Chapter 1

Prologue

1.1 An Example

As an elementary but motivating example, let $y(t)$ be the cash at hand of a corporation at time $t \geq 0$. Suppose the corporation invests its cash into a project which guarantees a positive interest rate r so that

$$\frac{dy}{dt} = ry, \ t \geq 0. \tag{1.1}$$

What is the cash at hand of the corporation at any time $t > 0$ given that $y(0) = 1$?

One way to solve this problem in elementary analysis is to assume that $y = y(t)$ is a "power series function" of the form

$$y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots ,$$

then we have

$$a_0 = y(0) = 1.$$

By formally operating the power series $y(t)$ term by term, we further have

$$y'(t) = a_1 + 2a_2 t + 3a_3 t^2 + \cdots ,$$

and

$$ry(t) = ra_0 + ra_1 t + ra_2 t^2 + \cdots .$$

In view of (1.1), we see that

$$a_1 + 2a_2 t + 3a_3 t^2 + \cdots \equiv ra_0 + ra_1 t + ra_2 t^2 + \cdots .$$

By comparing coefficients on both sides, we may proceed formally and write

$$a_1 = r, \ 2a_2 = ra_1, \ 3a_3 = ra_2, \ ...,$$

This yields

$$a_1 = r, \ a_2 = \frac{r^2}{2}, \ a_3 = \frac{r^3}{3 \cdot 2}, \ ..., \ a_n = \frac{r^n}{n!}, ...,$$

so that

$$y(t) = 1 + rt + \frac{r^2}{2!} t^2 + \frac{r^3}{3!} t^3 + \cdots , \tag{1.2}$$

which is a "formal power series function".

In order that the formal solution (1.2) is a true solution, we need either to show that $y(t)$ is meaningful on $[0, \infty)$ and that the operations employed above are legitimate, or, we may show that $y(t)$ is equal to some previously known function and show that this function satisfies (1.1) and $y(0) = 1$ directly. If these can be done, then a power series solution exists and is given by (1.2).

Such solutions often reveal important quantitative as well as qualitative information which can help us understand the complex behavior of the physical systems represented by these equations.

In this book, we intend to provide some elementary properties of power series functions and its applications to finding solutions of equations involving unknown functions and/or their associated functions such as their iterates and derivatives.

1.2 Basic Definitions

Basic concepts from real and complex analysis and the theory of linear algebra will be assumed in this book. For the sake of completeness, we will, however, briefly go through some of these concepts and their related information. We will also introduce here some common notations and conventions which will be used in this book.

First of all, sums and products of a set of numbers are common. However, empty sums or products may be encountered. In such cases, we will adopt the convention that an empty sum is taken to be zero, while an empty product will be taken as one.

The union of two sets A and B will be denoted by $A \cup B$ or $A + B$, their intersection by $A \cap B$ or $A \cdot B$, their difference by $A \backslash B$, and their Cartesian product by $A \times B$. The notations $A^2, A^3, ...,$ stand for the Cartesian products $A \times A$, $A \times A \times A, ...,$ respectively. It is also natural to set $A^1 = A$. The number of elements in a set Ω will be denoted by $|\Omega|$.

The set of real numbers will be denoted by $\mathbf{R}$, the set of all complex numbers by $\mathbf{C}$, the set of integers by $\mathbf{Z}$, the set of positive integers by $\mathbf{Z}^+$, and the set of nonnegative integers by $\mathbf{N}$. We will also use $\mathbf{F}$ to denote either $\mathbf{R}$ or $\mathbf{C}$.

It is often convenient to extend the real number system by the addition of two elements, ∞ (which may also be written as $+\infty$) and $-\infty$. This enlarged set $[-\infty, \infty]$ is called the set of extended real numbers. In addition to the usual operations involving the real numbers, we will also require $-\infty < x < \infty$, $x + \infty = \infty$, $x - \infty = -\infty$ and $x/\infty = 0$ for $x \in \mathbf{R}$; $x \cdot \infty = \infty$ and $x \cdot -\infty = -\infty$ for $x > 0$; and

$$\infty + \infty = \infty, \ -\infty - \infty = -\infty, \ \infty \cdot (\pm\infty) = \pm\infty, \ -\infty \cdot (\pm\infty) = \mp\infty, \ 0 \cdot \infty = 0.$$

In the sequel, the equation

$$\frac{1}{u} = v$$

will be met where $v \in [0, \infty]$. The solution u will be taken as ∞ if $v = 0$ and as 0 if $v = \infty$.

