# Chapter 1

# Introduction

Many systems in science and engineering can be modeled by an explicit ordinary differential equation (ODE) of the form

$$u' = f(u, t), \tag{1.1}$$

where $f \in C^1(\Omega, \mathbb{R}^r)$, the set $\Omega$ being open in $\mathbb{R}^{r+1}$. This equation is often said to define a *state space model* of the system, and $u$ is referred to as a *state variable.* Explicit ODEs such as (1.1) have a long-term mathematical history, and a large number of analytical and numerical tools have been developed for their study.

However, in many cases such an explicit state space model for the dynamics of a given system is not available. The system may instead be described by an *implicit* ODE

$$F(t, x, x') = 0, \tag{1.2}$$

where $F : \mathbf{\Omega} \to \mathbb{R}^m$ is defined on an open set $\mathbf{\Omega} \subseteq \mathbb{R}^{2m+1}$. For reasons detailed later, the equation (1.2) is called a *semistate model* of the system and, accordingly, $x$ is said to be a *semistate variable.*

If the algebraic (in the sense of non-differential) problem $F(t, x, p) = 0$ arising from (1.2) can be globally and uniquely solved for $p$, a global state model for the system can be derived, and the implicit formulation would not make any difference with the explicit ODE setting. From a local point of view, if $F$ is a $C^1$ mapping and the matrix of partial derivatives $F_p(t, x, p)$ is nonsingular at a given zero of $F$, then by the implicit function theorem the set defined by $F(t, x, p) = 0$ can be locally described as $p = f(x, t)$ for a $C^1$ map $f$, so that $x' = f(x, t)$ defines a *local* state space model for (1.2).

Broadly speaking, the attention in this book will be mainly focused on the implicit ODE (1.2) under the assumption that the derivative $F_p(t, x, p)$ is everywhere singular. In many cases, some of the relations within (1.2) do not involve at all the time derivative $x'$, hence being purely algebraic equations. This motivates calling (1.2) a *differential-algebraic equation* (DAE) or a *differential-algebraic system*. From the modeling perspective, algebraic constraints within the DAE (1.2) make some components of the $m$-dimensional semistate variable $x$ redundant; the elimination of this redundancy may lead to a state space model of the form (1.1), in which the dimension of the state variable $u$ is $r < m$.

For the sake of completeness, explicit ODEs such as (1.1), as well as implicit systems (1.2) in which $F_p$ is everywhere nonsingular, are framed in the differential-algebraic context as *index zero* DAEs. Certain problems in which $F_p$ is singular only on a subset (typically, a hypersurface) of the semistate space will also be addressed. In this setting, the points where $F_p$ is singular are particular instances of what will be called a *singularity*; note, however, that the notion of a singularity will also appear in problems with an everywhere singular matrix $F_p$.

In this Chapter we present an overview of DAE theory. Some notes on the origin of DAEs, from a mathematical perspective but also regarding application fields where they show up, can be found in Section 1.1. Section 1.2 introduces several frameworks developed for the analysis of DAEs. Dynamical features and, in less detail, numerical aspects are also discussed. The role of DAEs in system modeling is examined in Section 1.3. Several issues concerning DAE formulations, as well as the structural forms that arise more often in applications, are compiled in Section 1.4. All this background will make it possible to define, in Section 1.5, the goals and contents of this monograph in a more detailed manner.

## 1.1 Historical remarks: Different origins, different names

The origins of DAE theory can be traced back to the work of K. Weierstrass and L. Kronecker on parametrized families of bilinear forms [144, 301]. In terms of matrices, these *pencils* were applied to the analysis of linear systems of ordinary differential equations with a possibly singular leading coefficient matrix by F. R. Gantmacher [90, 91]; see specifically Section 7 in Chapter XII of [91].

Another milestone is the work of P. Dirac on generalized Hamiltonian systems [79–81]. The key ideas supporting what nowadays is known as the differentiation index of a semiexplicit DAE can be found in these references. A geometric approach to the study of so-called *constrained sys-*

*tems* stemmed from the work of Dirac, mainly motivated by applications in mechanics [17, 101, 102, 170, 171, 205, 208, 261–263]. From different perspectives, mechanical systems have driven much investigation on DAEs; cf. additionally [83, 122, 163, 227] and the bibliography therein.

A large amount of research on differential-algebraic equations has also been motivated by applications in circuit theory. The differential-algebraic form of circuit equations is naturally due to the combination of differential equations coming from reactive elements with algebraic (non-differential) relations modeling Kirchhoff laws and device characteristics. The term 'algebraic-differential system' was already used in the circuit context by Brown in 1963 [32]. In the electrical circuit literature, these models are often referred to as *semistate systems*. The word 'semistate', which is due to Dziurla (see p. 31 in [136]), appeared for the first time in the joint work of Dziurla and Newcomb [82]; see also [210]. In spite of their different origins, the terms 'semistate' and 'differential-algebraic' can be understood as synonyms hereafter. DAEs are nowadays pervasive in nonlinear circuit analysis and design, specially because of their appearance in nodal analysis methods used to set up network equations in circuit simulation programs [85, 87, 112–116, 194, 253, 292, 293]. The reader is referred to Chapters 5 and 6 for extensive discussions of the role of DAEs in circuit modeling.

Control theory has also been the focus of considerable attention in the DAE literature. Differential-algebraic equations are often called in this context *descriptor systems*, after the seminal work of Luenberger [168, 169]. Linear and nonlinear, possibly time-varying control systems have been the focus of an increasing interest from the differential-algebraic perspective in the last three decades, optimal control problems playing an important role; see [12, 14, 30, 50, 53, 73, 97, 109, 146, 150–154, 282, 306, 307] and references therein.

