

## Leonhard Euler – 300 Years On

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2007 marked the 300th anniversary of the birth of Leonhard Euler. This article presents a selection of his achievements.

Euler was the most prolific mathematician of all time. He wrote more than 500 books and papers during his lifetime, with 400 further publications appearing posthumously. His collected works and correspondence, still not completely published, fill over seventy large volumes comprising tens of thousands of pages. He worked in an astonishing variety of areas, ranging from pure mathematics (the theory of numbers, the geometry of a circle and musical harmony), via infinite series, logarithms, the calculus and mechanics, to practical topics (optics, astronomy, the motion of the Moon and the sailing of ships). He originated so many ideas that his successors have been kept busy trying to follow them up ever since. Indeed, his influence was such that several concepts were later named after him: Euler's constant, Euler's polyhedron formula, the Euler line of a triangle, Euler's equations of motion, Eulerian graphs, Euler's pentagonal formula for partitions, to name but a few.

His life can be divided into four periods. He was born in Basel, Switzerland, on 15 April 1707, where he grew up and went to university. At the



age of 20 he went to Russia, to the St Petersburg Academy, where he became head of the mathematics division. In 1741 he went to Berlin, where he stayed for twenty-five years. In 1766 he returned to St Petersburg where he spent the rest of his life, dying on 7 September 1783.

## **Basel, Switzerland**

Leonhard Euler's father was a Calvinist pastor who wished his son to follow him into the ministry. On entering the University of Basel at the age of 14, the young Euler duly studied theology and Hebrew, law and philosophy. While there, he encountered Johann Bernoulli, possibly the finest mathematician of his day, who was impressed with his mathematical abilities and gave him private teaching every Saturday, quickly realising that his pupil was highly talented. Euler also became close friends with Johann's sons, Daniel and Nicholas.

Euler took his Master's degree in 1724, at the age of 17, and entered divinity school to train for the ministry, but made little progress since mathematics was proving to be such a distraction. Eventually, Bernoulli persuaded Euler's reluctant father that his son was destined to become a great mathematician, and Euler abandoned his theological training.

Euler's first significant mathematical achievement occurred when he was just 20. The Paris Academy had proposed a prize problem involving the placing of masts on a sailing ship in order to combine speed with stability. Euler's memoir, while not gaining the prize, received an honourable mention; later, he won the Paris prize on twelve occasions.

Euler next applied for the Chair of Mathematics at the University of Basel. Because of his young age he was unsuccessful, but meanwhile Daniel Bernoulli had taken up a position at the St Petersburg Academy in Russia, and invited Euler to join him there. The only available position was in medicine and physiology, but jobs were scarce so Euler learned these subjects, and his study of the ear led him to investigate the mathematics of sound and the propagation of waves.

## **St Petersburg, Russia**

Unfortunately, on the very day that Euler arrived in Russia, Empress Catherine I, who had set up the Academy, died. Her heir was still a boy, and the faction that ruled on his behalf regarded the Academy as a luxury. Euler quietly got on with his work, while working closely with Daniel Bernoulli.

In 1733, Daniel Bernoulli had had enough of the problems of the

Academy and returned to an academic position in Switzerland. Euler, still aged only 26, replaced him in the Chair of Mathematics, and determined to make the best of a difficult situation. The 1730s were indeed very productive years for him, with substantial advances in number theory, the summation of series, and mechanics. At the same time he was acting as a scientific consultant to the government – preparing maps, advising the Russian navy, testing designs for fire engines, and writing textbooks for the Russian schools.

An area to which Euler contributed throughout his life was the theory of numbers. In December 1729, he received a letter from his St Petersburg colleague Christian Goldbach, who is best remembered for the still-unproved *Goldbach conjecture* that every even number can be written as the sum of two prime numbers. Goldbach's letter was concerned with the *Fermat numbers*:

$$2^1 + 1 = 3, \quad 2^2 + 1 = 5, \quad 2^4 + 1 = 17, \quad 2^8 + 1 = 257, \quad 2^{16} + 1 = 65,537, \dots$$

Are *all* such numbers prime? Fermat had conjectured that they are, but Euler found an ingenious argument to prove that the next one ( $2^{32} + 1$ ), a ten-digit number, is divisible by 641. Since then, *no* other 'Fermat number' has been shown to be prime, so Fermat's conjecture was unfortunate.