The imaginary number $\sqrt{-1}$ in $\mathbf{C}$ will be denoted by $\mathbf{i}$. The symbols $0!$ and 0^0 will be taken as 1. Given a complex number z and an integer n, the n-th power of z is defined by $z^0 = 1$, $z^{n+1} = z^n z$ if $n \geq 0$ and $z^{-n} = (z^{-1})^n$ if $z \neq 0$ and $n > 0$.

Recall also that for any complex number $z = x + \mathbf{i}y$ where $x, y \in \mathbf{R}$, its real part is $\Re(z) = x$, its imaginary part is $\Im(z) = y$, its conjugate is $z^* = x - \mathbf{i}y$ and its modulus or absolute value is $|z| = \left(x^2 + y^2\right)^{1/2}$. We have $|z + w| \leq |z| + |w|$, $|zw| = |z|\,|w|$ and $(zw)^* = z^* w^*$ for any $z, w \in \mathbf{C}$.

Given a nonzero $z = x + \mathbf{i}y \in \mathbf{C}$, if we let θ be the angle measured from the positive x-axis to the line segment joining the origin and the point (x, y), then we see that

$$z = |z|\,(\cos\theta + \mathbf{i}\sin\theta).$$

We define an argument of the nonzero z to be any angle $t \in \mathbf{R}$ (which may or may not lie inside $[0, 2\pi)$) for which

$$z = |z|\,(\cos t + \mathbf{i}\sin t),$$

and we write $\arg z = t$. A concrete choice of $\arg z$ is made by defining $\arg_0 z$ to be that number t_0, called the principal argument, in the range $(-\pi, \pi]$ such that

$$z = |z|\,(\cos t_0 + \mathbf{i}\sin t_0)\,.$$

We may then write

$$\arg_0(zw) = \arg_0 z + \arg_0 w \pmod{2\pi}.$$

It is also easy to show that for any $z \neq 0$, given any positive integer n, there are exactly n distinct complex numbers $z_0, z_1, ..., z_{n-1}$ such that $z_i^n = z$ for each $i = 0, 1, ..., n-1$. The numbers $z_0, z_1, ..., z_{n-1}$ are called the n-th roots of z. The geometric picture of the n-th roots is very simple: they lie on the circle centered at the origin of radius $|z|^{1/n}$ and are equally spaced on this circle with one of the roots having polar angle $\frac{1}{n}\arg_0 z$.

Given a real or complex number α, and any real or complex valued functions f and g, we define $-f$, αf, $f \cdot g$, and $f + g$ by $(-f)(z) = -f(z)$, $(\alpha f)(z) = \alpha f(z)$, $(f \cdot g)(z) = f(z)g(z)$ and $(f + g)(z) = f(z) + g(z)$ as usual, while $|f|$ is defined by $|f|\,(z) = |f(z)|$. If no confusion is caused, the product $f \cdot g$ is also denoted by fg.

The zeroth power of a function, denoted by f^0, is defined by $f^0(z) = 1$, while the n-th power, denoted by f^n, is defined by $f^n(z) = (f(z))^n$.

The composition of f and g is denoted by $f \circ g$. The iterates of f are formally defined by $f^{[0]}(z) = z$, $f^{[1]}(z) = f(z)$, $f^{[2]}(z) = f(f(z)), ...,$ and $f^{[n]}$ is called the n-th iterate of f. Note that $f^{[n]}$ may not be defined if the range of $f^{[n-1]}$ does not lie inside the domain of f.

The n-th derivative of a function is defined by

$$f'(z) = f^{(1)}(z) = \lim_{w \to 0} \frac{f(z + w) - f(z)}{w}$$

and $f^{(k)}(z) = (f^{(k-1)})'(z)$ for $k \geq 2$. As is customary, we will also define $f^{(0)}(z) = f(z)$.

