# Chapter 1

# Basic Notions

## 1.1 Relativistic quantum field theory

### 1.1.1 *Quantum fields*

Aim of *Quantum Field Theory* (QFT) is to give a description of particles in agreement with the principles of Quantum Mechanics and Special Relativity (see Refs.[1],[2],[3] for complete expositions). A first attempt to get a coerent description of relativistic quantum particles was done by Dirac, who proposed an equation for the wave function of a relativistic fermion, the Dirac equation (with the convention $\hbar = c = 1$)

$$(i\bar{\gamma}_\mu \partial_\mu + m)\psi = 0 \tag{1.1}$$

where $x_\mu = (x_0, \vec{x})$, $\psi$ is a four dimensional spinor and $\bar{\gamma}_\mu$ are $4 \times 4$ matrices such that $\{\bar{\gamma}^\mu, \bar{\gamma}^\nu\} = 2g^{\mu,\nu}$, where $g^{\mu,\nu} = 0$ if $\mu \neq \nu$ and $1 = \bar{g}^{0,0} = -\bar{g}^{1,1} = -\bar{g}^{2,2} = -\bar{g}^{3,3}$.

The Dirac equation admits solutions with arbitrary negative energy so that apparently nothing prevent the particles to loose their energy indefinitely, for instance by interaction with the electromagnetic fields. This problem was solved by recasting the relativistic quantum mechanics as a many body problem, and assuming that the infinite states with negative energy levels are filled up in the vacuum state; according to Pauli exclusion principle, fermions cannot occupy the already filled states so that stability of states with positive energy is ensured. Moreover a photon can give a positive energy to a fermion in the sea, leaving an "hole" which appear as a fermion with opposite charge, a positron in the case of an electron, which was later one experimentally observed. The presence of infinities, the fact that the number of quantum particles can be created or annihilated and the analogy between the two apparently unrelated areas of relativistic quantum physics and quantum many body theory are features of the Dirac analysis which remained in all the subsequent developments.

As a single particle description appears to be not suitable for quantum relativistic particle for the possibility of creation or annihilation of pairs, one has to adopt a description in terms of *quantum fields.* They are operators acting on the Hilbert space of physical states and verifing a number of properties, see Ref.[4], among which is the *relativistic covariance* and the *microcausality*, which means that the fields must commute or anticommute if their support is space-like separated. If the fields are commuting they describe quantum particles called *bosons*, while if they are anticommuting they describe *fermions.* Fields with integer spins are bosons, and fields with semi-integer spins are fermions; this is the content of the *spin-statistics* theorem.

Fields describing *free* quantum particles can be explicitely constructed; important examples are the Dirac field, describing a relativistic fermion with spin 1/2, or the scalar field, describing a spin zero boson particle. The expectation values of time ordered product of field operators, called *Green functions*, can be exactly computed; they are expressed in terms of sum of products of the 2-point functions, a property called Wick rule. For instance the 2-point function for the Dirac field is given by

$$g(\vec{x}, x_0) = -i \int dk_0 d\vec{k} e^{ik_\mu x_\mu} \frac{\bar{\gamma}_\mu k_\mu - m}{-\vec{k}^2 + k_0^2 - m^2} \tag{1.2}$$

The properties of non-interacting quantum fields are then essentially understood. On the other hand, one is mostly interested in what happens when particles interact, that is to *interacting fields*; however, except in very rare situation, their are impossible to construct explicitely.

The *Reconstruction theorem* says that, if we have a set of functions veryfing a certain number of properties called *Wightman axioms*, such functions *completely define* a relativistic quantum field theory, in the sense that they are the expectation values of relativistic quantum fields. It turns out that it is much more convenient, from a mathematical point of view, to pass to immaginary times with the replacement $x_0 \to ix_0$ (and $k_0 \to -ik_0$). The $n$-point functions with imaginary time are called Euclidean Green functions or *Schwinger functions*, and they can be also used to completely reconstruct a Quantum Field Theory in real time, if they verify a set of properties called *Osterwalder-Schrader* axioms [5].

The rule of the game are then essentially given; one has to find a set of Schwinger functions verifying the Osterwalder-Schrader axioms, and from them a QFT is completely determined. Of course one has, among all the possible QFT models, to find the ones really describing the interaction of quantum particles in the physical word; in the same way that in classical physics one can study the motion of a particles with a generic force, but of course only specific type of forces are realized in nature.

There is no indications in the axioms how to compute the expectations. What physicists have done is to try to get correlations verifying the axioms starting from suitable functional integrals; this procedure has been very successful, as the prediction obtained in this way have been verified with the maximum possible precision in the experiments, but of course other ways of deriving a set of Schwinger functions may be in principle possible.

### 1.1.2 *Functional integrals*

QFT models can be constructed starting from functional integrals, by generalizing the minimal action principle in classical mechanics. Such principle, due to the Hamilton, says that the trajectory of a particle follows the path minimizing a functional called the *action*, at least for small time intervals.

An extension of this principle holds also in quantum mechanics, and it is particularly suggestive; it says that the transition probability from one state to another is found summing over all the possible trajectories or paths, each of them weighted by a phase factor $e^{iS}$, where $S$ is the action. This principle can be still generalized to quantum fields, either bosonic or fermionic, considering functional integrals over all the possible field configurations weighted by $e^{iS}$ with $S = \int dx_0 d\vec{x}\mathcal{L}$, where $\mathcal{L}$ is the lagrangian density.

To give a mathematical meaning to such functional integrals is extremely difficult. One can expect a great semplification considering imaginary times with the replacement $x_0 \to ix_0$, so that one has to integrate over all field configurations weighted by the exponential $e^{-S}$ instead that with the oscillating factor $e^{iS}$. To have an idea of the simplification one can think to the integral $\int_{-\infty}^{\infty} dx e^{ix^2}$ with respect to $\int_{-\infty}^{\infty} dx e^{-x^2}$.

Historically among the first QFT which were considered is *Quantum Electrodynamics* in four dimensions (QED4), describing the interaction of a fermionic particle, like an electron, with a quantized electromagnetic field. The gauge invariance of classical Electrodynamics leads to well known difficulties in its quantization, and the standard method to overcome them consists to add to the action a gauge-fixing term. The Schwinger functions of QED can be written, if $\mathbf{x} = (x_0, \vec{x})$

$$< \psi_{\mathbf{x}_1}; ...; \psi_{\mathbf{x}_n}; A_{\mu_1,\mathbf{y}_1}; ...; A_{\mu_1,\mathbf{y}_1} >= \frac{\partial^n}{\partial\phi_{\mathbf{x}_1}...\partial\phi_{\mathbf{x}_n}} \frac{\partial^m}{\partial J_{\mathbf{y}_1}...\partial J_{\mathbf{y}_m}} \mathcal{W}(\phi, J)|_{\phi=0} \tag{1.3}$$

where $\mathcal{W}$ is the *formal* functional integral

$$\mathcal{W} = \log \int \mathcal{D}\psi\mathcal{D}A e^{-\int d\mathbf{x}\mathcal{L}(A,\psi)+\int d\mathbf{x}(\phi_\mathbf{x}\bar{\psi}_\mathbf{x}+\psi_\mathbf{x}\bar{\phi}_\mathbf{x}+J_\mu A_\mu)}$$
$$\mathcal{L}(A,\psi) = \frac{1}{2}F_{\mu,\nu}F_{\mu,\nu} + \frac{1}{2}(\partial_\mu A_\mu)^2 + \bar{\psi}_\mathbf{x}(\not\partial + m + e\gamma_\mu A_\mu)\psi_\mathbf{x} \tag{1.4}$$

and $/\partial = \gamma_\mu\partial_\mu$, $\gamma_\mu$ are euclidean $\gamma$-matrices $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu}$, $e$ is the electric charge, $m$ is the electron mass, $F_{\mu,\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $\frac{1}{2}(\partial_\mu A_\mu)^2$ is the gauge fixing term.

