Upsilon(4\mathrm{s}) \rightarrow B^+ + B^-. \tag{1.3}$$

This type of experiment would probably be done with colliding electron and positron beams in a storage ring. This type of reaction has a certain production cross section, which when multiplied by something called the luminosity gives the rate of production of the B mesons. The usual backhanded definition of luminosity is: the number, that when multiplied by the cross section yields the interaction rate. The units for cross section are [cm^2] or sometimes the [barn $= 10^{-24}$ cm^2]. The instantaneous luminosity (see Appendix B) has units of [cm$^{-2} \cdot$ s^{-1}], and is proportional to the overlap integral of the densities of the two beams,

$$\mathcal{L} = |\vec{v}_+ - \vec{v}_-| f_0 \iiiint \rho_+(\vec{x} - \vec{v}_+ t, t)\, \rho_-(\vec{x} - \vec{v}_- t, t)\, dx\, dy\, dz\, dt, \tag{1.4}$$

where v_+ and v_- are the velocities of the beams in the lab, f_0 is the frequency of beam crossings, and the integrations are carried out for a single crossing of bunches. (Note that for extremely relativistic head-on collisions in the center of mass, the factor $|\vec{v}_+ - \vec{v}_-| = 2c$. Simply put, it takes half as long for two beams to pass each other if they are moving towards each other than if one is stationary.) The densities, ρ_- and ρ_+ are the densities of the respective electron and positron bunches, and will vary in z with the shape of the beam envelope. The total number of such interactions in an experiment is given by the cross section times the integrated luminosity, which is defined as the instantaneous luminosity integrated over the total time of the experiment. If there is more than one bunch per beam, the number of bunch crossings at a given interaction point (i. e., for a single experiment) is increased by the number of bunches, N_b.

Of course for a useful number, we must account for any dead time of the experimental apparatus. In order to identify the B mesons in this experiment, the detector must identify the tracks of the particles from the decays of the B mesons. The detector takes a certain amount of time to accumulate and log the data, producing a period of time in which the detector is unable to identify a new event. This is called the dead time of the detector.

This type of event produces many daughter particles, and if there are two simultaneous events, the data is usually too confused to be useful. Because of this confusion, it is useless to have the average number of interactions per beam crossing greater than some value (usually one or less for most experiments.)

For an experiment with a single beam incident on a fixed target, the intensity is traditionally quoted as the number of beam particles hitting the target per second, rather than as a luminosity.

Two other terms used for defining intensities are used with synchrotron light sources: brightness and brilliance. Both are proportional to the number of photons hitting the target per second, but they have the added feature of being inversely proportional to the bandwidth, or energy spread of the photon beam. Brightness is defined as the number dn, of photons per time interval dt, passing through a solid angle $d\Omega$, and divided by 0.1% of the bandwidth $d\lambda/\lambda$,

$$d\Phi_\Omega = 1000\,\frac{d^4 n}{dt\,d\Omega\,(d\lambda/\lambda)}. \tag{1.5}$$

Brilliance is defined as the brightness per area, s, of the source,

$$B = \frac{d\Phi_\Omega}{ds}. \tag{1.6}$$

In the rest of this chapter we review some basic concepts and briefly discuss a few of the early types of particle accelerators, which illustrate the different techniques used to accelerate charged particles.

1.2 Direct–voltage accelerators

The simplest type of elementary particle accelerator is a source of electrons or ions, and a pair of electrodes, activated by a potential drop ΔV, as shown in Fig. 1.1.

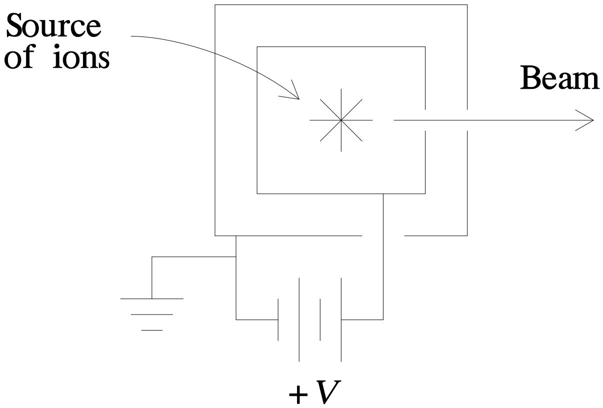

Figure. 1.1 A simple accelerator for charged ions of charge q. The kinetic energy of the beam is approximately qV.

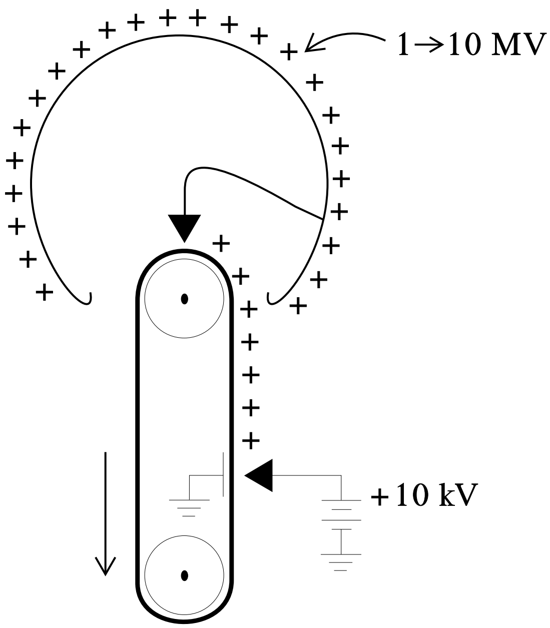

Figure. 1.2 A Van de Graaff generator. Electrons are pulled off the belt by a corona discharge at the bottom. The net positive charge moves up with the belt inside the dome, where electrons from the dome are pulled onto the belt through another corona discharge. As a result, the dome can reach a potential of several million volts relative to the lower corona points which are at ground.