Example 1.1. Recall that the identity function $f : \mathbf{F} \to \mathbf{F}$ defined by $f(t) = t$ for each $t \in \mathbf{F}$ is a polynomial function, so is any constant function $g : \mathbf{F} \to \mathbf{F}$ defined by $g(t) = c \in \mathbf{F}$. Any finite addition or multiplication of polynomial functions is also a polynomial function. For instance,

$$p(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_m t^m,\ c_0, ..., c_m \in \mathbf{F},$$

is a polynomial. In case a polynomial is obtained by finite addition or multiplication of the identity function and nonnegative (positive) constant functions, it is called a polynomial with nonnegative (positive) coefficients.

Example 1.2. The previous example defines polynomials with real or complex independent variable. Polynomials with a function as the independent variable can also be defined. More specifically, let f be a complex valued function. Given a polynomial $p(t)$, formally 'replacing' each t^i by the i-th power f^i of f will result in a polynomial in f, which is denoted by $p(f)$. For instance, given

$$p(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_m t^m,\ c_0, ..., c_m \in \mathbf{F},$$

we have

$$p(f) = c_0 f^0 + c_1 f^1 + c_2 f^2 + \cdots + c_m f^m,\ c_0, ..., c_m \in \mathbf{F}.$$

Note that $p(f)$ is a function such that

$$\begin{aligned} p(f)(z) &= c_0 f^0(z) + c_1 f^1(z) + c_2 f^2(z) + \cdots + c_m f^m(z) \\ &= c_0 + c_1 f(z) + c_2 \left(f(z)\right)^2 + \cdots + c_m \left(f(z)\right)^m . \end{aligned}$$

Another way to generate polynomials in f is to formally replace each t^i by the i-th iterate $f^{[i]}$ of f, resulting in $p[f]$. For instance, let p be the same polynomial above, then

$$p[f] = c_0 f^{[0]} + c_1 f^{[1]} + \cdots + c_m f^{[m]},\ c_0, ..., c_m \in \mathbf{F}.$$

As an example, let M be an n by n complex matrix, and $f(u) = Mu$ where $u \in \mathbf{C}^n$, then $f^{[0]}u = u$, $f^{[k]}(u) = M^k u$ for $k = 1, 2, ..., m$. Hence

$$p[f] = c_0 I + c_1 M + c_2 M^2 + \cdots + c_m M^m.$$

Example 1.3. Polynomials in several real or complex variables can also be defined in similar manners. More specifically, for each $i = 1, ..., \kappa$, let the projection function $f_i : \mathbf{F}^\kappa \to \mathbf{F}$ be defined by $f_i(t_1, t_2, ..., t_\kappa) = t_i$. Projection functions and constant functions are polynomials. Any finite addition or multiplication of polynomial functions is also a polynomial function. For instance,

$$p(t_1, t_2) = c_{00} + c_{10} t_1 + c_{01} t_2 + c_{20} t_1^2 + c_{11} t_1 t_2 + c_{02} t_2^2 + \cdots + c_{0m} t_2^m$$

is a polynomial in (t_1, t_2).

Example 1.4. The quotient of two polynomials is a rational function and is defined whenever its denominator is not zero. Any finite linear combination, products or quotients of rational functions are also rational functions.

Example 1.5. The exponential function exp of *a complex variable* is defined by

$$\exp(z) = e^x(\cos y + \mathbf{i} \sin y)$$

for each $z = x+\mathbf{i}y \in \mathbf{C}$. The value $\exp(z)$ is also written as e^z. Note that $e^x = \exp(x)$ for $x \in \mathbf{R}$ and $e^{\mathbf{i}y} = \cos y + \mathbf{i} \sin y$ for $y \in \mathbf{R}$. Furthermore, the function exp is $2\pi\mathbf{i}$-periodic and maps the strip $\{z \in \mathbf{C} | -\pi < \Im(z) \leq \pi\}$ one-to-one onto $\mathbf{C} \backslash \{0\}$.