Euler's prodigious calculating abilities were legendary. One day, two students were trying to sum a complicated progression and disagreed over the 50th decimal place: Euler simply calculated the correct value in his head to settle the argument. Another challenge given to Euler was to find four different numbers, the sum of any two of which is a perfect square: he produced the quartet: 18530, 38114, 45986 and 65570.

A different preoccupation in the 1730s was his work on infinite series. He first became interested in the 'harmonic series'

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots,$$

which does not converge, and noticed that the sum of the first  $n$  terms is very close to  $\log n$ . In particular, he proved that as  $n$  becomes large, the difference between them tends to a fixed number  $0.5772\dots$ , now known as *Euler's constant*.

Another problem on infinite series, known as the *Basel problem*, exercised many minds at the time. It was to find the sum of the reciprocals of the perfect squares:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$$

One of Euler's earliest achievements was to show that this sum is  $\pi^2/6$ , and this brought him international fame. He also extended his calculations to find the sum of the reciprocals of the 4th powers, the 6th powers, and so on, up to the 26th powers. This led him to investigate what is now known as the *Riemann zeta function*.

We next turn to a recreational puzzle that Euler solved in 1735: the problem of the *bridges of Königsberg*. The medieval city of Königsberg consisted of four areas of land linked by seven bridges, and the problem was to find a route crossing each bridge just once and returning to the starting point. Euler discovered a counting argument involving the number of bridges emerging from each land area, and proved that no such route exists. Furthermore, he obtained for any arrangement of land areas and bridges a corresponding rule for deciding when such a route is possible – namely,

- if there are no areas with an odd number of bridges, then a route exists starting anywhere and ending in the same place;
- if there are two areas with an odd number of bridges, then a route exists, starting in one area and ending in the other;
- if there are more than two such areas (as in Königsberg), then there is no such route.

Euler's solution of the Königsberg bridges problem is considered as the earliest contribution to graph theory, and is now solved by looking at a network with four points representing the land areas and seven lines representing the bridges. But Euler never did this – the network that represents this puzzle was not drawn for 150 years.

Around the same time, Euler published *Mechanica*, his first treatise on the dynamics of a particle. However, his most important work in this area came in 1750 with his work on the motion of rigid bodies – free, or rotating about a point. By choosing the point as the origin of coordinates, and with axes aligned along the principal axes of inertia of the body, he obtained what are now called *Euler's equations of motion*; the concept of *moment of inertia* was also due to him. Even later, in 1776, he proved that any rotation of a rigid body about a point is equivalent to a rotation about a line through that point. Much of his work in this area used differential equations, an area to which he had himself contributed a great deal.

In the late 1730s, Euler went blind in his right eye. Although he attributed this to overwork, it was probably due to an eye infection. However, this did not diminish his productivity: he continued to write on acoustics, musical harmony, ship-building, prime numbers, and much more besides.

## Berlin, Germany

In 1741, with his fame preceding him, Euler received an invitation from Prussia's Frederick the Great to join the newly vitalised Berlin Academy. With the political situation in Russia still uncertain, he accepted it and remained in Berlin for 25 years.

At first, he got on well with Frederick, but later, especially after the seven years war between Germany and Russia, things began to cool as Frederick started to take more and more interest in the workings of the Academy. Frederick considered himself cultured and witty, and found Euler unsophisticated; in return, Euler found Frederick pretentious, petty and rude.

Even so, Euler managed to work on a dazzling range of topics, writing works in the 1740s and 1750s on the theory of tides, the calculus of variations, the motion of the moon, hydrodynamics, and the wave motion of vibrating strings.

His most important work from this period was the *Introductio in Analysin Infinitorum* ('Introduction to the analysis of the infinite'), published in 1748. It was here that he presented some of his earlier work on the number  $e = 2.718281\dots$ , defined as

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \text{ or as } \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n,$$

and the related exponential function  $e^x$ .

In the *Introductio*, Euler expressed certain well-known functions as infinite series, and then introduced his great masterstroke. He knew how the functions  $e^x$ ,  $\sin x$  and  $\cos x$  could be expanded in powers of  $x$ , but at first they seem to have nothing in common. However, on introducing the complex number  $i$ , and manipulating the power series, he deduced the fundamental formula linking them,

$$e^{ix} = \cos x + i \sin x.$$

Although he did not explicitly write down the simple consequence  $e^{i\pi} = -1$ , often described as his most famous result, he would surely have known it.