We will discuss the exact meaning of an expression like (1.4) in the following section; for the moment, we just say that $A_\mu$ is a gaussian variables describing the photon and $\psi, \bar{\psi}$ are spinor Grasmann variables describing the fermions. What it is important to stress at this point is that there are well defined rules allowing to express the Schwinger functions as a power series in $e$, whose $n-th$ order is given by the sum of a large number of terms admitting a graphical representation in terms of *Feynman graphs.* To a graph is associated, according to well defined rules, a value which is expressed by integrals over all momenta of products of the fermionic Euclidean propagator

$$g(\mathbf{k}) = \frac{\not\mathbf{k} + im}{\mathbf{k}^2 + m^2} \tag{1.5}$$

and the photon propagators $v_{\mu,\nu}(\mathbf{k}) = \delta_{\mu,\nu}\mathbf{k}^{-2}$.

The Schwinger functions are related to physical observables (for instance one can compute from them the cross sections, which can be measured in accelerators); as there well defined rules to write the Schwinger functions as series, it is natural starting by truncating the perturbative series at some order (hoping that the contributions of the other orders is somewhat negligible), obtaining certain numbers and comparing them with the experiments.

However life is not so easy; while reasonable expressions are found at the very first orders in the series expansion, in going to higher orders one encounters Feynman graphs which are expressed by diverging integrals; this is the famous "problems of the infinities" in QFT. A closer look to the divergences of the integrals reveals that they are of two different types, called *ultraviolet* and *infrared.* The first are related to the fact that the integrands of the Feynman graphs do not decay fast enough at large monenta, while the second are due to the divergence of the integrand at zero momenta, and are present only with massless particles. The infrared divergences are connected to the low energy properties of the theory, while the infrared divergences are related to the high energy behaviour.

Even if the presence of the infinities seems to say that the above functional integrals are simply meaningless, we can forget for a moment such a problem and try instead to manipulate (1.4) *as* it would be a meaningful object. We can perform the *phase transformation*

$$\psi_{\mathbf{x}} \to e^{i\alpha_{\mathbf{x}}}\psi_{\mathbf{x}} \qquad \bar{\psi}_{\mathbf{x}} \to \bar{\psi}_{\mathbf{x}} e^{-i\alpha_{\mathbf{x}}} \tag{1.6}$$

so (1.12) becomes, assuming that the Jacobian of the transformation is 1

$$\mathcal{W} = \log \int \mathcal{D}\psi \mathcal{D}A e^{-\int d\mathbf{x}\mathcal{L}(A,\psi)}$$
$$e^{-\int d\mathbf{x}\alpha(\mathbf{x})\partial_\mu(\bar{\psi}_{\mathbf{x}}\gamma_\mu\psi_{\mathbf{x}})+\int d\mathbf{x}(e^{i\alpha_{\mathbf{x}}}\phi_{\mathbf{x}}\bar{\psi}_{\mathbf{x}}+e^{-i\alpha_{\mathbf{x}}}\psi_{\mathbf{x}}\bar{\phi}_{\mathbf{x}}+J_\mu A_\mu)} \tag{1.7}$$

and making a derivative with respect to $\alpha_{\mathbf{x}}, \phi_{\mathbf{y}}, \phi_{\mathbf{z}}$ one finds, if $j_{\mathbf{x},\mu} = \bar{\psi}_{\mathbf{x}}\gamma_\mu\psi_{\mathbf{x}}$

$$\partial_\mu\langle j_{\mathbf{x},\mu}\bar{\psi}_{\mathbf{y}}\psi_{\mathbf{z}}\rangle = \delta(\mathbf{x}-\mathbf{y})\langle\psi_{\mathbf{x}}\bar{\psi}_{\mathbf{z}}\rangle - \delta(\mathbf{x}-\mathbf{z})\langle\psi_{\mathbf{y}}\bar{\psi}_{\mathbf{z}}\rangle \tag{1.8}$$

or equivalently

$$i\mathbf{p}_\mu\langle A_{\mu,\mathbf{p}}\psi_{\mathbf{k}}\bar{\psi}_{\mathbf{k}-\mathbf{p}}\rangle = ev(\mathbf{p})[\langle\psi_{\mathbf{k}-\mathbf{p}}\bar{\psi}_{\mathbf{k}-\mathbf{p}}\rangle - \langle\psi_{\mathbf{k}}\bar{\psi}_{\mathbf{k}}\rangle] \tag{1.9}$$

and similar ones with any number of fields. This means that the Schwinger functions are not indipendent one from the other but are related by an infinite set of identities, called *Ward Identities* (WI); such relations somewhat replace the gauge invariance of the classical theory. The WI can be also obtained by Feynman graph expansions, from the relation (in the massless case)

$$g(\mathbf{k}-\mathbf{p}) - g(\mathbf{k}) = \not{\mathbf{p}}g(\mathbf{k})g(\mathbf{k}+\mathbf{p}) \tag{1.10}$$

obvious consequence of $g(\mathbf{k}) = (i\not{\mathbf{k}})^{-1}$. The WI (1.8) can be also derived from the conservation of the current $\partial_\mu j_{\mathbf{x},\mu} = 0$. Of course the Ward Identities (1.8), (1.9) are only formal as both the l.h.s.and the r.h.s. are infinite.

Other Ward Identities can be obtained, in the massless case $m = 0$, by the *chiral transformation* $\psi_{\mathbf{x}} \to e^{i\alpha\gamma_5}\psi_{\mathbf{x}}, \bar{\psi}_{\mathbf{x}} \to e^{-i\alpha\gamma_5}\bar{\psi}_{\mathbf{x}}$; it is found, if $j^5_{\mathbf{x},\mu} = \bar{\psi}_{\mathbf{x}}\gamma_\mu\psi_{\mathbf{x}}$

$$\partial_\mu\langle j^5_{\mathbf{x},\mu}\bar{\psi}_{\mathbf{y}}\psi_{\mathbf{z}}\rangle = \delta(\mathbf{x}-\mathbf{y})\gamma_5\langle\psi_{\mathbf{x}}\bar{\psi}_{\mathbf{z}}\rangle - \delta(\mathbf{x}-\mathbf{z})\gamma_5\langle\psi_{\mathbf{y}}\bar{\psi}_{\mathbf{z}}\rangle \tag{1.11}$$

The WI (1.11) can be also derived from the equation of conservation of the axial current $\partial_\mu j^5_{\mathbf{x},\mu} = 0$.

Several other QFT models have been introduced in addition to QED,considering also interactions involving only boson or fermions. An important model is the *Nambu-Jona Lasinio* model, known as *Thirring model* in $d = 2$, describing the self interaction of a massive fermion via a current-current interaction; the Schwinger functions are obtained by the derivatives of the following functional integral

$$\mathcal{W} = \log \int \mathcal{D}\psi e^{-\int d\mathbf{x}\mathcal{L}(\psi)+\int d\mathbf{x}(\phi_{\mathbf{x}}\bar{\psi}_{\mathbf{x}}+\psi_{\mathbf{x}}\phi_{\mathbf{x}})}|_{\phi=0} \tag{1.12}$$

with Lagrangian

$$\mathcal{L}(\psi) = \bar{\psi}_{\mathbf{x}}(\gamma_\mu\partial_\mu + m)\psi_{\mathbf{x}} + \lambda(\bar{\psi}_{\mathbf{x}}\gamma_\mu\psi_{\mathbf{x}})(\bar{\psi}_{\mathbf{x}}\gamma_\mu\psi_{\mathbf{x}}) \tag{1.13}$$

Also for such models the formal WI (1.8), (1.11) are valid. A variant of the model consist to add a colour index to the fermions, $\psi_{\mathbf{x},i}$, and $i = 1, .., N$; in such a case the model is called *Gross-Neveu model*.

Of course in real applications the dimension has to be 4, but it can be convenient consider QFT at lower dimensions, in which the analytical difficulties are much simpler and one can learn informations on general properties.

### 1.1.3 *Perturbative renormalization*

Let us return to the problem of infinities in the Feynman graph expansion of the functional integrals, which makes the theory apparently meaningless. The infrared divergences can be avoided assuming that all the particles have a mass; this is a good starting point, despite in nature neutrinos or photons have no mass and in any case the mass of electrons is very small with respect to the other quantities, in adimensional units.