In the 1970s, a mathematical approach to differential-algebraic systems, somehow independent of specific application fields, began to flourish. It is worth mentioning at this point the work of Gear [93], Takens [288] and, in the early 1980s, the books of Campbell [39, 40] as well as the papers by Petzold [216], Gear and Petzold [95], and Rheinboldt [238]. Numerical aspects were emphasized in many of these references. A variety of approaches to DAE analysis began to be developed in that period, through the work of the above-mentioned authors as well as Griepentrog, Hairer, März, Rabier, and Reich, among others; see subsection 1.2.1 below. After 1990, the attention has also been directed to so-called *singular DAEs*, extending in different ways the seminal work of Rabier [219] and Chua and Deng [61, 62] on impasse points. Singular problems will be presented in subsection 1.2.2 and extensively discussed in Chapter 4. Much recent research is focused on *partial differential-algebraic equations* (PDAEs) and stochastic DAEs, beyond the scope of this book; as a sample of the related literature, see [2, 25, 111, 172, 176, 232, 264, 293] and [259, 305], respectively.

Besides the ongoing mathematical interest on analytical and numerical aspects of differential-algebraic systems, quite a lot of attention on DAEs has remained associated with applications in the above-mentioned fields of mechanics, electrical circuits and control theory; applications of DAEs are also found in chemistry [146], power systems [11, 134, 175, 296, 298], magnetohydrodynamics [48, 49, 174], neural networks [55, 211, 256], fault diagnosis [138, 295], model identification and observer design [57, 212], and robotics [46, 211, 278, 279], among other fields. DAEs also arise from problems in other branches of mathematics, such as the discretization of PDEs [30, 167], root-finding [245, 247, 254, 255] or optimization [260].

Other names for DAEs, not linked with specific applications, are 'implicit systems', 'singular systems' and, restricted to quasilinear problems (cf. subsection 1.4.3 below) 'generalized vector fields'. Finally, DAEs should not be confused with 'algebraic differential equations', that is, problems of the form $x' = p(x)$ in which $p$ is a polynomial mapping.

## 1.2 DAE analysis

A $C^1$*-solution* of the differential-algebraic system (1.2) is a $C^1$ mapping $x : \mathcal{I} \to \mathbb{R}^m$, with $\mathcal{I} \subseteq \mathbb{R}$ an open interval, such that $(t, x(t), x'(t)) \in \mathbf{\Omega}$ and $F(t, x(t), x'(t)) = 0$ for all $t \in \mathcal{I}$. Other types of solutions for DAEs will arise in different contexts, for instance when using properly stated formulations, or in the presence of impasse points. For the moment the term 'solution' will mean '$C^1$-solution'.

A solution $x(t)$ which is *a priori* required to take a prescribed value $x(t_0) = x_0$ at a fixed time $t_0$ is said to have *initial value* $x_0$. Contrary to explicit ODEs (cf. (1.1)) defined by a $C^1$ map $f$, for DAEs it is usually the case that not every point admits a solution through it. A point $x_0$ for which there is indeed a solution of (1.2) with $x(t_0) = x_0$ is said to be a *consistent initial value* at $t_0$, and the set of pairs $(x_0, t_0)$ admitting a solution is called the *solution set*. Additionally, we say that the DAE has a unique solution through $x_0$ at $t_0$ if any two solutions $x(t)$ and $\tilde{x}(t)$ (defined on the open intervals $\mathcal{I}$ and $\tilde{\mathcal{I}}$, respectively, both of them comprising $t_0$)

such that $x(t_0) = \tilde{x}(t_0) = x_0$ verify $x(t) = \tilde{x}(t)$ for all $t \in \mathcal{I} \cap \tilde{\mathcal{I}}$. The solution is said to be locally unique if the condition $x(t_0) = \tilde{x}(t_0) = x_0$ implies $x(t) = \tilde{x}(t)$ on some neighborhood of $t_0$.

For autonomous DAEs (that is, problems of the form (1.2) with $F$ independent of $t$), the consistency of an initial value does not depend on the choice of $t_0$, and the solution set can be thought of as lying on $x$-space. In many cases, the solution set has a manifold structure (cf. Section 3.3), solutions are uniquely defined through every point on this manifold, and they yield a $C^1$-flow (see e.g. [3]) on it. This has led to the *solution manifold* concept and to several definitions in the literature around the notions of a *solvable DAE* and a *regular DAE*.

However, in many other problems the above-mentioned properties are not met. For instance, solutions though a given point may not be unique; examples can be found in (4.35) on p. 167, and (4.61), p. 185. Furthermore, the solution set may not be a manifold, as the same examples illustrate. To accommodate solutions at the backward impasse points discussed in subsection 4.4.1 we would need to deal with $C^0$-semiflows. Note that these phenomena are not due to weak smoothness assumptions, since all of them can be displayed even in analytic problems. There are too many situations to handle, so that the discussion of notions such as that of the solution manifold or the one of a regular DAE is postponed until Chapter 3. At this introductory level, it is enough for the moment to have in mind the above-presented definition of a $C^1$-solution and the notion of the solution set.

### 1.2.1 *Indices*

*The Kronecker index of regular linear time-invariant DAEs*

Linear time-invariant DAEs are systems of the form

$$Ax' + Ex = q(t), \tag{1.3}$$

where $A$ and $E$ belong to $\mathbb{R}^{m\times m}$ ($\mathbb{R}^{n\times p}$ denoting throughout the set of real matrices with $n$ rows and $p$ columns), and $q : \mathcal{J} \to \mathbb{R}^m$ is a sufficiently smooth mapping defined on an open interval $\mathcal{J} \subseteq \mathbb{R}$. As detailed in Section 2.1, the solutions of (1.3) can be completely characterized in terms of the Weierstrass-Kronecker canonical form of the matrix pencil $\{A, E\}$, provided that this pencil is a regular one; the *Kronecker index* (also called the *nilpotency index*) of the pencil will play a key role in this analysis. Needless to say, our interest will be directed to cases in which the leading matrix $A$ is singular, since otherwise the problem trivially amounts to a linear explicit ODE with constant coefficients.

