

*The modern physical developments
have required a mathematics
that continually shifts its foundations.*

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Preface

The theory presented in this book has four roots. The first lies in Dirac's seminal paper of 1931 [1] in which he pointed out that Maxwell's equations can accommodate magnetic monopoles, in spite of the vanishing divergence of the magnetic field, thanks to quantum mechanics. It is always possible to create a magnetic field emerging from a point by importing the field from far distance to the point through an infinitely thin magnetic flux tube. But it is only due to quantum mechanics, that such a flux tube can be made physically undetectable. This is true provided the famous *Dirac charge quantization condition* is fulfilled which states that all electric charges are integer multiples of $2\pi\hbar c/g$, where g is the total magnetic flux through the tube. The undetectable flux tube is called the *Dirac string*. From the endpoint of the string, magnetic field lines emerge radially outwards in the same way as electric field lines emerge from an electric point charge, so that the endpoint acts as a magnetic monopole. The shape of the undetectable string is completely irrelevant. It is a mathematical artifact. For this stunning observation, Pauli gave Dirac the nickname *Monopoleon*. The Dirac quantization condition was subsequently sharpened by Schwinger [2] who showed that the double-valuedness of the spin- $\frac{1}{2}$ wave functions of electron restricted the integer multiples to even multiples. Experimentally, no magnetic monopole was found in spite of intensive search, and the Dirac theory was put ad acta for a long time. It resurfaced, however, in the last 35 years, in the attempt to explain the phenomenon of quark confinement.

The second root in this book lies in the theory of the superfluid phase transition. Here the crucial papers were written by Berezinski [3] and by Kosterlitz and Thouless [4]. They showed that the phase transition in a film of superfluid helium can be understood by the statistical mechanics of vortices of superflow. Their description attaches to each point a phase angle of the condensate wave function which lies in the interval $(0, 2\pi)$. When encircling a vortex, this angle must jump somewhere by 2π . A jumping line connects a vortex with an antivortex and forms an analog of a "Dirac string", whose precise shape is irrelevant. If these ideas are carried over to bulk superfluid helium in three dimensions, as done in my textbook [5], one is led to the statistical mechanics of vortex loops. These interact with the same long-range forces as electric current loops.

The third root of the theory in this book comes from a completely different direction — the theory of plastic deformations, which is the basis of our understanding of work hardening of metals and material fatigue. This theory was developed after the discovery of dislocations in crystals in 1934 [6]. With the help of field-theoretic techniques, this theory was extended to a statistical mechanics of line-like defects in my textbook [7], where I explained the important melting transitions by the condensation of line-like defects.

The fourth root lies in the work of Bilby, Bullough, Smith [8], Kondo [9], and Kröner [10], who showed that line-like defects can also be described in geometric terms. Elastic distortions of crystals do not change the defect geometry, thus playing a similar role as Einstein's coordinate transformations. Crystals with defects form a special version of a *Riemann-Cartan space*. The theory of such spaces was set up in 1922 by Cartan who extended the curved Riemannian space by another geometric property: *torsion* [11]. Cartan's work instigated Einstein to develop a theory of gravitation in a Riemann-Cartan spacetime with teleparallelism [12].

Twenty years later, Schrödinger attempted to relate torsion to electromagnetism [13]. He noticed that the presence of torsion in the universe would make photons massive and limit the range of magnetic fields emerging from planets and stars. From the observed ranges of his time he deduced upper bounds on the photon mass which were, even then, extremely small. Further twenty years passed before Utiyama, Sciama, and Kibble [14, 15, 16] clarified the intimate relationship between torsion and the spin density of the gravitational field. A detailed review of the theory was given in my textbook [7]. The recent status of the subject is summarized by Hammond [17].

I ran into the subject in the eighties after having developed a disorder field theory of line-like objects in my textbook [5]. My first applications dealt with vortex lines in superfluids and superconductors, where the disorder formulation helped me to solve the long-standing problem of theoretically predicting where the second-order phase transition of a superconductor becomes first-order [18].

After this I turned to the application of the disorder field theory to line-like defects in crystals. The original description of such defects was based on functions which are discontinuous on surfaces, whose boundaries are the defect lines. The shape of these surfaces is arbitrary, as long as the boundaries are fixed. I realized that the deformations of the surfaces can be formulated as gauge transformations of a new type of gauge fields which I named *defect gauge fields*.

By a so-called *duality transformation* it was possible to reformulate the theory of defects and their interactions as a more conventional type of gauge theory. This brought about another freedom in the description which I named *stress gauge invariance*. The dual formulation can be viewed as a linearized form of yet another geometric Einstein-Cartan space in which the gauge transformations are a combination of Einstein's local translations and a local generalization of *Lorentz invariance*.

The relation between the dual and the original description of defects in terms of jump surfaces is completely analogous to the well-known relation between Maxwell's theory of magnetism formulated in terms of a gauge field, the vector potential, and

an alternative formulation in which the magnetic field is the gradient of a multivalued scalar field.

While the above developments were in progress, field theorists were searching for a simple explanation of the phenomenon of quark confinement by color-electric field lines. Here the physics of superconductors became an important source of inspiration. Since London's theory of superconductivity [19] it was known that superconductors would confine magnetic charges if they exist. The reason is the *Meissner effect*, which tries to expel magnetic flux lines from a superconductor. As a consequence, flux lines emerging from a magnetic monopole are compressed into flux tubes of a fixed thickness. The energy of such tubes is proportional to their length implying that opposite magnetic charges are held together forever. From the BCS theory of superconductivity [20] we know that this effect is caused by a condensate of electric charges, the famous *Cooper pairs* of electrons.