Indeed such an apparatus is the prototype of a class of devices (cathode ray tubes, electron microscopes, etc.) which come under the branch of "Electron Optics." All direct current accelerators are variations on this theme, e. g., the electrostatic generator constructed by Van de Graaff[2] and the cascade generator developed by Cockcroft and Walton,[3] who first succeeded in disintegrating nuclei with accelerated particles.

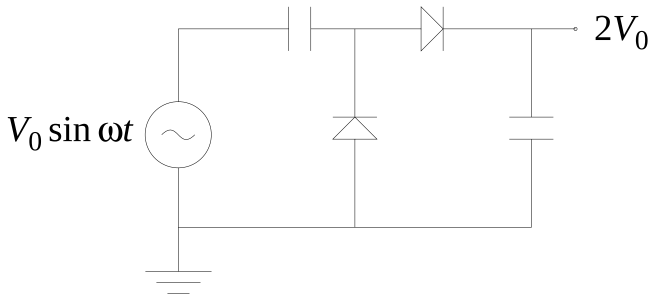

Figure. 1.3 A simple cascade circuit for doubling the voltage of an input generator.

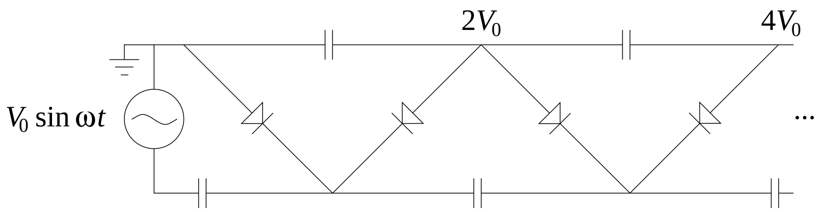

Figure. 1.4 A multistage cascade Cockcroft–Walton circuit, which rectifies and multiplies the input voltage.

Fig. 1.2 shows the sketch of the Van de Graaff electrostatic generator: a belt of insulating material runs between ground and a high-voltage generator ($\simeq$ 10 kV); corona discharge provides charge to the belt which, in its turn, induces electrostatic charges in the "hot" terminal (1—10 MV); another corona discharge neutralizes the belt. Notice how the drive of the engine is one of the best examples of electromotive force!

Mixtures of high pressure gases (N_2 and CO_2, for example) provide insulation and material for corona discharges. This machine can accelerate charged particles of either polarity.

A further improvement is the tandem generator, where negative ions are accelerated from the ground to the terminal, then are stripped of most of their electrons by a thin foil, hence the resulting positive ions are accelerated back to ground potential. In principle the energy gain can be increased (twice for protons) with respect to a simple acceleration.

The cascade generator is an extension of the doubling circuit, shown in Fig. 1.3, where two rectifying diodes and two capacitors, applied to an ac generator $V(t) = V_0 \sin \omega t$, give an almost dc voltage, $2V_0$.

Fig. 1.4 shows a sketch of a multistage cascade generator, capable of reaching a few million volts of potential. In both these high voltage generators the voltage must be distributed along the accelerating tube via either capacitive or resistive partitions, in order to avoid electric disruptions.

1.3 A review of relativistic particle motion

Particle accelerator physics is a realm of applied special relativity. In this section we review the most important and useful relations. Following the standard method we define the relativistic velocity β, and the Lorentz factor γ as

$$\beta = \frac{v}{c} \tag{1.7}$$

$$\gamma = (1 - \beta^2)^{-\frac{1}{2}} \tag{1.8}$$

with v being the velocity of the particle, and c being the velocity of light in a vacuum. Rearranging this gives

$$\beta\gamma = \sqrt{\gamma^2 - 1}, \quad \text{and} \quad \gamma^2 = (\beta\gamma)^2 + 1. \tag{1.9}$$

The total energy, momentum, and kinetic energy for a particle of rest mass, m, are, respectively:

$$U = \gamma mc^2, \tag{1.10}$$

$$p = \beta\gamma mc = \beta\frac{U}{c}, \quad \text{and} \tag{1.11}$$

$$W = (\gamma - 1)mc^2. \tag{1.12}$$

The relation between energy and momentum is

$$U = \sqrt{(pc)^2 + (mc^2)^2} = \sqrt{p^2 + m^2}, \tag{1.13}$$

where we have used the ever popular set of units, with $c = 1$, in the last expression. The most frequently accelerated particles are the electron with mass $m_e = 0.510999$ MeV, and the proton with mass, $m_p = 938.272$ MeV.