*Linear time-varying and nonlinear DAEs*

More difficult is the treatment of the linear time-varying counterpart of (1.3), namely

$$A(t)x'(t) + E(t)x(t) = q(t), \tag{1.4}$$

as well as that of nonlinear problems of the form $F(t, x, x') = 0$ depicted in (1.2). From an analytical point of view, usually the solutions are not explicitly sought in terms of (1.2) or (1.4). Instead, most strategies aim at constructing a related mathematical object from which the solutions of the DAE can be described; this object may be an explicit ODE, a vector field on a manifold, or a canonical form for the equation. These strategies have led to different analytical frameworks, organized around different index notions which generalize the nilpotency index of a regular matrix pencil. These approaches are commented on below.

*The differentiation index*

The idea behind the *differentiation index* framework is, roughly speaking, to define the index of (1.2) as the number of differentiations needed to write $x'$ in terms of $(x, t)$. As illustrated in Sections 3.1 and 3.2, the constraints are differentiated in order to realize an explicit *underlying ODE* for which the solution set of the DAE is an invariant manifold. A more general discussion can be found in Section 3.7. When applied in particular to a linear time-invariant equation of the form (1.3) having a regular pencil, this notion yields the nilpotency index of the pencil.

The differentiation index approach has been developed mainly by S. Campbell; important contributions are also due to C. W. Gear and L. Petzold, among other authors. The reader is referred to the book [30] for a detailed introduction to this framework; see also [10, 42, 43, 45, 46, 54, 94].

*The tractability index and projector-based methods*

A different generalization of the nilpotency index of a regular matrix pencil comes from the *tractability index* notion, which is the key concept in the projector-based framework developed by R. März. Within this approach, the behavior of linear DAEs such as (1.3) or (1.4) is unveiled by means of an *inherent ODE* which, speaking again in general terms, is obtained from the projection of the equation and the problem variables into certain characteristic subspaces. Allowing for mild smoothness assumptions, this framework provides explicit solution characterizations for DAEs with arbitrarily high index, as well as precise functional input-output descriptions of the system behavior in the original problem setting.

The main references concerning projector techniques directed to linear time-varying problems of the form (1.4) are [107, 108, 179, 182, 185]; recently, a different formulation for DAEs, discussed in subsection 1.4.2 below, has allowed for a substantial improvement of this approach, regarding both linear and nonlinear problems; see [16, 188, 189, 191, 192, 190, 193] and references therein, as well as the forthcoming title [157]. Projector techniques for linear DAEs are extensively discussed in Chapter 2; Chapter 4 addresses singularities of these linear problems, whereas nodal models of electrical circuits are analyzed via projector-based methods in Chapter 5.

*The geometric index and reduction methods*

Reduction methods, based on the so-called *geometric index,* describe the behavior of an autonomous DAE in terms of a vector field defined on the solution set, provided that it has a manifold structure. This vector field defines a differential equation on this manifold and induces a flow on it. Using local parametrizations, the ODE on the solution manifold locally leads to an explicit *reduced ODE* on an open subset of $\mathbb{R}^r$; note that, in contrast to the underlying ODE arising in the differentiation index framework, the state dimension of this reduced ODE is strictly lower than that of the original DAE. In the original problem coordinates, the reduction process can be roughly described as the elimination of certain variables by solving the constraints. The solution manifold and the reduced ODE are computed in an iterative manner, and the name of iteration steps needed for the algorithm to stabilize defines the index.

The geometric reduction approach has a key reference in the paper [238] by W. Rheinboldt, and has been later developed by S. Reich [229–231] and in the joint research of P. Rabier and Rheinboldt [220, 221, 228]. Reduction methods for nonlinear problems are the focus of Chapter 3; special attention will be paid to quasilinear DAEs (cf. subsection 1.4.3 below), including nonautonomous cases. Their application to linear time-varying DAEs (1.4) is briefly addressed in Section 2.4. Singularities of quasilinear problems are tackled via reduction methods in Chapter 4 (Section 4.4).

Reduction techniques will be applied to the analysis of certain electrical circuit models in Chapter 6.

*Perturbation, strangeness and structural indices*

A salient feature of linear time-invariant DAEs (1.3) with an index $\nu \geq 2$ regular pencil is that solutions will depend explicitly on the derivatives of the excitation $q(t)$ up to order $\nu - 1$. This dependence, which is displayed (in a more subtle way) also by linear time-varying and nonlinear problems, supports the *perturbation index* concept [121], which reflects the sensitivity of the DAE solutions to perturbations. See also [45, 122].

The reader is referred to the book [151] for an extensive analysis of the *strangeness index* notion and related canonical forms for DAEs introduced by P. Kunkel and V. Mehrmann. Other references on this topic are [147–150, 152]; some relations with the tractability and geometric indices are discussed in [157] and [228], respectively.

Finally, different aspects of the so-called *structural index* are addressed in [215, 237].

The paragraphs above give an overview of the main frameworks developed for the analysis of DAEs. This book will be mainly focused on projector-based and reduction methods, supported on the tractability and geometric indices, respectively. In these contexts, when the assumptions supporting the index notion are met the DAE is called *regular*. The failing of these assumptions will lead to *singular* problems, presented below.

### 1.2.2 *Dynamics and singularities*

As sketched above, when the conditions supporting the tractability or geometric index notions are met, the behavior of the DAE (1.2) can be described in terms of a related (inherent or reduced, respectively) explicit ODE. For the moment, we will informally refer to these systems as *regular*. In spite of the mathematical challenges raised by the unveiling of such an explicit ODE, from a dynamical point of view a regular autonomous DAE does not lead to new qualitative phenomena. For instance, in problems with a well-defined geometric index the dynamical behavior of the DAE is given by the flow induced by a vector field on a lower-dimensional manifold. Locally, the solutions of a reduced ODE are embedded onto those of the DAE, and therefore the flows of both are locally diffeomorphic.