The idea of Tomonaga,Schwinger and Feynman was to try to absorb the "ultraviolet divergences" in the parameters appearing in the lagrangian $\mathcal{L}$, according to a procedure called *renormalization*. The physical idea behind such procedure is that the parameters appearing in the Lagrangian (called *bare* parameters), like the electron mass or charge, are not the ones really observed, the *dressed* or physical parameters; in QED, for instance, their values are deeply modified by the interaction with the electromagnetic field.

Hence the Schwinger functions, which are written initially as functions of the bare parameters, must be re-expressed in terms of the dressed ones in order to compare them with real experiments. One has to consider the bare QED Lagrangian

$$\mathcal{L}_B = \frac{1}{2}Z_3 F_{\mu,\nu}F_{\mu,\nu} + Z_2\bar{\psi}_{\mathbf{x}}\gamma_\mu\partial_\mu\psi_{\mathbf{x}} + Z_4 m\bar{\psi}_{\mathbf{x}}\psi_{\mathbf{x}} + eZ_1\bar{\psi}_{\mathbf{x}}\gamma_\mu\partial_\mu\psi_{\mathbf{x}}A_\mu \tag{1.14}$$

where $Z_2$ and $Z_3$ are the fermionic and bosonic bare wave function normalization,

$$e\frac{Z_1}{Z_2\sqrt{Z_3}} = e_0 \tag{1.15}$$

is the bare electric charge and $mZ_4$ the bare mass.

The idea is then to choose the bare parameters so that the dressed ones have the observed values, that is the charge is $e$, the mass $m$ and the normalizations are equal to 1; in order to accomplish this goal it is convenient to write $\mathcal{L}_B$ as $\mathcal{L} + \delta\mathcal{L}$, with $\mathcal{L}$ was given by (1.4) and

$$\delta\mathcal{L}(A,\psi) = \frac{1}{2}\delta z_3 F_{\mu,\nu}F_{\mu,\nu} + \delta z_2 \bar\psi_{\mathbf{x}}\gamma_\mu\partial_\mu\psi_{\mathbf{x}} + \delta z_4 m\bar\psi_{\mathbf{x}}\psi_{\mathbf{x}} + e\delta z_1\bar\psi_{\mathbf{x}}\gamma_\mu\partial_\mu\psi_{\mathbf{x}}A_\mu \tag{1.16}$$

where $Z_3 = \sqrt{(1+\delta z_3)}$, $Z_2 = \sqrt{(1+\delta z_2)}$, $Z_1 = (1+\delta z_1)$ and $Z_4 = 1+\delta z_4$; the $\delta z_i$ are called counterterms. One considers then the (formal) generating function

$$\log \int \mathcal{D}\psi\mathcal{D}A e^{-\int d\mathbf{x}(\mathcal{L}+\delta\mathcal{L})+\int d\mathbf{x}(\phi_{\mathbf{x}}\bar\psi_{\mathbf{x}}+\psi_{\mathbf{x}}\bar\phi_{\mathbf{x}}+J_\mu A_\mu)} \tag{1.17}$$

which can be still written as sum of Feynman graphs, containing diverging expressions; in order to manage such divergences the Feynamnn graphs need to be *regularized*. There are several possible regularizations, each one with their proper advantages and disadvantages; at a purely perturbative level, the best regularizations are the one preserving the symmetries of the classical theory (and respecting (1.10)), like the *dimensional regularization*. One writes the *counterterterms* $\delta z_i$ as power series in $e$

$$\delta z_i(\Lambda) = e\delta z_{i,1}(\Lambda) + e^2\delta z_{i,2}(\Lambda) + ... \tag{1.18}$$

where $\Lambda$ is a parameter depending on the regularization, and try to choose the coefficients $\delta z_i(\Lambda)$ so the Schwinger functions are expressed by power series (in $e$) which are order by order *finite* removing the regularization ($\Lambda \to \infty$): this means that the infinities in the expansion are exactly compensated by the counterterms and that the resulting theory remain order by order finite when the regularization is removed.

Of course it is not obvious at all that such a procedure is successfull; there is apparently no reason *a priori* for which by adding a $\delta\mathcal{L}$ of the form (1.16) (that is exactly of the form of $\mathcal{L}$) one can exactly compensate all the infinities arising in the graph expansion. Indeed this can be checked explicitely at lowest orders but it was only through a great analytical effort that it was proved that, by choosing properly the $\delta z_i$ (diverging as $\Lambda \to \infty$), the theory is finite *at any order*. This property is called *perturbative* *renormalizability* is it is verified only by a few QFT models; it was proved for QED by Weinberg, Bogolubov, Zimmermann, Hepp and others.

At the end, it turns out that the *renormalized* Schwinger functions can be written as an expansion in $e$ order by order finite; all the infinities are absorbed in bare parameters which are of course diverging removing the cut-offs, but this is not important as they are by definition unobservable. On the contrary, the physical quantities, expressed in terms of the dressed parameters, are in excellent agreement with experiments.

Moreover, by using regularizations respecting (1.10), like the dimensional one, the Ward Identities (1.8) are valid for the *renormalized* Schwinger functions, as identities valid order by order in the perturbative expansion; one says that the WI are *preserved* by the renormalization procedure. The validity of the WI implies that the counterterms are not indipendent one from the other; for instance it holds the identity

$$Z_2 = Z_1 \tag{1.19}$$

as order by order statement. It says that $e_0 = eZ_3^{\frac{-1}{2}}$, that is the renormalization of the charge depends only from the photon renormalization.

While the WI based on the total phase transformation are preserved by the renormalization, quite surprisingly the WI based on a chiral transformation are instead *not preseverd*; the analogue equation for the axial current (1.11) is not true in the renormalized theory but it appears an extra term

$$\frac{\alpha}{4\pi}\langle F_{\mu,\nu}F_{\mu,\nu}; \bar{\psi}_{\mathbf{y}}\psi_{\mathbf{z}}\rangle \tag{1.20}$$

where $\alpha = e^2/4\pi$. The presence of the additional term in the Ward Identity is called a *quantum anomaly.* Note also that the anomaly coefficient is linear in $\alpha$, that is there are no higher orders corrections; this property is called *anomaly renormalization* or Adler-Bardeen theorem, and it is again a perturbative order by order statement. In QED2, anomaly also appears which are linear in the charge.

As we said, a peculiarity of $QED4$ is that the counterterms one needs to eliminate the singularities are exactly of the form of the parameters appearing in the original lagrangian; this property is true only for a small number of models, said *renormalizable.* For instance it is not true in the Gross Neveu or Thirring model in $d = 4$, while it is true if such models are considered in $d = 2$. Some models are indeed *superrinormalizable*; there is no need of counterterms (there are no divergences) or the divergence are only in a a finite number of Feynman graphs so that the series in (1.18) is just a finite sum; this is the case of QED2.

In addition to QED, the Weak and the Strong interactions (responsible respectively, for instance, of the decay of a neutron and of the formation of atomic nuclei) have been described in terms on renormalizable QFT models in the celebrated *Standard Model.* Actually such a theory provides an understanding of all the fundamental forces in nature (except gravity) and it has been tested in experiments with a precision never reached by any other physical theory. It provides an explanation of an enormous range of phenomena, from the properties of sub-atomic particles to the properties of stars. It is however important to recall that such computations are purely *perturbative*, that is obtained by arbitrary truncation series expansions, whose convergence cannot be proved (most of them are probably not convergent at all); hence there is no mathematical proof that a consistent QFT corresponding to such theory really exists.

A different point of view on renormalization, mainly due to Wilson Ref.[6], has also emerged in more recent years, and it is know as Renormalization Group or effective action approach, see Refs.[7],[8]. One introduces an ultraviolet momentum cut-off at some large momentum scale, that is the energies greater than some value $\Lambda$ are forbidden. One then starts from an action at scale $\Lambda$ and integrate iteratively the fields of decreasing energy scale, obtaining a sequence of effective actions. The bare parameters appearing in the action at the scale $\Lambda$ are chosen so that the effective action at energy corresponding to our experiments have the correct values. This method has some disadvantage, as it is more complex to prove the validity of the WI in the renormalized theory; for finite values of the cut-offs, the momentum regularizations violate (1.10) and produce additional terms in the WI, so that one has to show that WI are finally restored removing the cut-offs.