The following divisions are frequently used

$\gamma \simeq 1$	Non-relativistic	N. R.
$\gamma > 1$	Relativistic	—
$\gamma \gg 1$	Ultra-Relativistic	U. R.

The non-relativistic case can be checked by expanding Eq. (1.8) for small β and obtaining $\gamma \simeq 1 + \frac{1}{2}\beta^2$, which when inserted into Eq. (1.12) produces

$$W \simeq \frac{1}{2}mc^2\beta^2 = \frac{1}{2}mv^2. \tag{1.14}$$

In the ultra-relativistic case the mass becomes negligible and Eqs. (1.10, 1.11, and 1.12) collapse into the simpler relation $U \simeq W \simeq pc$.

Now a set of relations, particularly useful to accelerator physics, will be deduced. Differentiating respectively (1.8) and (1.9), we obtain:

$$d\gamma = \beta(1-\beta^2)^{-\frac{3}{2}} d\beta = \beta\gamma^3 d\beta \tag{1.15}$$

$$d(\beta\gamma) = \gamma d\beta + \beta d\gamma = \gamma(1+\beta^2\gamma^2)d\beta = \gamma^3 d\beta = \frac{d\gamma}{\beta}. \tag{1.16}$$

Squaring and differentiating Eq. (1.13) gives $2UdU = 2pdp$ or

$$\frac{dU}{U} = \frac{p^2}{U^2}\frac{dp}{p} = \beta^2\frac{dp}{p}. \tag{1.17}$$

By dividing Eq. (1.15) by $\beta^2\gamma^3$, the fractional change in velocity can be found as

$$\frac{d\beta}{\beta} = \frac{1}{(\beta\gamma)^2}\frac{d\gamma}{\gamma} = \frac{1}{(\beta\gamma)^2}\frac{dU}{U} = \frac{1}{\gamma^2}\frac{dp}{p}. \tag{1.18}$$

For acceleration in one dimension, Newton's second law becomes

$$F = \frac{dp}{dt} = mc\frac{d}{dt}(\beta\gamma) = \gamma^3 m\frac{dv}{dt} = m^*\frac{dv}{dt}, \tag{1.19}$$

having considered Eq. (1.16), and defining as effective mass,

$$m^* = \frac{dp}{dv} = \frac{d(\gamma mv)}{dv} = m\gamma^3. \tag{1.20}$$

The electromagnetic force is what accelerates charged particles, and is described mathematically by the Lorentz equation,

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}). \tag{1.21}$$

If there is no electric field and only a uniform magnetic field, then the force equation may be written as

$$\vec{F} = q\vec{v}\times\vec{B} = \frac{d}{dt}(\gamma m\vec{v}) = m(\gamma\frac{d\vec{v}}{dt} + \frac{d\gamma}{dt}\vec{v}) = \gamma m\frac{d\vec{v}}{dt}, \tag{1.22}$$

since $\beta = |\vec{\beta}|$ is a constant, which implies that $(d\gamma/dt) = 0$. The velocity $\vec{v} = \vec{\omega}\times\vec{\rho}$, with the angular velocity, $\vec{\omega}$ being constant for a central force of constant magnitude.

The *cyclotron radius* ρ is just the radius of the particle's orbit. Eq. (1.22) now becomes

$$q\vec{v} \times \vec{B} = \gamma m \vec{\omega} \times \frac{d\vec{\rho}}{dt} = \gamma m \vec{\omega} \times \vec{v}, \tag{1.23}$$

or for a particle moving in a plane perpendicular to $\vec{B}$,

$$qvB = \gamma m \omega v = \gamma m \frac{v^2}{\rho}, \tag{1.24}$$

i. e., the Lorentz force is the centripetal force which keeps the particle of charge q and mass m on a circular orbit. Dividing Eq. (1.24) by v/ρ, we get a relation for the momentum in terms of the orbit radius, magnetic field and charge of the particle:

$$p = \beta\gamma mc = qB\rho. \tag{1.25}$$

For a particle with same charge as the electron, it is useful to remember

$$p[\mathrm{GeV/c}] \simeq 0.3B[\mathrm{T}]\ \rho[\mathrm{m}]. \tag{1.26}$$

Another popular formula is the one for the angular velocity or cyclotron frequency:

$$\omega = \frac{qB}{\gamma m}. \tag{1.27}$$

It may also be useful to remember the Lorentz transformations of the electromagnetic field from the lab system to the rest system:

$$\vec{E}^*_\perp = \gamma(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp), \tag{1.28}$$

$$\vec{E}^*_\parallel = \vec{E}_\parallel, \tag{1.29}$$

$$\vec{B}^*_\perp = \gamma(\vec{B}_\perp - \vec{v} \times \vec{E}_\perp), \quad \text{and} \tag{1.30}$$

$$\vec{B}^*_\parallel = \vec{B}_\parallel, \tag{1.31}$$

where the $\parallel$ designates the component of the field parallel to the boost velocity $\vec{v}$, $\perp$ indicates the component perpendicular to the boost, and the asterisks indicate quantities in the rest system.