Things change substantially in the presence of *singularities*. Singular points of a DAE will be defined as those where the assumptions supporting the definition of an index fail, and indeed yield several types of dynamical behavior not displayed by explicit ODEs. The best-known phenomenon associated with singular DAEs is the one occurring at *impasse points* [61, 62, 219, 222, 223, 233]. Other singular phenomena in autonomous DAEs have been addressed only for particular structures, mainly quasilinear problems with a dense subset of regular points [165, 202, 203, 236, 254, 276, 277, 309] and semiexplicit DAEs [22, 23, 173, 236, 249]. Last-step singularities (cf. Chapter 4) were considered by Rabier and Rheinboldt in [228]. From a different perspective, several results concerning singularities of scalar problems have been discussed in [7, 34, 72, 289]. Singular bifurcations are tackled in [19–21, 166, 241–243, 246, 275, 296–298]. In the linear time-varying context singularities have been studied in [196, 197]; some results can also be found in [139, 225, 228].

A systematic approach to the local analysis of singular DAEs is presented in Chapter 4; no previous framework for the general study of singularities in differential-algebraic systems is known to the author. An outline of the main ideas follows. After providing invariant definitions of singular points in linear time-varying and quasilinear autonomous DAEs, the "regular" analysis frameworks will be modified in order to accommodate singular problems. In both contexts, this will result in an implicit ODE which can be rewritten in explicit form on an open dense subset of its domain, the leading term of which captures the singularities of the original DAE. In particular, this approach extends the scope of the results of Rabier and Rheinboldt for quasilinear systems. Broadly speaking, the working scenarios allowing for this replace constant rank assumptions by the requirement that certain characteristic spaces admit a continuation through the singularity, thereby relaxing the hypotheses which support the analysis in regular settings.

### *Qualitative properties of DAEs*

Qualitative results for DAEs are of major importance in different applications. For regular autonomous DAEs, even though the local dynamics amount to that of explicit ODEs, it is necessary to have tools allowing one to assess qualitative properties directly in the differential-algebraic setting, since an explicit state space description of the system dynamics may not be feasible in many practical cases. The qualitative theory of DAEs makes such an explicit ODE description unnecessary. In this regard it is important to examine, in DAE terms, stability properties of invariants such as equilibrium points or periodic trajectories, as well as the associated bifurcations characterizing the dependence of these properties on system parameters.

Although a general discussion of qualitative aspects of differential-algebraic systems would exceed the scope of this book, we will address some issues in this direction. Linear stability properties of regular equilibria in quasilinear DAEs will be characterized in terms of matrix pencils in Section 3.5; from different points of view, this type of results has been discussed in [92, 183, 184, 186, 228, 231, 240, 281, 291]. The reduction framework seems to be well-suited for characterizing the situations in which the matrix pencil which results from the DAE linearization describes the stability properties of equilibria. The electrical counterpart of these results is tackled in Section 6.3, where a key point will be to distinguish the topological conditions characterizing the hyperbolicity or exponential stability of equilibrium points from those allowing for the formulation of a state space model for the network, a distinction which is not always clear in the circuit literature.

Other qualitative aspects of differential-algebraic systems have been analyzed in the last two decades. Local normal forms are discussed in [21–23, 67, 68, 202, 236, 277, 309]. Regarding periodic DAEs and periodic solutions, several results concerning Floquet theory for DAEs can be found in [158, 159]; related aspects involving the analysis of nonlinear oscillations in semistate models of electrical circuits are addressed in [74]. Finally, the reader is referred to Chapter 4 and the bibliography cited there for the discussion of qualitative phenomena at singularities, in particular at singular equilibria and pseudoequilibria.

### 1.2.3 *Numerical aspects*

The numerical treatment of differential-algebraic systems has driven a great deal of research on this field. Most monographs on DAEs are partially (or, in some cases, totally) devoted to numerical methods: see [10, 30, 107, 121, 122, 151, 157, 228]. Essentially, all the frameworks discussed in subsection 1.2.1 have a numerical counterpart. This book will be focused on analytical properties of DAEs, and hence the reader is referred to the above-mentioned titles for detailed discussions of computational issues. Without any attempt to be exhaustive, other important references are [8, 38, 44, 47, 51, 52, 96, 129–133, 140, 141, 182, 216, 217, 267, 270].

Regarding the problem of computing consistent initial values, see [30, 33, 86, 97, 124, 126, 156, 162, 215] and the bibliography therein. Numerical techniques for the analysis of singular DAEs are discussed in [9, 199, 207, 218, 223, 228, 275, 304], among other references.

## 1.3 State vs. semistate modeling

The dichotomy between DAEs and explicit ODEs is also relevant from the point of view of system modeling. In fact, much attention has been directed to DAEs because of its chance to model easily systems in which certain constraints are imposed among the variables appearing within a given set of differential equations. This is very clear for instance in circuit theory, where time-domain models of electrical networks combine differential equations arising from capacitors and inductors with algebraic ones coming from Kirchhoff laws and the characteristics of devices (cf. Sections 5.2 and 6.1).

These constraints mean that there exist some *redundancy* among the model variables. For this reason differential-algebraic models are sometimes called *semistate models* [82, 210]; the idea is that, in contrast to *state space models* based on explicit ODEs of the form (1.1), where an initial value can be freely assigned to every component of the state variable $u$, in a DAE such as (1.2) an initial value can be specified only for some components of the vector $x$ for a solution to be well-defined. The dynamic degree of freedom of a given DAE, which is a local quantity, will be called the (local) state dimension of the problem. More details in this regard can be found in the reduction framework of Chapter 3.

From the modeling perspective, the elimination of redundant variables can be seen as a *model reduction* process. This includes, in particular, the derivation of a state space description of a given system by means of the reduction of an initial DAE model; in this direction, the state formulation problem for electrical circuits will be tackled in Chapter 6 as a reduction of certain semistate models. On the other hand, in many practical cases only some of the variables are eliminated from a given semistate equation, yielding another DAE which may be easier to analyze while still capturing the essential features of the system. Again, the electrical circuits discussed in Chapters 5 and 6 will provide several examples of this, since for instance augmented nodal analysis or multiport models can be seen as intermediate formulations between a tableau or a branch-oriented model, in which no variable has been eliminated and therefore redundancy is maximal, and a state space model, displaying no redundancy among variables.