On the other hand, this approach has the advantage to be suitable in principle for going beyond a purely perturbative approach, see Refs.[3], [9],[10],[11], and indeed using it the existence of several nontrivial QFT models, mainly in $d = 2$, can be rigorously established at a non-perturbative level, as we will see in the first part of this book. While such models are still far from the realistic ones, they are important to show that QFT based on functional integrals and the renormalization procedure can really provide a coherent mathematical understanding of all fundamental interactions (except possibly gravity).

## 1.2 Classical statistical mechanics

### 1.2.1 *Phase transitions*

The aim of statistical mechanics is to compute the macroscopic properties of systems composed by a huge number of atoms or molecules, given only a knowledge of the microscopic forces between the components (for detailed expositions, see Refs.[12],[13]). More exactly, starting from the hamiltonian of a system (taking into account of all the interaction between atoms or molecules) one can compute, according to the postulates of statistical physics, thermodinamic quantities like the pressure or the entropy. Typical phenomena one is interested in statistical mechanics are the *phase transitions*; a well known example in everyday life is for intance the transition from water to vapour. Another important example of phase transition, somewhat simpler to study, happens when a magnetic field is applied to a magnet; at temperatures lower than $T_c$ the magnets developes a magnetization which persists even when the magnetization is turned off (ferromagnetic phase), while above $T_c$ there is no spontaneous magnetization (paramagnetic phase).

Experimentally in corrispondence of a phase transition the physical observables (like the magnetization or the specific heat) have some singularity, typically of the form $O(|T - T_c|^{-\alpha})$, with $\alpha$ called a *critical index*.

Can phase transitions be explained in the statistical mechanics framework? Singularities in the thermodinamic functions, signaling phase transitions, can possibly appear only in the so-called thermodinamic limit, consisting in taking the limit of infinite volume and particle number taking the density fixed. For a long time it was thougth that such singularities do not appear, and that phase transition cannot be understood starting from the forces between the molecules componing a material. It was Onsager the first who showed a microscopic model exhibiting a phase transition: the 2-dimensional, nearest neighbor *Ising model*, in which the thermodinamic functions can be explicitely computed.

The Ising model is a paradigmatic model for statistical mechanics. It describes a magnet as a lattice made up of molecules with a magnetic dipole wich either points in some direction or in the opposite one (of course the magnetic moment of a molecule should be a vector pointing in any direction, hence such a description is rather crude). To each point $\mathbf{x}$ of the lattice a spin $\sigma_{\mathbf{x}} = \pm 1$ is then associated, and the energy interaction, due to the dipole force, of two spins is $J_{\mathbf{x},\mathbf{x}'}$ if the two spins have the same value, and

$-J_{\mathbf{x},\mathbf{x}'}$ if they have different values, so that the hamiltonian is

$$H_I = -\sum_{\mathbf{x},\mathbf{x}'} J_{\mathbf{x},\mathbf{x}'}\sigma_{\mathbf{x}}\sigma_{\mathbf{x}'} \tag{1.21}$$

Typically $J_{\mathbf{x},\mathbf{x}'}$ is chosen as a short-range interaction; particularly important is the case when $J_{\mathbf{x},\mathbf{x}'}$ is different from zero only if $\mathbf{x}, \mathbf{x}'$ are nearest neighbor, as in this case several semplifications in the mathematical analysis appear.

The partition function at inverse temperature $\beta$ is given by

$$Z = \sum_{\{\sigma_{\mathbf{x}}=\pm\}_{\mathbf{x}\in\Lambda}} e^{-\beta H_I} \tag{1.22}$$

and, if $\beta = (\kappa T)^{-1}$, $\kappa$ is the Boltzmann constant and $N$ the number of point in $\Lambda$,

$$f(\beta) = -\beta \lim_{N\to\infty} N^{-1}\log Z \tag{1.23}$$

is the *free energy* for site; the limit $N \to \infty$ is called thermodinamic limit. The *specific heat* is given by

$$C_v = -\frac{\partial}{\partial T}\frac{\partial}{\partial \beta}(\beta^{-1} f(\beta))] \tag{1.24}$$

and in a similar way are defined all the other thermodinamic functions.

Even with so many simplifying features,the computation of the thermodynamic functions corresponding to (9.1) is quite difficult. Explicit values for the critical indices can be obtained quite easily in the so called *mean field approximation*, but such values are in general quantitatively not correct.

If one considers only nearest neighbor interactions the Ising model con be solved in $d = 1$ (where it does not exhibit phase transitions) and in $d = 2$, through the remarkable exact solution found by Onsager; in $d = 2$ the hamiltonian of the nearest-neighbor Ising model is given by, if $\mathbf{x} = (x_0, x)$

$$H_{n.n.} = \sum_{\mathbf{x},\mathbf{x}'\in\Lambda} [\sigma_{x,x_0}\sigma_{x,x_0+1} + \sigma_{x,x_0}\sigma_{x+1,x_0}] \tag{1.25}$$

and there is a phase transition at the critical temperature

$$\tanh \beta_c = \sqrt{2} - 1 \tag{1.26}$$

The thermodinamic quantities corresponding to (1.25) are singular at $\beta = \beta_c$; for instance the specific heat is given by

$$C_v = -C_1 \log|\beta - \beta_c| + C_1 \tag{1.27}$$

with $C_1, C_2$ constants. The above result should be compared with what is found in the mean field approximation, in which the specific heat is discontinuous. Other thermodynamic variables can be considered; a remarkable one is the spontaneous magnetization, from which one can see that the system has at $\beta = \beta_c$ a phase transition from a paramagnetic to a ferromagnetic phase; the spontaneous magnetization vanishes as $O(|\beta - \beta_c|^{\frac{1}{2}})$.

The solution of the $2d$ nearest-neighbor Ising model was followed by the solutions of other lattice spin models, see [14], like the *Ice models*, with a physical meaning within the theory of the idrogen bond, or the *Vertex models*, and a lot of important informations were achieved from them, which in several cases were also experimentally verified.

It should be noted however that the exact solvability requires a quite special structure, and it is immediately destroyed even by apparently innocuous modifications. For instance in the case of the Ising model, if one includes also a next to nearest interaction (there is physically no reason for which only nearest neighbor spins should interact) the exact solvability is immediately lost. Moreover, while the Ising model in $1d$ or $2d$ is solvable, there is no exact solution for the $3d$ Ising model.

### 1.2.2 *Universality and non-universality*

A crucial role in theory of phase transition is played by the principle of *universality.* Let us consider an hamiltonian of the form

$$H = H_0 + \lambda V \tag{1.28}$$

where $H_0$ is some simple hamiltonian, whose thermodynamic quantities can be computed, $V$ is a complicated perturbation and $\lambda$ is a parameter measuring its strenght.

The natural question is if the critical properties are modified or not by the presence of $\lambda V$. According to the *universality hypothesis*, the singularities in the thermodinamic functions, in particular the critical indices should be insensitive to perturbations as long as symmetry and some form of locality are retained.

The most natural model in which universality can be investigated is the $d = 2$ Ising model; we can choose $H_0$ as (1.25) and $V$ is some complex term involving many spin interaction. What the universality hypothesis says in this case is that, while the thermodinamical quantities (and the critical temperature) depend in general from the perturbation, the critical indices would be identical to the one of the Ising model.

The importance of such hypothesis is clear; if universality holds, one can use extremely simplified and higly idealized models instead of more realistic but extremely complicated ones, and the critical properties should be the same; models with the same critical behaviour are called to be in the same "universality class". Indeed universality seems verified in experiments: for instance carbon dioxide, xenon and the $d = 2$ Ising model appears to be in the same universality class.