This phenomenon suggested the presently accepted viewpoint on quark confinement. The vacuum state of the world is imagined to contain a condensate of color-magnetic monopoles. This condensate acts upon color-electric fields in the same way as the Cooper pairs in a superconductor act upon the magnetic field, causing a Meissner effect and confinement of color-electric charges. Models utilizing this confinement mechanism were developed by Nambu [21], Mandelstam [22], 't Hooft [23], and Polyakov [24], and on a lattice by Wilson [25].

In studying this phenomenon I observed the close mathematical analogies between Dirac's magnetic monopoles and the above defect structures. Dirac used a vector potential with a jump surface to construct an infinitely thin magnetic flux tube with a magnetic point source at its end. Thus the world line of a monopole in spacetime could be viewed as a kind of "vortex line" in a Maxwell field. Knowing how to construct a disorder theory of vortex lines it was easy to set up a disorder field theory of monopole worldlines, which presently serves as the simplest model of quark confinement [26].

When extending the statistical mechanics of vortex lines to defect lines in the second volume of the textbook [7], I used the dual description of defect lines, and expressed it as a linear approximation to a geometric description in Riemann-Cartan space. This suggested to me that it would be instructive to reverse the development in the theory of defects from multivalued fields to geometry and reformulate the theory of gravity, which is conventionally treated as a geometric theory, in an alternative way with the help of jumping surfaces of translation and rotation fields. In the theory of plasticity, such singular transformations are used to carry an ideal crystal into crystals with translational and rotational defects. Their geometric analogs carry a flat spacetime into a spacetime with curvature and torsion. The mathematical basis expressing the new geometry are *multivalued* tetrad fields $e^a{}_\mu(x)$.

In the traditional literature on gravity with spinning particles, a special role is played by single-valued vierbein fields $h^a{}_\mu(x)$. They define local nonholonomic coordinate differentials dx^a . These are reached from the physical coordinate differentials dx^μ by a transformation $dx^a = h^a{}_\mu(x)dx^\mu$. Only infinitesimal vectors dx^a are defined, and the transformation cannot be extended over finite domains. For

the description of spinning particles, such an extension is not needed since the infinitesimal nonholonomic coordinates dx^α are completely sufficient to specify the transformation properties of spin in Riemannian spacetime.

The theory in terms of multivalued tetrad fields to be presented here goes an important step further, leading to a drastic simplification of the description of non-Riemannian geometry. The key is the efficient use of a set of completely new non-holonomic coordinates dx^a which are more nonholonomic than the traditional dx^α . To emphasize this one might call them *hyper-nonholonomic coordinates*. They are related to dx^α by a *multivalued* Lorentz transformation $dx^a = \Lambda^a_\alpha(x)dx^\alpha$, and to the physical dx^μ by the above *multivalued* tetrad fields as $dx^a = e^a_\mu(x)dx^\mu \equiv \Lambda^a_\alpha(x)h^\alpha_\mu(x)dx^\mu$. The gradients $\partial_\mu e^a_\nu(x)$ determine directly the full affine connection, and their antisymmetric combination $\partial_\mu e^a_\nu(x) - \partial_\nu e^a_\mu(x)$ determines the torsion. This is in contrast to the curl of the usual vierbein fields $h^\alpha_\mu(x)$ which determines the object of anholonomy, a quantity existing also in purely Riemannian spacetime, i.e., in curved spacetime without torsion.

One of the purposes of this book is to make students and colleagues working in electromagnetism and gravitational physics appreciate the many advantages brought about by the use of the multivalued tetrad fields $e^a_\mu(x)$. Apart from a simple intuitive reformulation of Riemann-Cartan geometry, it suggests a new principle in physics [27], which I have named *multivalued mapping principle* or *nonholonomic mapping principle*, to be explained in detail in this book. Multivalued coordinate transformations enable us to transform the physical laws governing the behavior of fundamental particles from flat spacetime to spacetimes with curvature and torsion. It is therefore natural to postulate that the images of these laws describe correctly the physics in such general affine spacetimes. As a result I am able to make *predictions* which cannot be made with Einstein's construction method based merely on the postulate of covariance under ordinary coordinate transformations, since those are unable to connect different geometries.

It should be emphasized that it is not the purpose of this book to propose repeating all geometric calculations of gravitation with the help of multivalued coordinate transformations. In fact, I shall restrict much of the discussion to almost flat auxiliary spacetimes. This will be enough to derive the general form of the physical laws in the presence of curvature and torsion. At the end I shall always return to the usual geometric description. The intermediate auxiliary spacetime with defects will be referred to as *world crystal*.

The reader will be pleased to see in Subsection 4.5 that the standard minimal coupling of electromagnetism is a simple consequence of the multivalued mapping principle. The similar minimal coupling to gravity will be derived from this principle in Chapter 17.

At the end I shall argue that torsion fields in gravity, if they exist, would lead quite a hidden life, unless they are of a special form. They would not be observable for many generations to come since they could exist only in an extremely small neighborhood of material point particles, limited to distances of the order of the Planck length 10^{-33} cm, which no presently conceivable experiment can probe.

The detailed development in this book in gravity with torsion is thus at present a purely theoretical endeavor. Its main merit lies in exposing the multivalued approach to Riemann-Cartan geometry, which has turned out to be quite useful in teaching the geometrical basis of gravitational physics to beginning students, and to explain what is omitted in Einstein's theory by assuming the absence of torsion.

The definitions of parallel displacements and covariant derivatives appear naturally as nonholonomic images of truly parallel displacements and ordinary derivatives in flat spacetime. So do the rules of minimal coupling.

Valuable insights are gained by realizing the universality of the multivalued defect description in various fields of physics. The predictions based on the multivalued mapping principle remain to be tested experimentally.

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H. Kleinert
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