It is important to notice that a reduction process leading to a state space model of a given system is not always feasible in practice. When this is the case, the thoroughly discussed mathematical tools for the analysis of dynamical systems formulated in terms of explicit ODEs become unavailable. These tools include, in particular, qualitative theory and numerical integration methods. In these cases the DAE-oriented techniques referred to in subsections 1.2.2 and 1.2.3 above, which apply without recourse to a state space reduction, become very relevant in system analysis and simulation.

Finally, a particularly important role in system modeling is played by so-called *semiexplicit* DAEs (cf. subsection 1.4.3 below), having the form

$$y' = h(y, z, t) \tag{1.5a}$$

$$0 = g(y, z, t). \tag{1.5b}$$

These systems arise in applications when algebraic (that is, non-differential) constraints are explicitly added to a set of differential relations, but also as the reduced equation of the *singular perturbation* problem

$$y' = h(y, z, t) \tag{1.6a}$$

$$\varepsilon z' = g(y, z, t) \tag{1.6b}$$

where, typically, $0 < \varepsilon \ll 1$. Conversely, the way from (1.5) to (1.6) is usually called a *regularization* of the DAE [10, 30, 142, 143, 214]. Semiexplicit DAEs also arise as the *enlarged* system

$$x' = p \tag{1.7a}$$

$$0 = F(t, x, p) \tag{1.7b}$$

of the fully implicit problem (1.2).

## 1.4 Formulations

When using DAEs to describe physical systems, several options arise in the formulation of the model. In particular, there are different ways to capture the variables which are required to be differentiated. If the derivatives of all the components of the semistate vector $x$ are involved in the model, as it happens in explicit ODEs, we are faced with a *standard form DAE*. Nevertheless, this presents certain disadvantages which are discussed, in terms of input-output system descriptions, in subsection 1.4.1 below. This will motivate the introduction of so-called *properly stated DAEs*, presented in subsection 1.4.2. Finally, in subsection 1.4.3 we discuss some structural forms for DAEs which are often displayed in applications.

### 1.4.1 *Input-output descriptions*

*Explicit ODEs*

Let us consider from a functional point of view the linear constant coefficient ODE

$$x' + Ex = q(t), \tag{1.8}$$

where $E$ is an $m \times m$ real matrix, together with an initial condition

$$x(t_0) = x_0 \in \mathbb{R}^m. \tag{1.9}$$

The initial value problem (IVP) defined by (1.8) and (1.9) is known to yield a *smoothing* operator

$$\begin{array}{rcl} C^k(\mathcal{J}, \mathbb{R}^m) & \longrightarrow & C^{k+1}(\mathcal{J}, \mathbb{R}^m) \\ q(t) & \longrightarrow & x(t), \end{array}$$

where $x(t)$ is the unique solution of the ODE (1.8) which satisfies (1.9), in the understanding that $t_0 \in \mathcal{J}$. This means that the initial value problem induces a mapping between the functional spaces $C^k(\mathcal{J}, \mathbb{R}^m)$ and $C^{k+1}(\mathcal{J}, \mathbb{R}^m)$, for any $k \geq 0$; this point of view is important for instance in system theory or in electrical engineering, where $q(t)$ and $x(t)$ are viewed as input and output signals of a system defined by the matrix $E$ and with initial state $x_0$.

Moreover, the ODE (1.8) alone defines a *bijection*

$$\begin{array}{rcl} C^k(\mathcal{J}, \mathbb{R}^m) \times \mathbb{R}^m & \longrightarrow & C^{k+1}(\mathcal{J}, \mathbb{R}^m) \\ (q(t), x_0) & \longrightarrow & x(t), \end{array}$$

with an inverse given by $x(t) \to (x'(t) + Ex(t),\ x(t_0))$. The bijective nature of this mapping somehow indicates that $C^k$-spaces are the right ones to accommodate excitations and solutions of the problem. Note that the same remarks apply if we replace the constant matrix $E$ in (1.8) by a time-varying one $E(t) \in C^k(\mathcal{J}, \mathbb{R}^{m \times m})$.

*Linear time-invariant DAEs*

In the differential-algebraic context one is often interested in obtaining similar functional descriptions of the solution behavior and, specially, in providing functional spaces where excitations and solutions are mapped into one another bijectively. As discussed below, and unlike the ODE case, neither the DAE formulations handled so far nor the use of $C^k$ spaces are appropriate in this regard.

Indeed, consider the linear time-invariant DAE (1.3), and assume that the matrix pencil $\{A, E\}$ is regular with Kronecker index one (cf. Section 2.1). It will shown that in this situation there exist nonsingular matrices $G$, $H$ such that the solutions of (1.3) can be obtained from those of

$$u' + Wu = \tilde{q}_1(t) \tag{1.10a}$$

$$v = \tilde{q}_2(t), \tag{1.10b}$$

for a certain matrix $W$, by means of the transformation $x = Hw$. Here the variable $w$ stands for $(u, v)$, whereas the excitation $\tilde{q}(t) = (\tilde{q}_1(t), \tilde{q}_2(t))$ in (1.10) is defined by $\tilde{q}(t) = Gq(t)$.

Let us assume that $q(t)$ (and thereby $\tilde{q}(t)$) is continuous. System (1.10) certainly has a solution $w(t) = (u(t), v(t))$, defined by a $C^1$-solution $u(t)$ of (1.10a) and a $C^0$-mapping $v(t)$ given by (1.10b). But if the excitation $q(t)$ (or, more precisely, $\tilde{q}_2(t)$) is not differentiable, then neither are $v(t)$, $w(t)$, nor $x(t) = Hw(t)$; in this situation it does not make sense to differentiate $x$ in (1.3). This raises the problem of the sense in which $x(t) = Hw(t)$ can be considered a solution of (1.3) in cases in which $q(t)$ is just a continuous map. We are additionally faced with the characterization of the functional spaces of (i) solutions onto which continuous excitations are mapped, and (ii) excitations yielding $C^1$-solutions.

We might agree to restrict the attention to $C^1$-excitations $q(t)$. This is however unsatisfactory, not only because continuous excitations seem to be acceptable in the light of (1.10), but also when looking for the space of solutions $x(t)$ onto which $C^1$-excitations are mapped.