There were many other interesting things in the *Introductio*. Over the 100 years since Descartes there had been a gradual movement from geometry towards algebra, and this reached its climax when Euler actually defined the conic sections (ellipse, parabola and hyperbola) by their algebraic equations, rather than geometrically as sections of a cone. In particular, starting with the equation  $y^2 = \alpha + \beta x + \gamma x^2$ , he showed that we get an ellipse if  $\gamma < 0$ , a parabola if  $\gamma = 0$ , and a hyperbola if  $\gamma > 0$ . He then extended

his algebraic arguments to three dimensions, to the seven types of *quadrics*, and discovered the *hyperbolic paraboloid* in the process.

Yet another interesting topic in the *Introductio* is *partitions*, or ‘divisions of integers’, as Leibniz had called them in a letter to Bernoulli. In how many ways can we split up a positive integer  $n$  into smaller ones?

Let  $p(n)$  be this number – for example,  $p(4) = 5$ , corresponding to the five partitions 4, 3 + 1, 2 + 2, 2 + 1 + 1 and 1 + 1 + 1 + 1 (the order doesn’t matter). So we can draw up a table of values – but how can we show that  $p(200) = 3,972,999,029,388$ ?

To investigate partitions, Euler introduced the generating function

$$p(x) = 1 + p(1)x + p(2)x^2 + p(3)x^3 + \dots,$$

and used it to derive what is now called *Euler’s pentagonal number formula* involving the ‘generalised pentagonal numbers’  $\frac{1}{2}k(3k \pm 1)$ :

$$p(n) = p(n - 1) + p(n - 2) - p(n - 5) - p(n - 7) + p(n - 12) + \dots$$

This yields  $p(n)$  by iteration, and is still the most efficient way of finding  $p(n)$ .

A particularly nice result on partitions, which appears in the *Introductio*, concerns odd and distinct partitions. In an *odd partition* all the parts are odd, and in a *distinct partition* all of them are different: for example, 9 has eight odd partitions (9, 7 + 1 + 1, 5 + 3 + 1, etc.) and eight distinct partitions (9, 8 + 1, 7 + 2, etc.). Using generating functions, Euler proved that for *any* number, the number of odd partitions is the same as the number of distinct partitions.

Another preoccupation was mentioned in a letter to Goldbach, in 1750. Euler had been looking at *polyhedra*, and observed that the numbers of vertices, edges and faces are always related by the formula:

$$(\text{no. of faces}) + (\text{no. of vertices}) = (\text{no. of edges}) + 2.$$

This formula, now known as *Euler’s polyhedron formula*, has been incorrectly credited to Descartes, who did not have the terminology or motivation to derive it: indeed, it was Euler who introduced the concept of an *edge*. However, Euler’s proof was deficient – a complete proof was not given until 40 years later, by the algebraist and number-theorist Legendre.

Euler’s most popular and best-selling book was his *Letters to a German Princess*. Euler was always an extremely clear writer, and this was a multi-volume masterpiece of exposition that he produced when he was asked to give elementary science lessons to the Princess of Anhalt-Dessau. The resulting collection had over 200 ‘letters’ that Euler wrote on a range

of scientific topics, including gravity, astronomy, light, sound, magnetism, logic, and much else besides. He wrote about why the sky is blue, why the moon looks larger when it rises, and why the tops of mountains are cold (even in the tropics). It was one of the best books ever written on popular science.

The last of Euler's Berlin books was his 1755 massive tome on the differential calculus. This contained all the latest results, many due to him, and presented the calculus in terms of the basic idea of a function – indeed, it was Euler who introduced the notation  $f$  for a function. Other notations he introduced at various times were  $\sum$  (for summation),  $i$  (the square root of  $-1$ ) and  $e$  (the exponential number). He also popularised the notation for  $\pi$ , although that had actually been introduced by William Jones in 1706. He followed his book on differentiation in 1768–1770 with a three-volume treatise on the integral calculus.

### St Petersburg, revisited

After his difficulties with Frederick the Great, Euler must have felt very relieved when in 1766, at the age of 59, he received an invitation from Catherine the Great of Russia to return to St Petersburg. Things there had improved greatly, thanks to the enlightened Empress, and he was received royally.