Example 1.6. The logarithm function of a real variable is

$$\ln(x) = \int_1^x \frac{1}{t} dt, \ x > 0,$$

and the exponential function exp of *a real variable* is defined to be the inverse function of log. Thus $y = \exp(x)$ if $x = \ln(y)$. If z is a nonzero complex number, then there exist complex numbers w such that $e^w = z$. We define $\log z$ to be any number w such that $e^w = z$. Therefore

$$\log z = \ln |z| + \mathbf{i} \arg z, \ z \neq 0.$$

Note that one such w is the complex number $w = \ln (|z|) + \mathbf{i} \arg_0(z)$ and any other such w must have the form

$$\ln (|z|) + \mathbf{i} \arg_0(z) + 2\pi n\mathbf{i}, \ n \in \mathbf{Z}.$$

The complex number $\ln (|z|) + \mathbf{i} \arg_0(z)$ will be called the principal logarithm of z and denoted by $\log_0(z)$. Thus the function $\log_0$ defined on $\{z \in \mathbf{C} | -\pi < \Im(z) \leq \pi\}$ is the inverse of exp.

Example 1.7. If $z, w \in \mathbf{C}$ and $z \neq 0$, we define

$$z^w = e^{w \log_0(z)}.$$

Note that if $n \in \mathbf{Z}$, then $z^0 = e^0 = 1$ and $z^{n+1} = e^{(n+1)\log_0(z)} = e^{n \log(z)} e^{\log_0(z)} = z^n z$ so that our definition here is compatible with the definition of the n-th power of z. Also, since

$$(z^{1/n})^n = \left(e^{\frac{1}{n} \log_0(z)}\right)^n = e^{\log_0(z)} = z, \ z \neq 0, n \in \mathbf{Z}^+,$$

$z^{1/n}$ is an n-th root of z.

Example 1.8. Some elementary functions are defined in terms of the exponential function:

$$\sin z = \frac{1}{2i}\left\{e^{\mathbf{i}z} - e^{-\mathbf{i}z}\right\}, \cos z = \frac{1}{2}\left\{e^{\mathbf{i}z} + e^{-\mathbf{i}z}\right\},$$

$$\sinh z = \frac{1}{2}\left\{e^z - e^{-z}\right\}, \cosh z = \frac{1}{2}\left\{e^z + e^{-z}\right\}.$$

Note that when z is real, these functions coincide with the usual definitions of cosine, sine, hyperbolic sine and hyperbolic cosine. Basic properties of these functions can be found in standard text books.

A (univariate) sequence is a function defined over a set S of (usually consecutive) integers, and can be denoted by $\{u_k\}_{k\in S}$ or $\{u(k)\}_{k\in S}$. When S is finite and, say, equals $\{1,2,...,n\}$, a sequence is also denoted by $\{u_1,...,u_n\}$. Bivariate or multivariate sequences are functions defined on subsets Ω of $\mathbf{Z}^2$ or $\mathbf{Z}^\kappa$ respectively. There are many different ways to denote bivariate or double sequences. One way is to denote a bivariate sequence by $\{u_{i,j}\}$. However, we may also denote it by $\{u_{ij}\}$ if no confusion is caused. Another way is by $\{u_i^{(j)}\}$. In general, when the independent variables have different interpretations, the latter notation is employed. For instance, $u_i^{(t)}$ may represent the temperature of a mass placed at the integral position i and in the time period t. For multivariate sequences, it is cumbersome to denote them by writing $\{u_{i,j,...,k}\}$. For this reason, we may employ the following device. First, an element in a subset of $\Omega \subseteq \mathbf{Z}^\kappa$ has the form $v = (v_1, v_2, ..., v_\kappa)$. Therefore, we may write $\{u_v\}_{v\in\Omega}$ for a multivariate sequence, and v is naturally called a *multi-index*. When v is treated as a multi-index, it will be convenient to use the standard notation $|v|_1 = v_1 + v_2 + \cdots + v_\kappa$, and $v! = v_1!v_2!\cdots v_\kappa!$. $|v|_1$ is usually called the order of v.