This introductory digression shows that, for index one DAEs, there is no hope for a bijective transformation $C^k \to C^{k+1}$ or $C^k \to C^k$ of excitations onto solutions; the cases considered above correspond to $k = 0, 1$, but the same is true for $k \geq 2$. Note that for the sake of simplicity we are deliberately vague concerning initial values.

A solution can be devised by rewriting $Ax' = (Ax)'$ and then considering, instead of (1.3), the reformulation

$$(Ax)' + Ex = q(t), \tag{1.11}$$

which comprises all $C^1$-solutions of (1.3) but allows for the search of additional ones in the larger space

$$C_A^1(\mathcal{J}, \mathbb{R}^m) = \{x \in C^0(\mathcal{J}, \mathbb{R}^m) \ / \ Ax \in C^1(\mathcal{J}, \mathbb{R}^m)\}.$$

Continuous excitations $q(t)$ can now be shown to yield a solution in the space $C_A^1$, whereas $C^k$ excitations lead to solutions within

$$C_A^{k+1}(\mathcal{J}, \mathbb{R}^m) = \{x \in C^k(\mathcal{J}, \mathbb{R}^m) \ / \ Ax \in C^{k+1}(\mathcal{J}, \mathbb{R}^m)\}.$$

This type of reformulations will also lead to a characterization of the excitations which are bijectively mapped onto $C^k$-solutions.

The discussion above is restricted to linear time-invariant DAEs with nilpotency index one. Using the Kronecker canonical form (see Section 2.1), these remarks can be extended to linear time-invariant problems with higher index, under stronger smoothness requirements on $q(t)$. However, in the linear time-varying framework defined by (1.4), and also in nonlinear contexts, this kind of results are more involved. These issues can be tackled in time-varying and/or nonlinear settings by means of a recently proposed formulation for the leading term of DAEs, detailed below. The reader is referred to subsections 2.1.3.2 and 2.2.1 for additional details concerning input-output functional descriptions.

### 1.4.2 *Leading terms*

As indicated above, rewriting the leading term of (1.3) as $(Ax)'$ yields several advantages from the analytical point of view. The same can be achieved by recasting the leading term as $A(Px)'$, where $P$ is a projector along $\ker A$, based on the identity $A = AP$ (note that $A(I - P) = 0$).

The key remark at this point is that these ideas can be extended to the linear time-varying context. Indeed, several benefits will follow from the consideration of DAEs of the form

$$A(t)(D(t)x)' + B(t)x = q(t), \tag{1.12}$$

instead of (1.4); see the seminal work [16] and [157, 188, 189, 191, 192] together with the references therein. The leading term of (1.12) will be said to be *properly stated* when the matrix mappings $A(t)$ and $D(t)$ are well-matched in a certain sense (cf. Definition 2.2 on p. 38). The above-mentioned input-output functional descriptions are better tackled in the framework defined by (1.12), which allows for a precise formulation of smoothness requirements on the excitation $q(t)$ and the matrix mappings $A(t)$, $D(t)$ and $B(t)$. Additionally, the leading term of (1.12) arises in this form in different circuit and control applications and, in particular, displays nice symmetry properties when considering adjoint formulations [15, 16, 187]. Several advantages are also met from the numerical point of view [130–132, 157, 293].

Certainly, a lot of DAE literature is directed to the form (1.4). We will refer to these problems as *standard form* linear DAEs. Under mild requirements on the leading matrix $A(t)$, a standard form DAE can be rewritten in the properly stated form (1.12), and therefore the results coming from the properly stated framework can be applied to standard form problems; see, in this regard, Sections 2.3 and 4.3.

Properly stated formulations can be also used in nonlinear settings; some instances can be actually found in subsection 5.3.1. This involves, however, some additional complexities, and most readers will be unfamiliar with this type of formulation in the nonlinear context. Due to the intrinsic difficulties of nonlinear problems, in their analysis we will try to keep at a minimum certain technicalities. Therefore, the attention in Chapter 3 will be restricted to standard form DAEs, and precise smoothness requirements will be disregarded; that is, all objects will simply be assumed to be smooth enough, and the $C^\infty$ setting will be often invoked for that. In any case, we believe that the results of Chapter 2 should be enough to give a reader a solid understanding of the benefits of properly stated formulations, and we refer him/her to the forthcoming title [157] for an extensive discussion of nonlinear, properly stated DAEs.

### 1.4.3 *Semiexplicit, semilinear and quasilinear DAEs*

Many DAEs in applications display some kind of structure. In most practical cases the equation is linear in the derivative $x'$; depending on the specific form of the matrix map in front of this derivative, these *quasilinear* or *linearly implicit* systems have received several names in the literature. The semilinear and quasilinear problems here presented can be thought of as being located between linear DAEs and the "nonlinear" system (1.2). Note that these families are not mutually exclusive; quite on the contrary, linear DAEs are a particular case of semilinear problems (mind, in this regard, that semiexplicit problems may well be linear), which in turn are instances of quasilinear DAEs with a leading matrix independent of $x$. Similarly, the form (1.2) comprises quasilinear problems. Actually, referring to (1.2) as a 'nonlinear DAE' is a terminological abuse since it may stand in particular for linear cases. System (1.2) is often called a *fully implicit* DAE.

#### 1.4.3.1 *Semiexplicit and semilinear DAEs*

In an autonomous setting, *semiexplicit* DAEs are defined by a system of the form

$$y' = h(y, z) \tag{1.13a}$$

$$0 = g(y, z), \tag{1.13b}$$

where for the moment we disregard smoothness requirements on the maps $h : W_0 \to \mathbb{R}^r$ and $g : W_0 \to \mathbb{R}^p$. Here $W_0$ is an open set in $\mathbb{R}^{r+p}$. The form depicted in (1.13) makes it possible to distinguish the $r$-dimensional vector $y$ of *dynamic variables* from the $p$-dimensional one $z$ of *algebraic* ones. The nonautonomous counterpart of (1.13) is displayed in (1.5).