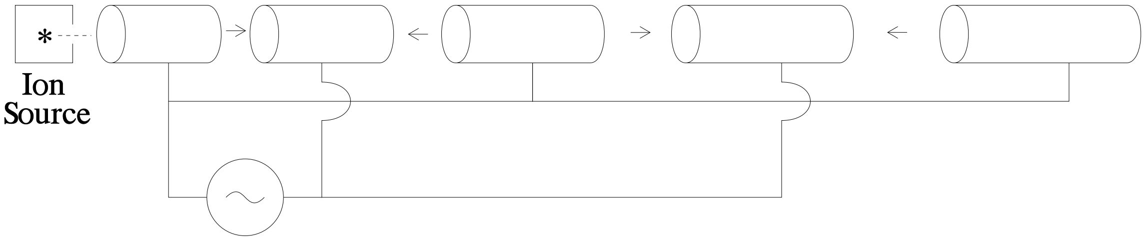

Figure. 1.5 A schematic of the Wideröe linac structure. The arrows indicate the directions of the accelerating electric field at a given instant of time.

1.4 Linear accelerators with oscillating electric fields

Since it is very difficult to produce dc voltages more than a few million volts, it was necessary to find a new method for acceleration to energies beyond a few MeV. In 1928 Wideröe proposed an accelerating structure using a series of cylindrical tubes, called drift tubes, which were alternately connected to a high frequency oscillator, as shown in Fig. 1.5. Charged particles from the source are accelerated in the gaps between tubes. They then drift in the field free region inside the tube. While the particles are inside the tube the direction of the field is reversed so that when the particles reach the next gap, they again see an accelerating electric field. If a constant frequency generator is used, the tubes must increase in length as the particle velocity increases, so that the particles will always arrive at the next gap with the correct phase of the accelerating voltage in the gap. Particles will only leave the source when the voltage in the first gap has the correct sign to accelerate them, thus the accelerated beam will have a pulsed structure to it.

A cell is usually defined as the region from the midplane of one drift tube to the next. (Sometimes it is convenient to offset this slightly, but the length of the cell remains the same.) The Wideröe structure is called a $\beta\lambda/2$ or π-mode structure, since the electric field configuration repeats every two cells. The product $\beta\lambda$ is the distance that the particle travels during one rf cycle. As the particles' velocities increase, the cell lengths must increase. Bunches of particles cannot be accelerated in every gap during a half cycle, but must be spaced with a free gap between every pair of bunches.

The Alvarez structure, shown in Fig. 1.6, is a $\beta\lambda$ or 2π-mode structure and can accelerate particles simultaneously in each gap. Since this is a 2π-mode structure, the rods connecting the tank and drift tubes are only necessary for support. In this structure the charges oscillate between the ends of the drift tubes. Unlike the

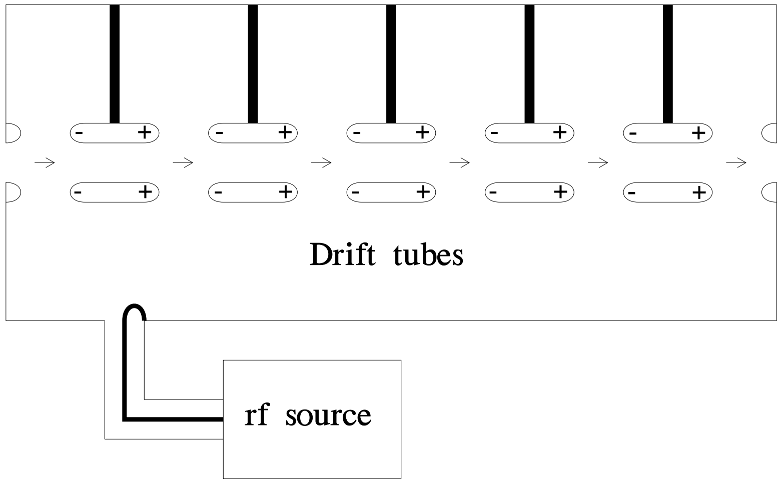

Figure. 1.6 The Alvarez drift tube structure. This is a 2π-mode structure with the field pattern repeating in every cell. The arrows indicate the direction of the electric field at one instant.

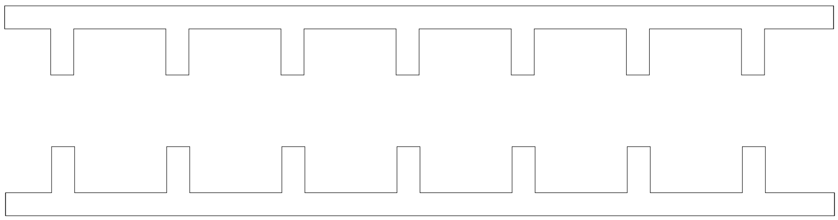

Figure. 1.7 A traveling wave disk and washer structure.

Wideröe structure with the charges actually traveling from one tube to the next by passing through the rf generator, the Alvarez structure is a resonant structure which is inductively coupled through a transformer consisting of a one turn primary inserted through the wall of the resonant tank containing the drift tubes.