On the other hand, a too naive extension of the notion of univesality can be incorrect. The 8-vertex model is equivalent to two Ising models coupled by a quartic interaction, but for such a model universality does not hold; the critical indices of such a model can be explicitely computed ond one sees that they are different with respect to the ones of the Ising model.

How it is possible to check if the universality holds in a model? How it is possible to compute critical indices, when exact solutions are lacking? The more promising technique is the Wilson Renormalization Group, based an an iterative integration leading to a sequence of effective theories. A very important achievement of this method was the computation of the non-universal critical indices in the $3d$ Ising model which, contrary to the $2d$ case, is not solvable. The indices (different from the ones computed in the $d = 2$ case) can be written in terms of a series expansion and are in remarkable agreement with experimental data, see Ref. [6]; unfortunately, a proof of the convergence of such series is still lacking.

A simpler applications of such ideas, which can be instead performed with a full mathematical rigor, can be done for the computation of the critical indices in $2d$ lattice spin models, like non nearest neighbor Ising model, or Vertex or Askhin-Teller models. In such cases in fact one can exploit the remarkable mapping of the Ising model is a system of free fermions in $d = 2$ dimensions, very similar to the ones for $d = 2$ QFT seen before; the mass of the fermions corresponds to $|T - T_c|$ so that the critical point corresponds to massless fermions. Consequently, models which are perturbations of the Ising model can be mapped in fermionic interacting systems; the lack of solvability is reflected in the fact that the Lagrangian is not quadratic.

It turns out then that many thermodynamic quantities (like the specific heat) of several spin lattice models can be written as a fermionic functional integral of the form

$$\int \mathcal{D}\psi e^{-S_0(\psi)-S_1(\psi)} \tag{1.29}$$

where $S_0(\psi)$ is the lattice regularization of the euclidean action of Dirac fermions in $d = 2$ and $S_1(\psi)$ corrsponds to the interacting part. The lattice functional integrals introduced in QFT in $d = 1 + 1$ somewhat artifically to cure the ultraviolet divergences, naturally appear in statistical physiscs as perturbations of the Ising model. Note however that in QFT the continuum limit has to be taken while in statistical physics the lattice is fixed; one has only the infrared problem to face.

The representation (1.29) is very useful as it allows to apply the methods of renormalization developed in QFT to several 2d lattice spin models. In the second part of this book we will see that such methods allow to give a proof of universality for certain classes of perturbed Ising model, in the sense that the behaviour of physical quantities is the same as of the Ising model up to a renormalization of the critical temperature. The same methods allow also the rigorous computations of several non universal critical indices in solvable or non solvable models, like Vertex or Askin-Teller models, equivalent to Ising models coupled by quartic interactions.

## 1.3 Condensed Matter

### 1.3.1 *Electrons in a crystal*

Condensed matter is concerned with the average properties of a system composed by a large number of quantum particles (see Refs. [15],[16]). A crystal can be described as a lattice of atoms in which the valence electrons are lost by the atoms (which become ions) and move freely in the metal; they are responsible of the conduction properties of the crystal. The conduction electrons, whose number is enormous, interact either with the ions and between each other, in a way which depend from the relative positions; the final effect of all such interactions is of course terribly complicated and the macroscopic properties of the crystal, like its conductivity or specific heat, depend from it.

We recall that, according to quantum mechanics, particles are described by a complex, square integrable *wave function* $\Psi(\vec{x}_1, ..., \vec{x}_N)$ with $|\Psi|^2$ representing the probability density of finding $N$ particles at positions $\vec{x}_1, \vec{x}_2, ..., \vec{x}_N$, which we will assume in a $d$-dimensional square box with side $L$ and periodic boundary conditions. The time evolution of the wave-function is driven by the Schroedinger equation

$$-i\frac{\partial}{\partial t}\psi = H_N\psi \tag{1.30}$$

where $H_N$ is the *Hamiltonian operator*, and the choice of such operator is determined by the physical system one wants to describe.

In order to understand the properties of the conduction electrons in a metal one should determine an an antisymmetric wave function verifing the Schroedinger equation (1.30) with an hamiltonian of the form

$$H_N = \sum_{i=1}^{N}[-\frac{\partial^2_{\vec{x}_i}}{2m} + u\, c(\vec{x}_i)] + \sum_{i<j} \lambda\, v(\vec{x}_i - \vec{x}_j) \tag{1.31}$$

in which the first term represents the non relativistic kinetic energy of the electrons ($m$ is the mass), $u\, c(\vec{x})$ is a periodic potential due to the ions in the lattice ($c(\vec{x}) = c(\vec{x} + \vec{R})$ with $\vec{R} = (n_1 a_1, .., n_d a_d)$, $n_i \in \mathbb{Z}$) and $\lambda\, v(\vec{x} - \vec{y})$ is a two body interaction potential, which is described by short range potential to take into account, phenomenologically, the electrostatic screening. Finally $\lambda$ and $u$ are couplings which measure the "strength" of the corresponding interaction. Much more complicated and "realistic" Hamiltonians could be considered; for instance one can add an interaction with a stochastic field to take into account impurities in the lattice, or with a boson field to take into account the dynamics of the ions, and so on. Note also that one can study not only three dimensional Fermi systems ($d = 3$), but also $d = 2$ or $d = 1$ systems; they can describe the conduction electrons of crystals so anisotropic to be considered as bidimensional or one dimensional system.

The *Second quantization* formalism introduces some technical simplification in the analysis of interacting systems. One introduces the Hilbert space of states of a system of $N > 1$ fermions as the space $\mathcal{H}_N$ of all the complex square integrable antisymmetric functions $\Psi(\vec{x}_1, ..., \vec{x}_N)$. Let be $\{\phi_{\vec{k}}(\vec{x})\}_{\vec{k}\in R^d}$ be a *basis* for $\mathcal{H}_1$ (the one particle Hilbert space of all the complex square integrable functions $\Psi(\vec{x}_1)$), where $\vec{k}$ is an index called *quantum number*.

Usually the set of $\phi_{\vec{k}}(\vec{x})$ is chosen as the *eigenfunctions* of the single particle Hamiltonian

$$-\frac{\partial^2_{\vec{x}}}{2m} + u\, c(\vec{x}) \tag{1.32}$$

with eigenvalue $\varepsilon(\vec{k})$. If $u = 0$ then $\phi_{\vec{k}}(\vec{x}) = \frac{1}{L^{d/2}} e^{i\vec{k}\vec{x}}$ with $\vec{k}$ representing the *momentum*, and $\varepsilon(\vec{k}) = \frac{|\vec{k}|^2}{2m}$; due to periodic boundary conditions $\vec{k}$ has the form $\vec{k} = \frac{2\pi}{L}\vec{n}$, $\vec{n} = n_1, ..n_d$ with $n_i$ integer and $-[L/2] \le n_i \le [(L-1)/2]$.

If we call $|\vec{k}_1, ..\vec{k}_N\rangle$ the normalized antisymmetrization of

$$\phi_{\vec{k}_1}(\vec{x}_1)\phi_{\vec{k}_2}(\vec{x}_2)...\phi_{\vec{k}_N}(\vec{x}_N) \tag{1.33}$$

we have that the set of all possible $|\vec{k}_1, ..\vec{k}_N\rangle$ is a *basis* for $\mathcal{H}_N$; $|\vec{k}_1, ..\vec{k}_N\rangle$ describes a state in which the $N$ fermions have quantum numbers $\vec{k}_1, .., \vec{k}_N$.

One can introduce the *creation or annihilation operators* $a^+_{\vec k}, a^-_{\vec k}$: they are *anticommuting operators*

$$\{a^+_{\vec k}, a^-_{\vec k'}\} = \delta_{\vec k,\vec k'} \qquad \{a^+_{\vec k}, a^+_{\vec k'}\} = \{a^-_{\vec k}, a^-_{\vec k'}\} = 0 \tag{1.34}$$

such that $a^+_{\vec k}|\vec k_1, ..\vec k_N\rangle = |\vec k, \vec k_1, ..\vec k_N\rangle$ if $\vec k \neq \vec k_i$,$i = 1, ...N$ and 0 otherwise; $a^-_{\vec k}$ is the *adjoint* of $a^+_{\vec k}$. The state $|0\rangle$ such that $a^-_{\vec k}|0\rangle = 0$ for all $\vec k$ is called *vacuum state* and it represents a state with zero particles. The *Fock space* is defined as the direct sum of the Hilbert spaces with any number of particles, and all the elements of the Fock space can be generated by superposing linearly products of creation operators acting over the vacuum state.