It will be necessary to list the components of a sequence in a linear order. For this purpose, we will order the multi-indices in a linear fashion. We say that a mapping $\Psi : \mathbf{N} \to \Omega \subseteq \mathbf{Z}^\kappa$ is an ordering for Ω if Ψ is one to one and onto. For example, let $\Omega = \mathbf{N}\times\mathbf{N}$, a well known ordering for Ω is the mapping $\tilde{\Psi}$ defined by

$$\begin{aligned}&\tilde{\Psi}(0) = (0,0),\ \tilde{\Psi}(1) = (1,0),\ \tilde{\Psi}(2) = (0,1),\ \tilde{\Psi}(3) = (2,0),\\ &\tilde{\Psi}(4) = (1,1),\ \tilde{\Psi}(5) = (0,2),\ ...\end{aligned} \tag{1.3}$$

In terms of an ordering Ψ for Ω, a rearrangement or enumeration of a multivariate sequence $\{f_v\}_{v\in\Omega}$ is the sequence $\{g_i\}_{i\in\mathbf{N}}$ such that $g_i = f_{\Psi(i)}$.

The notation l^Ω will denote the set of all real or complex sequences defined on Ω. In particular, $l^\mathbf{N}$ denotes the set of all real or complex sequences of the form $\{f_k\}_{k\in\mathbf{N}}$. We will call f_k the k-th term of the sequence f. There are several common sequences in l^N which will be useful. First, for each $m \in \mathbf{N}$, $\hbar^{\langle m\rangle} \in l^\mathbf{N}$ denotes the Dirac sequence defined by

$$\hbar_k^{\langle m\rangle} = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases},$$

and $\mathbf{H}^{(m)} \in l^\mathbf{N}$ denotes the jump (or Heaviside) sequence defined by

$$\mathbf{H}_k^{(m)} = \begin{cases} 0 & 0 \le k < m \\ 1 & k \ge m \end{cases}.$$

Let $\alpha \in \mathbf{F}$, the sequence $\{\alpha, 0, 0, ...\}$ will be denoted by $\overline{\alpha}$ and is called a *scalar sequence*, and the geometric sequence $\left\{1, \alpha, \alpha^2, \alpha^3, ...\right\}$ will be denoted by $\underline{\alpha}$. Thus $\underline{z}_n = z^n$ for $n \in \mathbf{N}$. The sequence $\{0, 0, ...\}$ can be denoted by $\overline{0}$ (but it is also commonly denoted by 0), and $\{1, 0, 0, ...\}$ can be denoted by $\overline{1}$ or $\hbar^{\langle 0\rangle}$. The 'summation' sequence $\{1, 1, 1, ...\}$ will be denoted by $\boldsymbol{\sigma}$ which is equal to $\mathbf{H}^{(0)}$, and the 'difference' sequence $\{1, -1, 0, 0, ...\}$ by δ. The sequence $\{1/0!, 1/1!, 1/2!, 1/3!, 1/4!, ...\}$ will be denoted by ϖ. It is also convenient to write $\hbar$ instead of $\hbar^{\langle 1\rangle}$ and this practice will be assumed for similar situations in the sequel.

For any $z \in \mathbf{F}$, the sequence $\lfloor z \rfloor \in l^{\mathbf{N}}$ is defined by

$$\lfloor z \rfloor = \{1, z, z(z-1), z(z-1)(z-2), ...\}.$$

Thus $\lfloor z \rfloor_0 = 1$ and

$$\lfloor z \rfloor_m = z(z-1)(z-2)\cdots(z-m+1),\ m \in \mathbf{Z}^+.$$

Note that $\lfloor n \rfloor_n = n!$, $\lfloor 0 \rfloor_0 = 0! = 1$, and $0 = \lfloor n \rfloor_{n+1} = \lfloor n \rfloor_{n+2} = \cdots$ for $n \in \mathbf{N}$. Therefore, the sequence $\{1, -3, 3, -1, 0, 0, ...\}$ can be written as $\{(-1)^k \lfloor 3 \rfloor_k / k!\}_{k\in\mathbf{N}}$, and the sequence $\{1, 2, 3, ...\}$ as $\{\lfloor k+1 \rfloor_1\}_{k\in\mathbf{N}}$.

For any $z \in \mathbf{F}$, the binomial sequence $C^{(z)} \in l^{\mathbf{N}}$ is defined by

$$C^{(z)} = \lfloor z \rfloor \cdot \varpi = \left\{\frac{1}{0!}, \frac{z}{1!}, \frac{z(z-1)}{2!}, \frac{z(z-1)(z-2)}{3!}, ...\right\}$$

so that $C_0^{(z)} = 1$ and

$$C_m^{(z)} = \frac{z(z-1)\cdots(z-m-1)}{m!},\ m \in \mathbf{N}, z \in \mathbf{C}.$$

In particular, for $i, j \in \mathbf{N}$ such that $j \le i$, $C_j^{(i)}$ is the usual binomial coefficient.