Clearly, as the particles become more relativistic, the length of a cell increases. To counteract this requires either a much longer cell, or a source of higher frequency. Conventional triode and tetrode tubes were unable to operate at high frequencies in the microwave regime. Another type of tube called a klystron[5] can produce very high power at frequencies from a few hundred megahertz to several tens of gigahertz. The klystron is really more like a small linear accelerator than an electron vacuum tube. It uses a driven rf cavity to modulate a dc beam by varying the velocity of the particles with respect to the time of passage through the cavity. The particles drift for some distance and accumulate into bunches which appear as a pulsed current at a second resonant output cavity. The output power can be coupled by an inductive loop to a waveguide that pipes the power to an accelerating structure.

At about the same time as the invention of the klystron, it was realized that

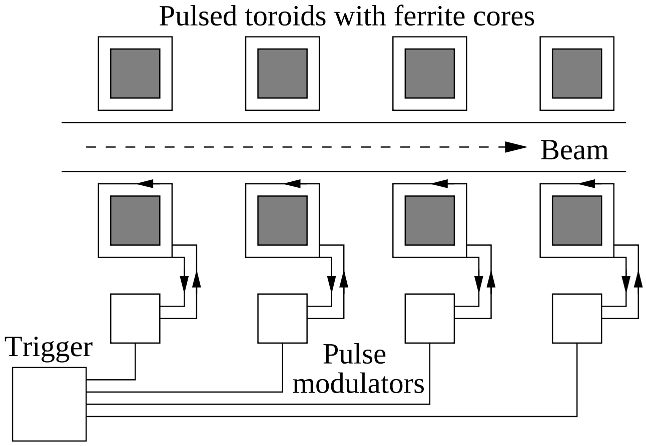

Figure. 1.8 An induction linac. This operates in the same manner as a transformer. The primary turns are toroidal pulsed electromagnets, and the secondary is the beam.

a traveling wave could be used to accelerate relativistic particles. A cylindrical waveguide propagates waves with phase velocities greater than the speed of light. Since the charged particles must be traveling at less than the speed of light, they will not obtain any net acceleration, because they cannot keep in phase with the wave. If the waveguide is loaded by corrugating its walls, so that the induced charges have a longer path length (as in Fig. 1.7), the phase velocity of the wave can be slowed down to a usable value (or even slower.) The particles may then "surf" along the wave with a phase yielding an accelerating force. This type of structure is called a *traveling wave structure*.

A *standing wave* structure is a structure which has two traveling waves moving in opposite directions. This type of structure is necessary for accelerating oppositely charged particles (e^+e^- or $p\bar{p}$) in opposite directions in the same accelerator.

Another type of linac is the induction linac (see Fig. 1.8.) This linac uses a series of toroidal electromagnets coaxially placed along the beam axis. By successively pulsing each magnet, large peak values of emf can be produced which will accelerate the beam. Typical currents of several kiloamperes can be achieved.

More recently, the radio frequency quadrupole (RFQ) has been developed for preliminary acceleration of protons and heavier ions. This uses four parallel electrodes around the beam axis as shown in Fig. 1.9. It is a resonant structure with adjacent electrodes having opposite charges. From the end, they look like an electric quadrupole. This arrangement of electric fields focuses the beam in one plane and defocuses the beam in the other plane. Since the electric field oscillates, a net focusing effect can be obtained. If the electrodes are scalloped with a curve somewhat like a sinusoid, and with the curves of the adjacent electrodes differing

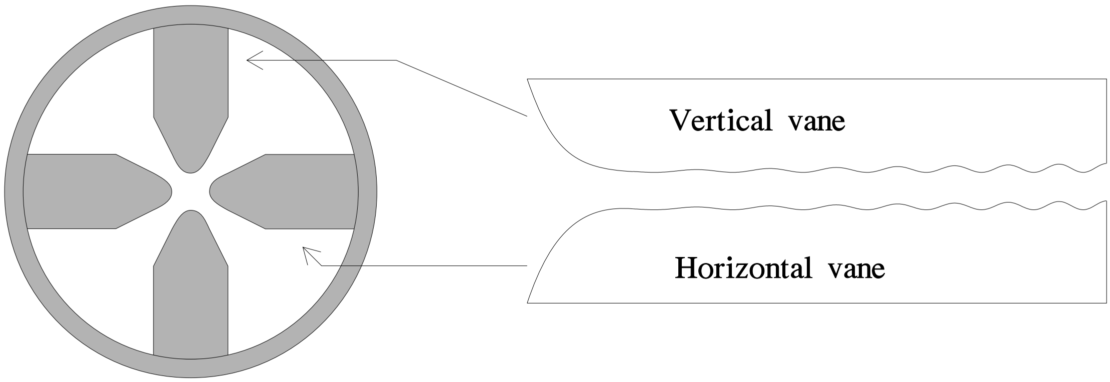

Figure. 1.9 A cross section of an RFQ structure. The longitudinal profile of adjacent vanes is shown a the right.