He continued to work enthusiastically, soon producing a delightful result in pure geometry. In any triangle three particular points of interest are the *orthocentre* (the meeting point of the perpendiculars from the vertices to the opposite sides), the *centroid* (the meeting point of the three lines joining a vertex to the midpoint of the opposite side), and the *circumcentre* (the centre of the circle passing through the vertices of the triangle). By calculating their coordinates, Euler proved the attractive result that these three points always lie in a straight line – now called the *Euler line* of the triangle – and that the centroid always lies exactly one-third of the distance between the other two.

Euler's life-long interest in number theory continued into his later years, when he extended some results associated with Fermat – in particular, *Fermat's last theorem* that, for any  $n > 2$ , there are no positive numbers  $a, b, c$  satisfying  $a^n + b^n = c^n$ . In his number theory book of 1770, Euler proved for the first time that the sum of two cubes cannot equal another cube ( $n = 3$ ).

Another connection with Fermat was *Fermat's little theorem*, which states that, if  $a$  is any number that is not divisible by a given prime number

$p$ , then  $p$  divides  $a^{p-1} - 1$ ; for example, on taking  $p = 29$  and  $a = 48$ , we deduce that  $48^{28} - 1$  is divisible by 29. In 1760 Euler extended this result to numbers other than primes, introducing the *Euler  $\varphi$ -function* and proving that, for any numbers  $a$  and  $n$ ,  $a^{\varphi(n)} - 1$  is always divisible by  $n$ .

Yet another result in number theory concerned *perfect numbers* – numbers whose proper divisors add up to the number itself; for example, 28 is a perfect number because its proper divisors are 1, 2, 4, 7 and 14, which sum to 28. In his *Elements*, Euclid had proved that every number of the form  $2^{n-1} \times (2^n - 1)$  is perfect, when  $2^n - 1$  is prime. Euler proved that every *even* perfect number has this form – but it is still not known whether any *odd* perfect numbers exist.

The last few years of Euler's life, though more peaceful than his earlier ones, saw many personal tragedies. In 1771 his house burned down, with the loss of his library and almost his life, but fortunately his manuscripts were saved. Shortly after this, his beloved wife died and he remarried. And finally he lost most of the sight in his other eye – but again, his productivity remained undiminished as he wrote on slates with his sons and friends as amanuenses; indeed, shortly after he went blind, he produced a 700-page volume on the motion of the moon.

In a book on recreational mathematics in 1725, Jacques Ozanam had shown how to lay out the sixteen court cards so that each row and column contains each suit and each value (J, Q, K, A). It is also possible to make a similar arrangement with 25 cards (five suits and five values), but what about 36 cards? In the year before he died, in a paper mainly on magic squares (another interest of his), Euler posed this as the 36 officers problem:

*Arrange 36 officers, one each of 6 ranks from 6 regiments, in a square array so that each row or column has one officer of each rank and one officer of each regiment.*

Euler believed that this cannot be done, and this was eventually confirmed around 1900 by Gaston Tarry, essentially by enumerating all the possibilities. Euler also claimed that the corresponding problem has a solution for any number of ranks and regiments, except when this number is of the form  $4k + 2$ : that is, 6, 10, 14, 18, . . . . He was right about 6, but wrong about *all* the others, although his error was not demonstrated until almost 300 years later.

## Conclusion

Euler worked until the very end. In a Eulogy by the Marquis de Condorcet, we read about his final afternoon:

*On the 7th of September 1783, after amusing himself with calculating on a slate the laws of the ascending motion of air balloons, the recent discovery of which was then making a noise all over Europe, he dined with Mr Lexell and his family, talked of Herschel's planet (Uranus), and of the calculations which determine its orbit. A little after, he called his grandchild, and fell a playing with him as he drank tea, when suddenly the pipe, which he held in his hand, dropped from it, and he ceased to calculate and to breathe. The great Euler was no more.*

## Acknowledgments

This article is adapted from the author's article 'Read Euler, read Euler, he is the master of us all', *Plus* (Online mathematics magazine), Mathematics Millennium Project (University of Cambridge, England), 42, March 2007.

## Further reading

A good introductory book on Euler's life and works is: William Dunham, *Euler: the master of us all*, Mathematical Association of America, 1999.

To celebrate the 300th anniversary of his death, the Mathematical Association of America has also published a series of five books in 2007 on the life and works of Leonhard Euler.

Further information can also be found in standard reference works, such as:

*The Dictionary of Scientific Biography*, Scribner, New York, 1970–1990.