It is worth remarking at this point that the term 'differential-algebraic equation' is sometimes used in the literature to mean the semiexplicit equation (1.13), and even to refer to (1.13) under the assumption that the matrix of partial derivatives $g_z$ is everywhere nonsingular. The nonsingularity of $g_z$ will be later defined as an *index one* requirement.

Autonomous *semilinear* DAEs are problems of the form

$$Ax' = f(x), \tag{1.14}$$

where $A$ is an $m \times m$ real matrix and $f : W_0 \to \mathbb{R}^m$ is a sufficiently smooth mapping, $W_0$ being open in $\mathbb{R}^m$. Semilinear equations comprise in particular the semiexplicit DAE (1.13), since the latter can be written as

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} h(y, z) \\ g(y, z) \end{pmatrix}$$

which is a system of the form (1.14) with $x = (y, z)$, $f = (h, g)$. Conversely, semilinear DAEs may written in semiexplicit form: denoting $\operatorname{rk} A = r$, there exist nonsingular matrices $G,\ H$ such that [90]

$$GAH = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Now, if we premultiply (1.14) by $G$ and perform the coordinate change $x = H\tilde{x}$, the semilinear system reads

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \tilde{x}' = Gf(H\tilde{x})$$

which, via the splitting $\tilde{x} = (y, z)$, is easily checked to be a semiexplicit equation.

The semiexplicit DAE (1.13) can be written in properly stated form as

$$\begin{pmatrix} I_r \\ 0 \end{pmatrix} \left[ \begin{pmatrix} I_r & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \right]' = \begin{pmatrix} h(y,z) \\ g(y,z) \end{pmatrix}, \tag{1.15}$$

where

$$A = \begin{pmatrix} I_r \\ 0 \end{pmatrix}, \ D = \begin{pmatrix} I_r & 0 \end{pmatrix},$$

define a well-matched pair of matrices yielding a properly stated leading term; cf. subsection 2.2.2 for additional details in this regard.

Finally, by allowing the above-introduced operators $A$ and $f$ depend explicitly on the time $t$, we obtain the nonautonomous analog of (1.14), namely,

$$A(t)x' = f(x,t).$$

### 1.4.3.2 *Hessenberg DAEs*

In several applications, the mappings $h$ and $g$ within the semiexplicit DAE (1.5) depend only on certain variables. This is often the case in mechanics or, as will be detailed in Chapter 6, in electrical circuit theory. In this direction, we present below so-called *Hessenberg DAEs* directly in the nonautonomous context.

When the constraints $g$ do not depend on the algebraic variable $z$, we are led to a *Hessenberg DAE of size two*. These systems can be written as

$$y' = h(y,z,t) \tag{1.16a}$$

$$0 = g(y,t). \tag{1.16b}$$

For simplicity, we assume that $h : \mathbb{R}^{r+p+1} \to \mathbb{R}^r$ and $g : \mathbb{R}^{r+1} \to \mathbb{R}^p$, although the domains of these mapping may be assumed to be open sets $W_0 \subseteq \mathbb{R}^{r+p+1}$ and $\hat{W}_0 \subseteq \mathbb{R}^{r+1}$. Labeling (1.16) as a 'size two' Hessenberg DAE without additional requirements comprises a terminological abuse, since this term is usually reserved to systems of the form (1.16) in which the product $g_y h_z$ is nonsingular [30, 70]. Later on we will elaborate further on the nonsingularity of $g_y h_z$, which will define an *index two* condition for (1.16); note that this requires, in particular, that $r \geq p$.

If, additionally, in the Hessenberg DAE (1.16) the mapping $g$ does only depend on $t$ and some of the dynamic variables $y$ (to be denoted by $y_2 \in \mathbb{R}^{r_2}$, with $r_2 < r$), and the corresponding components of $h$ do not depend on $z$, we are led to a *Hessenberg DAE of size three*, which may be written as

$$y_1' = h_1(y_1, y_2, z, t) \tag{1.17a}$$

$$y_2' = h_2(y_1, y_2, t) \tag{1.17b}$$

$$0 = g(y_2, t). \tag{1.17c}$$

Analogously, a *Hessenberg DAE of size* $k$ has the structure

$$y_1' = h_1(y_1, y_2, \ldots, y_{k-1}, z, t) \tag{1.18a}$$

$$y_i' = h_i(y_{i-1}, \ldots, y_{k-1}, t), \qquad 2 \leq i \leq k-1 \tag{1.18b}$$

$$0 = g(y_{k-1}, t), \tag{1.18c}$$

where $y_i \in \mathbb{R}^{r_i}$ for $i = 1, \ldots, k-1$, $z \in \mathbb{R}^p$. This DAE will be index $k$ if the product

$$\frac{\partial g}{\partial y_{k-1}} \frac{\partial h_{k-1}}{\partial y_{k-2}} \cdots \frac{\partial h_2}{\partial y_1} \frac{\partial h_1}{\partial z} \tag{1.19}$$

is nonsingular. This implies that $r_i \geq p$ for $i = 1, \ldots, k-1$. The above-mentioned terminological abuse applies also here, since the nonsingularity of (1.19) is usually required in order to call (1.18) a 'size $k$' Hessenberg DAE; this holds in particular for the case $k = 3$ displayed in (1.17).

The term *Hessenberg DAE* stands for a DAE which has a Hessenberg structure of size $k$ for some $k \geq 2$. Sometimes the semiexplicit DAE (1.13) itself is also considered as a Hessenberg DAE [10].

#### 1.4.3.3 *Quasilinear DAEs*

Finally, *quasilinear* autonomous DAEs are systems of the form

$$A(x)x' = f(x), \tag{1.20}$$

where $A(x)$ and $f(x)$ are sufficiently smooth mappings $W_0 \to \mathbb{R}^{m \times m}$ and $W_0 \to \mathbb{R}^m$, respectively; the set $W_0$ is open in $\mathbb{R}^m$. The pair $(A, f)$ is sometimes called a *generalized vector field* on $W_0$ [67, 68, 202, 203].