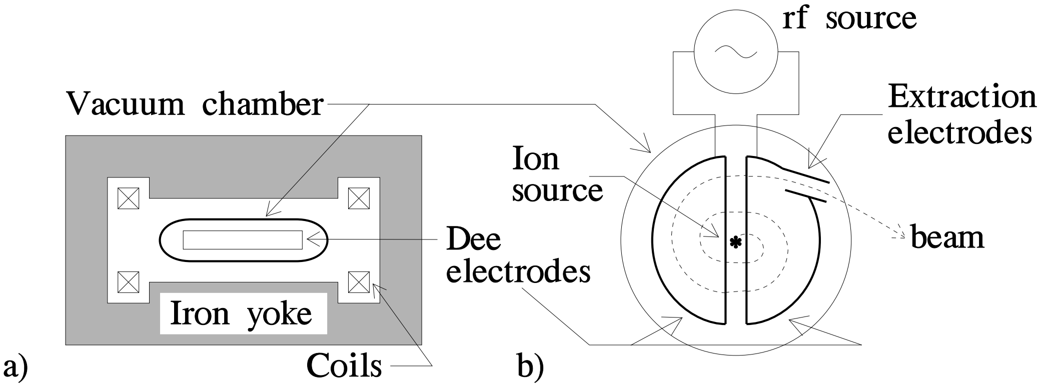

Figure. 1.10 a) Side view of a cyclotron. b) Top view of a cyclotron, with "D" electrodes shown inside the vacuum chamber.

by 180° in phase, then a component of longitudinal field will be produced, which may be used to accelerate the particles.

1.5 Circular machines

The cyclotron is the first example of a circular machine.[6,7] A homogeneous magnetic field, supplied by an H-shaped magnet, as in Fig. 1.10a, bends back the particles to the same rf gap between the two D-shaped electrodes shown in Fig. 1.10b, twice each period of the radio frequency oscillation. If the rf is set equal to the cyclotron frequency (a resonance condition) given by Eq. (1.27) with $\gamma = 1$ (N.R. ions or protons), the particles will continue to pass near the peak of the rf voltage twice per turn, gaining kinetic energy, and then increasing the radius of their orbits by Eq. (1.25), till reaching some extraction device.

Usually $\Delta E \leq 200$ keV/turn, then for $W_{\max} \simeq 20$—25 MeV, one can infer

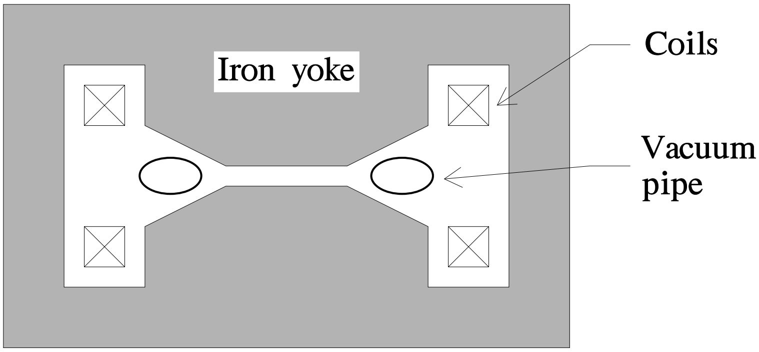

Figure. 1.11 Side cross section of a betatron.

that some 100 to 125 turns suffice to achieve the wanted acceleration. The required frequency can be calculated from

$$\frac{\nu_{\mathrm{rf}}}{B} = \frac{e}{2\pi m} \simeq 15\ \mathrm{MHz} \cdot \mathrm{T}^{-1}, \text{ for protons.} \tag{1.32}$$

Typical beam currents would be about 1mA.

However, Eq. (1.27) contains the Lorentz factor γ, which, as soon as the kinetic energy is significantly increased, begins to deviate from one, causing a decrease of the radio-frequency. A remedy to this consists of just varying the rf according to a $1/\gamma$ law; this means that only particles in phase with this varying rf can be accelerated. That is, a synchronous acceleration principle[8,9] is now required, and only one bunch of particles will reach the final kinetic energy (600—700 MeV for protons), due to technical limitations on magnet dimensions and rf modulation. The current is of the order of a few microamperes, but still a rather small number (a few thousand) of revolutions are required to accomplish the full cycle of acceleration.

This attention paid to the number of revolutions, run by particles from rest to their goal-energy, is dictated by the need of having orbits with limited dimensions. In fact, since particles leave the ion source with nonzero angles and energy spread to form a beam of nonzero size, a very high number of turns would imply a broadening of the beam, unless some focusing mechanism plays a role.

This problem first arose for the betatron,[10] a rather different machine based on acceleration of electrons by induction. The working principle of this accelerator will be illustrated briefly: the magnetic flux, crossing a circle of radius R, is $\phi = \pi R^2 \bar{B}$, where $\bar{B}$ is some average magnetic induction. If ϕ varies with time, an electron orbiting at this radius will experience a force,

$$F = -eE \simeq (-e)\frac{V}{2\pi R} = \frac{(-e)}{2\pi R}\left(-\frac{d\phi}{dt}\right) = \frac{1}{2} eR\frac{d\bar{B}}{dt}. \tag{1.33}$$

Newton's second law of dynamics gives

$$F = \frac{dp}{dt} = eR\frac{dB_{\rm g}}{dt}, \tag{1.34}$$

having considered Eq. (1.25) with the field $B_{\rm g}$ in the magnet gap (see Fig. 1.11). From both Eqs. (1.33, and 1.34), it is easy to infer the 2–to–1 rule or Wideröe condition,[11]

$$B_{\rm g} = \frac{1}{2}\bar{B}, \tag{1.35}$$

typical of betatrons, which gives rise to the rather cumbersome structure of these machines.

