MARKOV FIELDS ON GRAPHS

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1. Introduction

Markov fields play an important role in classical probability, in physics, in biological and neurological models and in an increasing number of technological problems such as image recognition.

It is quite natural to forecast that the quantum analogue of these models will also play a relevant role.

The papers [7,2,3] are a first attempts to construct a quantum analogue of classical Markov fields. These papers extend to fields the notion of *quantum Markov state* introduced in [5] as a sub–class of the *quantum Markov chains* introduced in [6]. As remarked in [7], the peculiarity of the former class of states with respect to the latter consists in the fact that they admit a Umegaki conditional expectation *into* rather than *onto* their range.

This small difference allows, when applied to states on infinite tensor products of C^*–algebras, to obtain nontrivial (i.e. non product) states while maintaining most of the simple algebraic properties related to classical Markovianity.

The prize one has to pay for this simplification is that the resulting class of states, although non trivial, has very poor entanglement properties so that they cannot exhibit some of the most interesting properties which distinguish the quantum from the classical world.

On the contrary the quantum Markov chains or, more generally, the *generalized quantum Markov states* in the sense of [10] may exhibit very strong entanglement properties. In particular the papers [9,4] show that this is indeed the case for the *entangled Markov chains* constructed in [1].

The above considerations naturally suggest the study of following two problems:

(i) the extension to fields of the notion of generalized Markov state (or Markov chain)

(ii) the extension to fields of the construction of entangled Markov chains produced in [1]

The present paper is a first step towards the solution of these problems. We introduce an hierarchy of notions of Markovianity for states on discrete infinite tensor products of C^*–algebras (section (4)) and for each of these notions we construct some explicit examples. We show that the construction of [1] can be generalized to trees (section (6)) and, in a very special case which corresponds to the condition of *maximal entanglement*, to general graphs (section (7)). It is interesting to notice that, in a different context and for quite different purposes, the special role of trees was already emphasized in [7].

A comment on the notion of generalized quantum Markov state introduced in Definition (4.1) may help understanding the logic leading to this definition and in particular condition (6) which otherwise might, at first sight, seem artificial.

The point is that, as we know from Dobrushin's seminal work [8], the natural localization for fields on a discrete set L is is given by the finite subsets of L and their complements. This localization, when restricted to the 1–dimensional case, does not lead to the usual probabilistic localization but, in a certain sense to its *dual* (or *time reversal*), corresponding to the conditioning of the past on the future rather than conversely. This leads to different structures of the Markov chains in the two cases, a fact already noted in [6] where these two types were called *Markov chains* and *inverse Markov chains* respectively.

In particular the role played by the time zero algebra in the usual Markov processes is played by the algebra at infinity in the multi–dimensional case.

But, while the the time zero algebra has a meaning independent of the state, the algebra at infinity can be (meaningfully) defined only in the GNS representation of the given state. Therefore, if one wants to give a constructive and local definition of a state one cannot make use of a global notion such as the algebra at infinity.

In the ergodic cases, corresponding physically to the *pure phases* in Dobrushin's theory, one expects that the algebra at infinity is trivial and that the sequence of conditional expectations appearing in (6) converges weakly to a single state (asymptotic independence of the boundary) so that the resulting state is in fact independent of the sequence of states $(\hat{\varphi}_{\Lambda_n^c})$ which plays the role of the single "state" $\hat{\varphi}_{L^c} = \hat{\varphi}_\infty$, not available at a C^*-level.

2. Graphs

Let $\mathcal{G} = (L, E)$ be a (non-oriented simple) graph, that is, L is a non-empty at most countable set and

$$E \subset \{\{x, y\}; x, y \in L, x \neq y\}$$

Elements of L and of E are called a *vertices* and *edges*, respectively. Two vertices $x, y \in L$ are called *adjacent*, or *nearest neighbors*, if $\{x, y\} \in E$, and in that case we also write $x \sim y$.

For each $x \in L$ the set of nearest neighbors of x will be denoted

$$N(x) := \{y \in L : y \sim x\}$$

The *degree* of $x \in L$, denoted by $\kappa(x)$, is by definition the number of vertices adjacent to x, namely,

$$\kappa(x) := |N(x)| = |\{y \in L\,;\, y \sim x\}|,$$

where $|\cdot|$ denotes the cardinality.