In terms of $a^+_{\vec x,\sigma} = L^{-d/2}\sum_{\mathbf k}\phi_{\vec k}(\vec x)a^+_{\vec k,\sigma}$ and of its adjoint $a^-_{\vec x,\sigma}$, the Hamiltonian can be written as an operator on the Fock space

$$\begin{aligned} H = &\sum_\sigma [\int_V d\vec x a^+_{\vec x,\sigma}\frac{-\partial^2_{\vec x}}{2m}a^-_{\vec x,\sigma} + \\ &+u\int_V d\vec x c(\vec x)a^+_{\sigma,\vec x}a^-_{\vec x,\sigma}] + \sum_{\sigma,\sigma'}\lambda\int_V d\vec x\int_V d\vec y v(\vec x-\vec y)a^+_{\vec x,\sigma}a^-_{\vec x,\sigma}a^+_{\vec y,\sigma'}a^-_{\vec y,\sigma'}\end{aligned} \tag{1.35}$$

In many cases, one gets a good description of the effects of the crystal lattice on the conduction electron considering the so called tight-binding approximation, in which electrons occupy sites of a lattice and can can jump from one lattice site to another one. The Hamiltonian, (called *Hubbard* hamiltonian for local repulsive interactions), is given by

$$H = \sum_{\vec x\in\Lambda}\sum_{\sigma=\uparrow\downarrow} a^+_{\vec x,\sigma}\big(-\frac{\Delta}{2}-\mu\big)a^-_{\vec x,\sigma} + \lambda\sum_{\vec x,\vec y\in\Lambda} v(\vec x-\vec y)a^+_{\vec x,\sigma}a^-_{\vec x,\sigma}a^+_{\vec y,\sigma'}a^-_{\vec y,\sigma'} \tag{1.36}$$

where $\Lambda \subset \mathbb{Z}^d$ is a square sublattice of $\mathbb{Z}^d$ with side $L$ and $\Delta$ is the discrete Laplacean.

As in classical statistical mechanics, one introduces the grand canonical partition function $Z = \mathrm{Tr}e^{-\beta(H-\mu N)}$, where $\mu$ is the chemical potential, $N$ is the particle number operator $N = \sum_\sigma\int d\vec x a^+_{\vec x,\sigma}a^-_{\vec x,\sigma}$ and Tr is the trace operation over the Fock space. Many macroscopic observables can be expressed in terms of $Z$,like the specific heat. The thermodynamical *average* of an observable $O$, typically expressed by a monomial in the $a^\pm$ operators, is given by

$$< O >= Z^{-1}\mathrm{Tr}[e^{-\beta(H-\mu N)}O] \tag{1.37}$$

The Schwinger functions are defined as, if $\mathbf{x} = (\vec{x}, x_0)$ and $x_{0,1} \geq x_{0,2} \geq ...x_{0,s}$, $s$ even

$$S_s(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_s) = \frac{\text{Tr} e^{-(\beta - x_{0,1})(H-\mu N)} a^{\varepsilon_1}_{\vec{x}_1} e^{-(x_{0,1}-x_{0,2})(H-\mu N)} a^{\varepsilon_2}_{\vec{x}_2} .....e^{-x_{0,s}(H-\mu N)}}{\text{Tr} e^{-\beta(H-\mu N)}} \tag{1.38}$$

with $\varepsilon_i = \pm$ and $-\beta/2 \leq t_i \leq \beta/2$.

The physical observables of interest at temperature $\beta^{-1}$ can be obtained from the Schwinger functions. For instance the averaged number of electrons with momentum $\vec{k}$ is given, in the infinite volume limit, by

$$< a^+_{\vec{k},\sigma} a^-_{\vec{k},\sigma} >= \int d\vec{x} e^{i\vec{k}\vec{x}} S(\vec{x}, 0^+; 0, 0) \tag{1.39}$$

Important physical quantities which can be obtained from the higher order Schwinger functions are the *response functions*, which measure the density of the system to a perturbation; for intance the density-density response function is given by, $x_0 > y_0$

$$\Omega(\mathbf{x}, \mathbf{y}) = \sum_{\sigma,\sigma'} \frac{\text{Tr} e^{-(\beta - x_0)(H-\mu N)} a^+_{\vec{x},\sigma} a^-_{\vec{x},\sigma} e^{-(x_0-y_0)(H-\mu N)} a^+_{\vec{y},\sigma'} a^-_{\vec{y},\sigma'} e^{-y_0(H-\mu N)}}{\text{Tr} e^{-\beta(H-\mu N)}} \tag{1.40}$$

### 1.3.2 *The free Fermi gas*

Computing the physical observables corresponding to the complete Hamiltonian (1.36) is a very difficult task. If there is no interaction $\lambda = 0$ one obtains a model called *free Fermi gas* which can be analytically investigated and which is very succesfull in understanding the properties of the conduction electrons in metals.

It holds that $|\vec{k}_1, \sigma_1, ..\vec{k}_N \, \sigma_N\rangle$ are eigenfunctions of $H$ with eigenvalue $\sum_{\vec{k},\sigma} \varepsilon(\vec{k}) n_{\vec{k},\sigma}$, where $n_{\vec{k},\sigma} = 0, 1$ is the eigenvalue of $a^+_{\vec{k},\sigma} a^-_{\vec{k},\sigma}$; $n_{\vec{k},\sigma} = 1$ if in the state there is a fermion with momentum $\mathbf{k}$ and spin $\sigma$ and it is zero otherwise.

The eigenfunction $|\Omega\rangle$ of $H$ with lowest energy is called *ground state*, and it determines the low temperatures properties of the system. In order to find the ground state $|\Omega\rangle$, one has to minimize $\sum_{\vec{k},\sigma} \varepsilon(\vec{k}) n_{\vec{k},\sigma}$ with the constraint that $n_{\vec{k},\sigma}$ can take only the values 0 or 1 and $\sum_{\vec{k},\sigma} n_{\vec{k},\sigma} = N$; if there are many solutions to this problem one says that the ground state is *degenerate*. In the case $u = 0$, in the limit $L \to \infty$ the ground state is such that $n_{\vec{k},\sigma} = 1$ if $\vec{k}$ is in a *sphere* of radius $k_F = (3\pi^2 \rho)^{\frac{1}{3}}$, if $\rho$ is the density.

The boundary of the sphere with radius $k_F$ in the space of momenta is called *Fermi surface* and it is a key notion in the theory of Fermi systems; if $d = 2$ it is replaced by a circle and in $d = 1$ by two points.

If $u \neq 0$ the Fermi surface is still given by the set $\vec{k} : \varepsilon(\vec{k}) = \varepsilon_F$ with $\varepsilon_F$ determined by the condition $\sum_{\mathbf{k}:\varepsilon(\vec{k})\leq\varepsilon_F} 1 = N$. However in this case the Fermi surface is not anymore a sphere in $d = 3$, but it is in general polyhedron of a very complex shape.

The averaged number of electrons with momentum $\vec{k}$ is given, in the infinite volume limit, by

$$< a^+_{\vec{k},\sigma} a^-_{\vec{k},\sigma} >= (1 + e^{\beta(\varepsilon(\vec{k})-\mu)})^{-1} \tag{1.41}$$

at zero temperature it reduces to $\vartheta(\varepsilon(\vec{k}) - \varepsilon_F)$ ($\mu = \varepsilon_F$ at $T = 0$), *i.e.* it has a *discontinuity* at the Fermi surface, while at high temperature it is very close to the Maxwell distribution $\simeq e^{-\beta(\varepsilon(\vec{k})-\mu))}$.