Working out a few realistic numbers, one obtains $W_{\rm max} \simeq 50$ MeV, which for $B_{\rm g} = 0.5$ T, gives $R = 33$ cm and $\bar{\tau}_{\rm rev} = (2\pi R)/\bar{v} \simeq 14$ ns, since $\bar{v} \simeq c/2$. For $\Delta t_{\rm cycle} = (1/4 f_{\rm main}) = 5$ ms, we get

$$N_{\rm rev} = \frac{\Delta t_{\rm cycle}}{\bar{\tau}} \simeq 357,000 \text{ turns}. \tag{1.36}$$

Now the problem of focusing the circulating beam, mildly tackled in cyclotrons, becomes of primary importance. In the next chapter, we will derive the trajectory equations for a charged particle traveling through various shaped magnetic fields. Also, we will see that several effects will cause focusing in the dimensions transverse to the beam.

1.6 Momentum compaction and the synchronous particle

If a particle of fixed energy moves along a closed trajectory, the integral of the curvature, $1/\rho$, around the closed path must be

$$\oint \frac{ds}{\rho} = 2\pi. \tag{1.37}$$

Since $1/\rho = qB_\perp/p$, this means that the momentum

$$p = \frac{q}{2\pi}\oint B_\perp\, ds. \tag{1.38}$$

Leaving the magnetic guide field unchanged, consider another particle also with a closed trajectory but a slightly different momentum. This second particle's closed orbit must differ from the path of the first particle. By varying the momentum, the path length, $L = \oint ds$, of the closed trajectory can be varied. The fractional deviation of this path length divided by the fractional deviation of the momentum is frequently called *momentum compaction*

$$\alpha_p = \frac{dL}{L} \Big/ \frac{dp}{p} = \frac{p}{L}\frac{dL}{dp}. \tag{1.39}$$

The name momentum compaction (see Appendix A) is reviled by some authors since an increase in momentum implies a lengthening of the orbit if $\alpha_p > 0$, i. e., a dilation rather than a compaction.

For example, it is easy to show that the momentum compaction for a satellite of mass m in a circular orbit* around the earth at a radius r has a momentum compaction of -2, ignoring elliptical orbits:

$$F = \frac{GMm}{r^2} = \frac{p^2}{mr}, \tag{1.40}$$

$$\frac{dr}{dp} = -\frac{2pr^2}{GMm^2}, \tag{1.41}$$

and thus

$$\alpha_p = \frac{p}{r}\frac{dr}{dp} = -\frac{2p^2 r}{GMm^2} = -2. \tag{1.42}$$

In order to understand the acceleration of a particle due to an rf field, let us first consider a proton synchrotron with a fixed magnetic guide field and a single rf cavity driven by an oscillating electric field of constant amplitude and frequency ω_{rf}. The electric field on the axis of the cavity can be described by

$$E_s(x=0, y=0, s, t) = E_0(s)\cos(\omega_{\mathrm{rf}} t), \tag{1.43}$$

where $E_0(s)$ is the s-component of the amplitude of the electric field along the axis. For this simple example, let us ignore Maxwell and approximate $E_0(s)$ by a constant inside the cavity and zero everywhere else. This constant would then be the maximum voltage across the gap divided by the length of the gap, V_0/g.

As a particle of velocity v crosses the gap it will experience a varying acceleration. The change in energy can be calculated by

$$\Delta U = \int_{t_0}^{t_0+T} \frac{eV_0}{g}\cos(\omega_{\mathrm{rf}} t)\, v\, dt, \tag{1.44}$$

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* Here, it is assumed the increment dp of momentum is applied in such a way as to keep the satellite in a circular orbit. If this were not done, then the momentum would not be constant, and our definition of momentum compaction would not be applicable.

where the particle enters the gap at time t_0 and position s_0. The transit time T through the cavity must satisfy the condition

$$g = \int_{t_0}^{t_0+T} v\,dt. \tag{1.45}$$

Particles arriving at different times will accrue different energy shifts on passing through the cavity, some positive, some negative. There will be a specific phase relative to the rf oscillations, for which $\Delta U = 0$. In this simple case, a *synchronous particle* is defined as a hypothetical particle that moves along the design trajectory and passes through the rf cavity with a phase, such that there is no change in energy. The synchronous particle must have a path length which is an integral number of $\beta\lambda$, where λ is the wavelength of the rf field, and $\beta = v/c$ for the particle.

When the accelerator is ramped up in energy (by ramping up the fields of the guide magnets), the meaning of the synchronous particle changes slightly. The ramping is assumed to be slow, so that the particles gain only a very small increase in energy per revolution. It is then possible to consider the ramping to increase as a step function, so that on each revolution the momentum corresponding to the closed orbit increases in steps. The synchronous particle is now defined as the hypothetical particle whose momentum increases exactly by this amount. Clearly the phase of this synchronous particle with respect to the rf must be shifted slightly from the case of no acceleration, i. e., the particle must see a net electric potential averaged over the time that it spends in the gap of the cavity.