A real function (including a real sequence, a real matrix, etc.) f is said to be nonnegative if $f(x) \ge 0$ for each x in its domain of definition. In such a case, we write $f \ge 0$. Similarly, given two real functions with a common domain of definition Ω, we say that f is less than or equal to g if $f(x) \le g(x)$ for each $x \in \Omega$. The corresponding notation is $f \le g$. Other monotonicity concepts for real functions (such as $f < g$, $f > 0$, etc.) are similarly defined.

The product set $\mathbf{F}^\kappa$, where κ is a positive integer, is assumed to be equipped with the usual vector operations and the usual Euclidean topology. In particular, the distance between two points $w = (w_1, ..., w_\kappa)$ and $z = (z_1, ..., z_\kappa)$ in $\mathbf{F}^\kappa$ is defined by

$$|w - z| = \left\{|w_1 - z_1|^2 + \cdots + |w_\kappa - z_\kappa|^2\right\}^{1/2}.$$

If $r > 0$ and $c = (c_1, ..., c_\kappa) \in \mathbf{F}^\kappa$, we will set

$$B(c; r) = \{z \in \mathbf{F}^\kappa |\ |z - c| < r\}$$

$$\bar{B}(c; r) = \{z \in \mathbf{F}^\kappa |\ |z - c| \le r\},$$

and

$$B'(c; r) = \{z \in \mathbf{F}^\kappa |\ 0 < |z - c| < r\}.$$

They are usually called the open ball, the closed ball and the punctured ball respectively with center at c and radius r. It is well known that the set of all open balls can be used to generate the Euclidean topology for $\mathbf{F}^\kappa$. In particular, a subset Ω of

$\mathbf{F}^\kappa$ is said to be open if every point in Ω is the center of an open ball lying inside Ω.

Besides the open balls, polycylinders are also natural in future considerations. By a polycylinder of polyradius $\rho = (\rho_1, \rho_2, ..., \rho_\kappa)$, where $\rho_1, ..., \rho_\kappa > 0$, and polycenter $w = (w_1, w_2, ..., w_\kappa) \in \mathbf{F}^\kappa$, we mean the set

$$\{(z_1, ..., z_\kappa) \in \mathbf{F}^\kappa |\ |z_j - w_j| < \rho_j,\ 1 \le j \le \kappa\}.$$

We remark that the boundary of the above polycylinder is described by the set of inequalities

$$|z_j - w_j| \le \rho_j,\ 1 \le j \le \kappa,$$

whereby at least one equality must hold. Thus for $\kappa = 2$, the boundary consists of those (z_1, z_2) for which

$$|z_1 - w_1| = \rho_1,\ \ |z_2 - d_2| \le \rho_2,$$

and those for which

$$|z_1 - w_1| \le \rho_1,\ \ |z_2 - d_2| = \rho_2.$$

A subset Ω of $\mathbf{F}^\kappa$ is said to be a domain if it is nonempty, open and pathwise connected (i.e., a nonempty open set such that any two points of which can be joined by a path lying in the set). We remark that a path in Ω from w to z is a continuous function γ from a real interval $[s, t]$ into Ω with $\gamma(s) = w$ and $\gamma(t) = z$. In this case, w and z are the initial and final points of the path.

In terms of the distance d and the open balls, we can then define as usual limits and continuity for complex-valued functions $f = f(z_1, z_2, ..., z_\kappa)$ defined on a domain Ω or a more general subset of $\mathbf{F}^\kappa$, we can also define partial derivatives, etc. More precisely, the limit

$$\lim_{h \to 0} \frac{f(c_1, ..., c_{i-1}, c_i + h, c_{i+1}, ..., c_\kappa) - f(c_1, ..., c_\kappa)}{h},$$

if it exists, is called the i-th partial derivative of f at $(c_1, ..., c_\kappa)$ and is denoted by

$$\frac{\partial f(c_1, ..., c_\kappa)}{\partial z_i}.$$

Higher partial derivatives of the form

$$\frac{\partial^{v_1}}{\partial z_1^{v_1}} \frac{\partial^{v_2}}{\partial z_2^{v_2}} \cdots \frac{\partial^{v_\kappa}}{\partial z_\kappa^{v_\kappa}} f(c_1, ..., c_\kappa)$$

are defined in recursive manners. Multi-indices can also be used to simplify such notations. Such simplifications are convenient and can be seen in our later sections.