Most cases of interest in applications will be defined by an everywhere singular matrix mapping $A(x)$. However, in the reduction framework of Chapter 3 an important role will be played by problems of the form (1.20) in which $A(x)$ is singular only on a codimension-one submanifold of $W_0$. These cases have received considerable attention in the literature [165, 166, 202, 203, 236, 240, 254, 276, 277, 309]; we will reserve the term *quasilinear ODE* to refer to problems of the form (1.20) in which $A(x)$ is nonsingular on a dense subset of $W_0$.

The nonautonomous counterpart of a quasilinear DAE is defined by the assumption that $A$ and/or $f$ depend on $t$, yielding an equation of the form

$$A(x,t)x' = f(x,t).$$

These systems arise for instance when setting up electrical circuit models using Modified Nodal Analysis (MNA); cf. Chapter 5.

## 1.5 Contents and structure of the book

With the background presented in previous Sections, we are now in a position to define the goals of the present book, and also to provide a more detailed description of its contents and structure. Broadly speaking, the main goal is to present several frameworks for the analysis of differential-algebraic systems and, subsidiarily, of semistate circuit models. Thus, the two parts of the book are respectively devoted to the discussion of analytical aspects of DAEs in general, and to different issues arising in the use of DAEs in electrical circuit modeling.

In Part I, we discuss in detail several analysis methods for DAEs. The main focus is on projector-based techniques and reduction methods. Both approaches certainly apply to linear time-varying and nonlinear DAEs; nevertheless, a detailed discussion of both frameworks in these two settings would be excessively long. For this reason, projector methods will be discussed in the context of linear time-varying systems, whereas reduction techniques will be addressed for nonlinear DAEs; special emphasis will be put on quasilinear problems. However, reduction techniques for linear problems are briefly presented in Section 2.4, whereas some remarks on the use of projector methods for the analysis of quasilinear DAEs arising in circuit modeling can be found in Section 5.3. More details can be found in [228] and [157], respectively.

Therefore, Chapters 2 and 3 address linear and nonlinear DAEs via projector-based and reduction techniques, respectively. Undoubtedly, the reader will find a certain gap between both approaches, and also with respect to the differentiation index framework (cf. Section 3.7) but, from the author's point of view, the different perspectives should provide a richer knowledge of the fundamentals of DAE analysis. In spite of the salient differences between the projector and reduction frameworks, the main ideas somehow go in parallel. In both cases we examine in detail the assumptions supporting the (tractability and geometric) index notions in which the methods are supported, arriving at two results (Theorems 2.3 and 3.2 on pages 51 and 107, respectively) which characterize the solutions of DAEs in regular contexts. The failing of these assumptions will lead to the analysis of singularities carried out in Chapter 4.

Part II tackles different analytical issues in electrical circuit theory using semistate (differential-algebraic) models. The use of time-domain formulations make the results applicable both to linear and nonlinear problems. Two model families are considered; those based on nodal analysis are considered in Chapter 5, whereas branch-oriented models define the scope of Chapter 6. In both cases, the focus is placed on index analyses.

As detailed in Chapter 5, nodal models have attracted quite a lot of attention in the DAE literature, specially regarding Modified Nodal Analysis (MNA) systems, widely used in circuit simulation programs. The tractability index and projector methods have succeeded in providing an accurate characterization of the index of these models in passive contexts; we will discuss these results and present as well several index characterizations for non-passive problems by means of tree-based techniques.

The branch-oriented models in Chapter 6 have been comparatively overlooked from the DAE point of view. We attempt to illustrate that the differential-algebraic formalism is certainly of interest also in the analysis of these circuit models. In particular, the geometric index framework makes it possible to recast the state formulation problem and the normal tree method as a reduction of a branch-oriented circuit model. Several advantages, regarding for instance qualitative properties of lumped circuits, will be derived from this approach.

We aim to discuss all these results in a self-contained manner, by means of the inclusion of background material on different topics. From a mathematical point of view, we compile some basic results coming from linear algebra (involving matrix pencils, projectors or Schur complements, as well as specific results such as the Cauchy-Binet formula), differential geometry and graph theory, which virtually reduce the technical prerequisites for reading this book to basic courses on differential calculus, linear algebra and ordinary differential equations.

Regarding circuit theory, Section 5.1 presents a detailed introduction to the fundamentals of electrical circuit analysis. Subsection 5.1.1 may be useful for readers interested in getting an introduction to elementary aspects of graph theory, regardless of specific applications. The material in Section 5.1, together with that of Sections 5.2 and 6.1, should allow readers without a background in circuit analysis to profit from the results discussed in Chapters 5 and 6.

*How to read this monograph*

Some readers will be mainly interested on certain parts of this book. There are several ways to go through the material here presented; some of them are proposed below.

Readers who aim to get an introductory vision on DAEs, or maybe trying to get a glimpse at the topic before going into details, should probably read this Chapter and then Sections 2.1.1, 2.1.2, 3.1, 3.2 and 4.1, before proceeding further.

Mathematically-oriented readers without a substantial background on DAEs may find of interest, from an analytical point of view, Sections 2.2 and 2.3 for linear time-varying DAEs, Sections 3.4 and 3.5 for quasilinear problems, and Chapter 4 for the study of singularities in both contexts.

Expert readers interested on specific approaches will find projector methods and the tractability index in Chapter 2, focused on linear systems. Sections 4.2 and 4.3 adapt the projector-based framework to singular linear time-varying problems, whereas some ideas concerning the tractability index of quasilinear DAEs arising in circuit theory can be found in Section 5.3. Reduction methods and the geometric index are discussed for linear problems in Section 2.4 and for nonlinear systems in Chapter 3; singularities of quasilinear DAEs are then addressed in Section 4.4. The differentiation index notion is presented in Section 3.7, after some introductory remarks in Sections 3.1 and 3.2.

Finally, readers interested on applications of DAEs will find a thorough discussion of differential-algebraic circuit models in Chapters 5 and 6, many parts of which can be read independently of the rest of the book.