This concept of synchronous particle may also be extended to a linac. A linac is designed to give a specific increase in energy for each cell. These increments typically vary from cell to cell. For the linac, the synchronous particle is defined as the hypothetical particle that obtains exactly the increment of energy of the design at each cell.

The next few chapters will study the transverse component of motion in beam lines and accelerators. The concept of phase stability and acceleration of particles will be discussed starting in Chapter 7.

Problems

1–1 a) If the bunches can be described by Gaussian ellipsoids with

$$\rho \propto \exp\left(-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} + \frac{z^2}{2\sigma_z^2}\right)\right),$$

show that the luminosity reduces to

$$\mathcal{L} = f_0 N_b \frac{N_+ N_-}{\pi(4\sigma_x \sigma_y)},$$

where it is assumed that the beams move along the z-axis. The number of particles per bunch in the electron and positron beams are N_- and N_+, respectively. The number of bunches in each beam is given by N_b. (Assume the bunches do not change shape either due to the accelerator optics or the interaction of the beams passing through each other.) b) An e^+e^- storage ring (CESR) operates at 5.3 GeV with 7 bunches of e^+ and 7 bunches of e^- orbiting in opposite directions. Assume the current per bunch is initially 8 mA, and the ring circumference is 768 m. For $\sigma_x = 8.4 \times 10^{-4}$ m, $\sigma_y = 3.5 \times 10^{-5}$ m, and $\sigma_z = 2.2$ cm, what is the initial luminosity in one of the experiments? What is the integrated luminosity of this experiment for a 3 hr run if the beam lifetimes are both 2 hr? (Assume that the beam currents decay exponentially.)

1–2 Calculate the brightness of a NdYAG laser with the following parameters:

$$\lambda = 1.064\mu \text{ m}$$
$$\text{Power} = 20 \text{ W}$$
$$\text{Bandwidth } \frac{\Delta\omega}{2\pi} = 120 \text{ GHz}$$
$$\text{Beam divergence} = 10 \text{ mrad.}$$

1–3 In a fixed Cartesian coordinates system, show that the equations of motion for a charged particle moving in a magnetic field may be written as

$$x'' = \frac{q}{p}(1 + x'^2 + y'^2)^{\frac{1}{2}}[y' B_z - (1 + x'^2)B_y + x'y'B_x], \quad \text{and}$$
$$y'' = -\frac{q}{p}(1 + x'^2 + y'^2)^{\frac{1}{2}}[x' B_z - (1 + y'^2)B_x + x'y'B_y],$$

where the primes denote derivatives with respect to z (i. e., $x' = dx/dz$). Here it has been assumed that the electric field is zero and that $dz/dt \neq 0$.

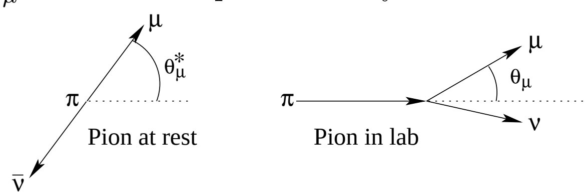
1–4 Consider a charged pion decaying into a muon plus an antineutrino:

$$\pi^- \to \mu^- + \bar{\nu}_\mu.$$

Use $M_{\pi^\pm} = 140$ eV/c^2, $m_\mu = 106$ MeV/c^2, and $m_{\bar{\nu}} = 0$.

a) In the rest system of the pion, what are the energies and momenta of the muon and antineutrino?

b) For a moving pion with total energy $U_\pi = \gamma M_\pi c^2$ find an expression for the direction, θ_μ of the muon relative to the pion in the lab in terms of the angle θ^*_μ in the in the pion's rest system.



1–5 The Tevatron collides protons ($m_p = 0.938$ GeV) at 1 TeV per beam. What is the equivalent proton beam energy required to produce the same center-of-mass energy with a stationary hydrogen target? How fast would you have to drive your new 1.3 ton VW Beetle to have the same kinetic energy as a bunch of 10^{13} protons with this energy? (The speed of sound in air is 330 m/s.)

1–6 HERA collides 920 GeV protons with 27.5 GeV electrons with zero crossing angle.

a) What is the center-of-mass energy?

b) What is the velocity of the center of mass in the lab system?

1–7 The Stanford Linear Accelerator is 3.05 km long and can accelerate electrons up to 50 GeV.

a) What is the average accelerating gradient of the rf cavities?

b) For bunches of 4×10^{10} electrons per bunch and a duty cycle of 100 Hz, what is the power transferred to the beam?

1–8 An experiment has a 10 cm long liquid hydrogen target with a density of

$$\rho = .063 \text{ g/cm}^3.$$

Estimate the interaction rate for p+p collisions for a beam of 10^{13} protons every two minutes. Assume the total cross section is 40 mb. Note: 1 barn = 10^{-24} cm^2 and Avogadro's number is 6×10^{23}.

References for Chapter 1

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