We will need to define integrals for functions $f : \mathbf{F} \to \mathbf{F}$. One such integral is the Cauchy (line) integral

$$\int_\Gamma f(z) dz$$

where Γ is a well behaved path. In this book, it suffices to consider paths Γ that are representable by 'piecewise smooth' functions $\gamma : [a, b] \to \mathbf{F}$, that is, there are points $t_0, t_1, ..., t_n$ with $a = t_0 < t_1 < \cdots < t_n = b$ such that γ' is continuous on each $[t_k, t_{k+1}]$ for $k = 0, ..., n-1$. Then by the standard theory of Riemann-Stieltjes integral, when f is continuous on the image $\Gamma([0,1]) \subset \mathbf{F}$,

$$\int_\Gamma f(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt.$$

In case Γ is the straight line segment joining the point $u = \gamma(a)$ to $v = \gamma(b)$, we will also write

$$\int_\Gamma f(z)dz = \int_u^v f(z)dz.$$

Note that when $\mathbf{F} = \mathbf{R}$, the above line integral is compatible with the usual Riemann integral of a real function.

Recall that Ω is a metric space if there is a metric $d : \Omega \times \Omega \to [0, \infty)$ which satisfies (i) for every pair of $x, y \in \Omega$, $d(x, y) = 0$ if, and only if, $x = y$, (ii) $d(x, y) = d(y, x)$ for $x, y \in \Omega$, and (iii) $d(x, z) \leq d(x, y) + d(y, x)$ for $x, y, z \in \Omega$. Ω is said to be complete if every Cauchy sequence in Ω converges to a point in Ω. $T : \Omega \to \Omega$ is a contraction if there is number λ in $[0, 1)$ such that $d(Tx, Ty) \leq \lambda d(x, y)$ for all $x, y \in \Omega$.

A large number of metric spaces are normed linear spaces, that is, linear spaces whose metrics are induced by norms. Recall that a norm $\|\cdot\|$ on a linear space Ω is a function that maps Ω into $[0, \infty)$ such that (i) for every $x \in \Omega$, $\|x\| = 0$ if, and only if, $x = 0$, (ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for any scalar α and $x \in \Omega$, and (iii) $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in \Omega$. When a normed linear space is also a complete metric space, it is called a Banach space.

A well known result for mappings defined on complete metric spaces is the Banach contraction mapping theorem: If Ω is a nonempty complete metric space and $T : \Omega \to \Omega$ a contraction mapping, then T has a fixed point in Ω.

1.3 Notes

There are several standard reference books on functional equations, see for examples, the books by Aczel [1], Aczel and Dhombres [2], Kuczma [104], Kuczma and Choczewshi [107], and the survey papers by Cheng [29], Kuczma [102], Li and Si [126], Zhang et al. [232]. In this book, we also treat differential equations as functional equations. The corresponding references are too many to list. The books by Bellman and Cooke [16], Coddington and Levinson [40], Driver [52], Friedrichs [66], Hale [73], Hille [78], Kamke [92], Sansone [167], etc., are related to some of our discussions.

There are also several text books which emphasize on analytic functions, see for examples, Balser [13], Krantz and Parks [99], Krantz [100], Smith [211], Sneddon [212], Valiron [216].

In this book, we do not use sophisticated mathematics beyond the usual material taught in courses such as Advanced Engineering mathematics. The reader may also consult text books in real and complex analysis such as Apostol [5], Fichtenholz [62, 63], Kaplan [94], Watson [223], Whittaker and Watson [224], etc.

We have introduced univariate sequences and discussed some of their properties. Further properties will be discussed in later chapters. A summary of their properties can be found in the Appendix.