The two point Schwinger function $g(\mathbf{x}_1 - \mathbf{x}_2)$ is given by, using that $e^{Hx_0}\psi^\pm_{\vec{k}}e^{-Hx_0} = e^{\pm(\varepsilon(\vec{k})-\mu)x_0}\psi^\pm_{\vec{k}}$ and calling $t = x_{0,1} - x_{0,2}$

$$\begin{aligned} g(\vec{k}, t) &= e^{(\varepsilon(\vec{k})-\mu)t} \frac{\mathrm{Tr} e^{-\beta(H-\mu N)} T(a^-_{\vec{k}} a^+_{\vec{k}})}{\mathrm{Tr} e^{-\beta(H-\mu N)}} \\ &= \frac{e^{-(\varepsilon(\vec{k})-\mu)t}}{1 + e^{\beta(\varepsilon(\vec{k})-\mu)}} [\vartheta(t) - e^{-\beta(\varepsilon(\vec{k})-\mu)}\vartheta(-t)] \end{aligned} \tag{1.42}$$

Note that $g(\vec{k}, t) = -g(\vec{k}, t + \beta)$; we can then write $g(\vec{k}, t)$ in Fourier series in the following form

$$g(\vec{k}, t) = \frac{2\pi}{\beta} \sum_{k_0=2\pi(n_0+1/2)\beta^{-1}} e^{-ik_0 t}\widehat{g}(\mathbf{k}) \tag{1.43}$$

and

$$\widehat{g}(\mathbf{k}) = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dt e^{ik_0 t} g(t, \vec{k}) = \frac{1}{-ik_0 + \varepsilon(\vec{k}) - \mu} \tag{1.44}$$

At finite temperature $\beta < \infty$ is not singular; only in the limit $\beta \to \infty$ it can be singular when $k_0 = 0$ and at the Fermi surface $\varepsilon(p_F(\vartheta)) = \mu$. Assume that the Fermi surface is sufficiently regular, and that it is possible to parametrize the Fermi surface $\varepsilon(\vec{k}) = \mu$ as $\vec{p}_F(\vec{\vartheta})$, where $\vec{\vartheta}$ is a angle (in $d = 2$) or a couple of angles (in $d = 3$); close to the singularity $\widehat{g}(\mathbf{k})$ has the form

$$\widehat{g}(\mathbf{k}) = \frac{1}{-ik_0 + \vec{v}_F^{(0)}(\vec{\vartheta}) \cdot (\vec{k} - \vec{p}_F^{(0)}(\vec{\vartheta})) + R(\vec{k})} \tag{1.45}$$

where $\vec{v}_F^{(0)}(\vec{\vartheta}) = (\partial\varepsilon_0/\partial\vec{k})\big|_{\vec{k}=\vec{p}_F(\vartheta)}$ is the *free Fermi velocity.* Moreover, near the Fermi surface, $|R(\vec{k})| \leq C\big|\vec{k} - \vec{p}_F^{(0)}(\vartheta)\big|^2$, for some positive constant $C$.

In presence of the interaction the Schwinger functions cannot be exactly computed as in the free case. The Schwinger functions (1.38) can be written as a fermionic functional integral; in the case of the Hubbard model, for instance, the Schwinger functions can be written as

$$S(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N) = \frac{\partial^N}{\partial\phi^{\varepsilon_i}_{\mathbf{x}_1}...\partial\phi^{\varepsilon_i}_{\mathbf{x}_N}}\mathcal{W}(\varphi)|_{\varphi=0} \tag{1.46}$$

where, if $\psi^{\pm}_{\mathbf{x}}$ are *Grassmann variables*, $\mathbf{x} = (x_0, \vec{x})$ and $\vec{x}$ are points on a square lattice $\Lambda$ and, if $v(\mathbf{x}-\mathbf{y}) = v(\vec{x}-\vec{y})\delta(x_0 - y_0)$

$$e^{\mathcal{W}(\phi)} = \int P(d\psi) \tag{1.47}$$

$$e^{-\lambda\sum_{\sigma,\sigma'}\int_{\frac{-\beta}{2}}^{\frac{\beta}{2}} dx_0 dy_0 \sum_{\vec{x},\vec{y}\in\Lambda} v(\mathbf{x}-\mathbf{y})\psi^+_{\mathbf{x},\sigma}\psi^-_{\mathbf{x},\sigma}\psi^+_{\mathbf{y},\sigma'}\psi^-_{\mathbf{y},\sigma'} + \sum_\sigma \int d\vec{x}\phi^+_{\mathbf{x},\sigma}\psi^-_{\mathbf{x},\sigma} + \phi^-_{\mathbf{x},\sigma}\psi^+_{\mathbf{x},\sigma}}$$

and, if $\mathcal{D}_L = \{\vec{k} = \frac{2\pi}{L}(n_1, n_2, .., n_d) \ : \ -[L/2] \leq n_1, n_2, .., n_d \leq [(L-1)/2]\}$ and $\psi^{\pm}_{\mathbf{x}} = \frac{1}{\beta L^d}\sum_{\mathbf{k}\in\mathcal{D}_{\beta,L}} e^{\pm i\mathbf{k}\mathbf{x}}\psi^{\pm}_{\mathbf{k},\sigma}$, the "fermionic measure" is given by

$$P(d\psi) = \mathcal{D}(\psi)\exp[\frac{(2\pi)^{d+1}}{(L^d\beta)}\sum_{\mathbf{k}}\psi^+_{\mathbf{k},\sigma}(-ik_0 + 2 - \sum_{i=1}^{d}\cos k_i - \mu)\psi^-_{\mathbf{k},\sigma}] \tag{1.48}$$

The above functional integrals are very similar to the ones of QFT seens before. The lattice is a natural ultraviolet cut-off, and the temperature plays the role of an infrared cut-off; no divergences are then present at finite volume and non zero temperature, and indeed at high temperature the functional integrals can be expressed by convergent series; the interacting Schwinger function has more or less the same properties of the free one. Things are however much different at very low or vanishing temperatures; in such case an explicit computation shows the Feynman integrals can be diverging or so large to make the power series not converging. This is a signal that the interacting Schwinger and the free one are not perturbatively close even if the coupling $\lambda$ is small.

### 1.3.3 *Fermi liquids*

If there is no interaction between the particles, the properties on the $N$-particle system can be deduced from the single particle ones. When an interaction is switched on, there is no reason a priori to expect this: the interaction between an (essentially infinite) number of particles can induce the emergence of a radically new behavior with respect to the free case. Nevertheless, it is an experimental matter of fact that the properties of the conduction electrons in a number of metals are well described by the free Fermi gas model, up to a redefinition (or *renormalization*) of the parameters. In other words, a description in terms of non interacting electrons is still valid for many interacting systems, modulo a renormalization of the the parameters like the mass or the wave function renormalization. A fermionic system with such property is called a *Landau Fermi liquid*, from Landau who introduced in the 50's such a notion.

For a number of years, physicists were very happy of the fortunate circonstance that, at least for temperatures not too low, metals were well described in terms of Fermi liquids. At very low temperatures the Fermi liquid description breaks down, and some sort of phase transition toward more complex states appear: before it, in their normal phase, the Landau Fermi liquid description was quite successfull in many metals. However in more recent years, the discovery of high $T_c$ superconductivity focus the attemption on a number of material which, in their normal phase, apparently do not behave as Fermi liquids. This leads people, see for instance Ref. [17], to reconsider the notion of Fermi liquid and to try to understand, starting from functional integrals of the form (1.47), how such behavior emerges (or does not emerge) from a microscopic model.

One calls *Landau Fermi liquid* a system whose interacting Schwinger functions are similar to the free one, up to a renormalization of the physical parameters. In a Fermi liquid the 2-point Schwinger function has the form

$$\widehat{S}(\mathbf{k}) = \frac{1}{Z(\vec{\vartheta})} \frac{1}{-ik_0 + \vec{v}_F(\vec{\vartheta}) \cdot \big(\vec{k} - \vec{p}_F(\vec{\vartheta})\big) + R(\mathbf{k})} \tag{1.49}$$

where $Z(\vec{\vartheta})-1, \vec{v}_F(\vec{\vartheta})-\vec{v}_F^{(0)}(\vec{\vartheta}), \vec{p}_F(\vec{\vartheta})-\vec{p}_F^{(0)}(\vec{\vartheta})$ essentially independent from the temperature, and in addition

$$|R(\mathbf{k})| \le C\big[|\vec{k} - \vec{p}_F(\vec{\vartheta})|^2 + k_0^2 + |\vec{k} - \vec{p}_F(\vartheta)||k_0|\big] \tag{1.50}$$

According to the above definition, the Schwinger functions of the interacting system are very similar to the Schwinger function of a free Fermi gas (1.45), and as a consequence the physical properties of the interacting system (which can be deduced from the Schwinger functions) are qualitatively very similar to the ones of the free Fermi gas, up to a renormalization of the parameters.

Not all systems are Fermi liquids; surely $d = 1$ interacting fermionic systems are not Fermi liquids, and the same is true at higher dimensions when the Fermi surface has some cusps or flat pieces, like in the Hubbard model in the half-filled band case. Moreover at very low temperature Fermi liquid behavior is generically absent as a consequence of phase transitions, for instance toward a superconducting state.

In general, to prove that a certain model has a Fermi liquid behavior is not an easy task; starting from the functional integral (1.47), one has to prove the convervence of very complicate expansions in Feynman graphs. The issue of convergence is not just a mathematical curiosity; indeed in the debate on high $T_c$ superconductivity it has been proposed that the apparent discrepancy between theoretical prediction and experimental data was due to the fact computation at lowest order give wrong results for the lack of convergence of the series.

In the third part of this book, we will see how the renormalization mehods allow to give a proof of Fermi liquid behavior in the 2d interacting fermionc systems, up to exponentially small temperatures $T \leq O(e^{\frac{-k}{\lambda}})$ and for free Fermi surfaces verifying suitable convexity properties (including for instance the Hubbard model not at half filling).

### 1.3.4 *Luttinger liquids and BCS superconductors*

What happens at lower temperatures, that is from exponentially small temperatures up to $T = 0$? The answer depends critically on the dimension.

In one dimension, systems have generically a non Fermi liquid behavior, that is their Schwinger function cannot be written as in (2.19) even above exponentially small temperatures. In order to describe $1d$ systems, the notion of *Luttinger liquids* has been introduced; such systems have with the Luttinger model the same relation that the Landau Fermi liquids have with the free Fermi gas.

The Luttinger model has unique peculiarity (in many body theory) to be interacting and exactly solvable in a strong sense,that is all its Schwinger function can be explicitely computed, see ref.[18].

The model describes a system of two kinds of interacting fermions in $d = 1$ described by a field $\psi^{+}_{\vec{x},\omega}$, $\omega = \pm$, with hamiltonian

$$H = \int d\vec{x} \sum_{\omega=\pm} a^{+}_{\vec{x},\omega}(i\omega\partial_x - p_F)a^{-}_{\vec{x},\omega} + \lambda \int d\vec{x}d\vec{y}v(\vec{x}-\vec{y})\rho_{+}(\vec{x})\rho_{-}(\vec{y}) \quad (1.51)$$

with $\rho_\omega(\vec{x}) = a^{+}_{\vec{x},\omega}a^{-}_{\vec{x},\omega}$ and $v(\vec{x}-\vec{y})$ is a short range interaction. The single particle energy $\varepsilon(\vec{k}) = \omega\vec{k}$ is not bounded from below, and, as in Dirac theory, one has to fill all the states with negative energy; this means that the operators $H$ and $\rho_\omega$ can be regarded as operators acting on the Hilbert space $\mathcal{H}$ constructed by completing the space given by the span of the vectors obtained applying finitely many creation or annihilaton operators over the state $|0>$ defined as $|0>=\prod_{\vec{k}\le 0} a^+_{\vec{k},+} a^+_{\vec{k},-}|vac>$.

In the Hilbert space,the operators $\rho_\omega(\vec{p})$ verify a bosonic comutation relation

$$[\rho_\omega(\varepsilon\vec{p}), \rho_\omega(-\varepsilon\vec{p})] = -\varepsilon\omega\vec{p}L/2\pi \tag{1.52}$$

and the hamiltonian can be diagonalized in terms of bosonic operators. In other words the Luttinger model, describing interacting fermions, can be mapped in a system of non-interacting bosons; this property is called *bosonization.*

The Schwinger functions can be then exactly computed; one sees that the 2-point functions behaves for large distances at $T=0$ as

$$S(\mathbf{x}) \simeq_{\mathbf{x}\to\infty} \frac{1}{i\omega x + x_0}\frac{1}{(x^2+x_0^2)^\eta} \tag{1.53}$$

with $\eta(\lambda) = a\lambda^2 + O(\lambda^3)$ is a *critical index*; it is easy to verify that the Fourier trasform diverges at the Fermi points as $O((|k_0| + ||\mathbf{k}| - p_F|)^{-1+\eta})$, that is, contrary to what happens in Landau Fermi liquids, the interaction changes qualitatively the singularity; it is still a power law but a different $\lambda$-dependent exponent.

Proceeding as for Landau Fermi liquids, one can introduce then the notion of Luttinger liquid for systems which behave qualitativey as the Luttinger model. The we say that a system is a *Luttinger liquid* is the Schwinger function has the form

$$\widehat{S}(\mathbf{k}) = \frac{[k_0^2 + v_F^2(\lambda)(|\vec{k}| - p_F(\lambda))^2]^{2\eta}}{-ik_0 + v_F(\lambda)[|\vec{k}| - p_F(\lambda)]}[1 + A_\lambda(\mathbf{k})] \tag{1.54}$$

where $p_F(\lambda) = k_F + O(\lambda)$ and $A_\lambda(\mathbf{k})$. As a consequence the physical properties are different with respect to a Fermi liquid; for instance the averaged number of electrons with momentum $\vec{k}$ is continuous at $\beta = \infty$

$$< a^+_{\vec{k},\omega} a^-_{\vec{k},\omega} > \simeq const + O(||\vec{k}| - p_F|^{2\eta}) \tag{1.55}$$

In general bosonization requires linear dispersion relation and a Dirac sea of fermions with negative energy, all features making the Luttinger model a quite unrealistic description for the conduction electrons in metals. The only way to establish Luttinger liquid behavior in more realistic models like the Hubbard model is to analyze the fermionic function integrals (1.48).

In the third part of this book, will show that the methods of non-perturbative renormalization allow an essential complete understaing for Luttinger liquids in $d = 1$, even in realistic models, like the repulsive Hubbard model; the attractive case,in which no Luttinger liquid behavior is present, is much less understood.

The behavior of system in $d = 2, 3$ is more difficult to analyze. If the temperature is low enough, it is expected that Fermi liquid behavior breaks down, as a consequence of quantum instabilities present in the systems. The most famous of such instabilities is given by the phenomenon of superconductivity. According to the theory of Baardeen, Cooper and Schrieffer (BCS theory) the interaction between fermions leads to the formation of a gap in the energy spectrum, below the critical temperature; it is found, under certain approximations, that for $T$ small enough

$$\lim_{L\to\infty} \widehat{S}(\mathbf{k}) = \frac{-ik_0 - \varepsilon(\vec{k}) + \mu}{k_0^2 + (\varepsilon(\vec{k}) - \mu)^2 + \Delta_\lambda^2} \tag{1.56}$$

where $\Delta$ is exponentially small in $\lambda$. The physical properties predicted by (1.56) are completely different with respect to the free case: the occupation number is continuous, there is an energy gap in the spectrum, the specific heat is $O(e^{-\Delta_\lambda T})$ and the phenomenon of superconductivity appears. At the moment, the theory of superconductivity, and the derivation of (1.56), are based on a *mean field* approximation, and a mathematical derivation is still lacking, despite it is reasonable to hope that the renormalization methods will allow to understand such important phenomena in the near future.