A graph can be equivalently assigned by giving the pair

$$\{L, \sim\}$$

of its vertices and the binary symmetric relation $\sim$.

A *path* or a *trajectory* or a *walk* connecting two points $x, y \in L$ is a finite sequence of vertices such that $x = x_1 \sim x_2 \sim \cdots \sim x_n = y$. In this case n is called the *length* of the walk. For two distinct vertices $x, y \in L$ the distance $d(x, y)$ is defined to be the shortest length of a walk connecting x and y. By definition $d(x, x) = 0$.

Throughout the paper we always assume that a graph is locally finite, i.e., $\kappa(x) < \infty$ for all $x \in L$, and is connected, i.e., for any pair of vertices there exists a walk connecting them. We will write

$$\Lambda \subseteq_{\text{fin}} L$$

to mean that Λ is a finite subset of L. Given $\Lambda \subseteq_{\text{fin}} L$ we define the external boundary of Λ by

$$\vec{\partial}\Lambda := \{x \notin \Lambda : \exists y \sim x \ , \qquad y \in \Lambda\}$$

the closure of Λ by

$$\overline{\Lambda} := \Lambda \cup \vec{\partial}\Lambda$$

We will write

$$\Lambda \subset\subset \Lambda_1$$

to mean that

$$\overline{\Lambda} \subset \Lambda_1$$

Notice that, by definition

$$\Lambda \cap \vec{\partial}\Lambda = \emptyset$$

$$\vec{\partial}\{x\} =: \partial x = N(x) \setminus \{x\}$$

3. Bundles on graphs

To each $x \in L$ it is associated an Hilbert space $\mathcal{H}_x$ of dimension $d_{\mathcal{H}}(x) \in \mathbb{N}$. In the present paper we will assume that

$$d := d_{\mathcal{H}}(x) = d_{\mathcal{H}} < +\infty \qquad \text{(independent of } x)$$

Given

$$\Lambda \subseteq_{\text{fin}} L$$

we define

$$\mathcal{H}_\Lambda := \otimes_{x\in\Lambda}\mathcal{H}_x$$

We fix, $\forall\, x \in L$, an o.n. basis of $\mathcal{H}_x$:

$$(e_j(x)) \equiv e(x) \quad ; \qquad j \in S(x) := \{1, \ldots, d_{\mathcal{H}}(x)\} = \{1, \ldots, d_{\mathcal{H}}\}$$

By definition $\pi_S : S \to L$ is the bundle whose fibers are the finite sets $\pi_S^{-1}(x) := S(x)$ and the sections of this bundle are the maps:

$$\mathcal{F}(\Lambda, S) := \{\omega_\Lambda : x \in \Lambda \to \omega_\Lambda(x) \in S(x)\} =: \Omega_\Lambda$$

A section ω_Λ is also called a *configuration* in the volume Λ. For $\omega_\Lambda \in \mathcal{F}(\Lambda, S)$, the vector e_{ω_Λ} is defined by

$$e_{\omega_\Lambda} := \otimes_{x\in\Lambda} e_{\omega_\Lambda(x)}(x) \in \mathcal{H}_\Lambda \tag{1}$$

and we will use the symbol E_{ω_Λ} for the corresponding rank one projection:

$$E_{\omega_\Lambda} := |e_{\omega_\Lambda}\rangle\langle e_{\omega_\Lambda}| = e_{\omega_\Lambda} e^*_{\omega_\Lambda} \tag{2}$$

Then the set

$$\{e_{\omega_\Lambda} \ : \ \omega_\Lambda \in \mathcal{F}(\Lambda, S)\} \tag{3}$$

is an o.n. basis of $\mathcal{H}_\Lambda$. Thus the generic vector of $\mathcal{H}_\Lambda$ has the form

$$\sum_{\omega_\Lambda \in \mathcal{F}(\Lambda,S)} \lambda_{\omega_\Lambda} e_{\omega_\Lambda}$$

We will use the notations

$$\mathcal{B}_\Lambda := \mathcal{B}(\mathcal{H}_\Lambda) \quad ; \qquad \forall \Lambda \subseteq_{\text{fin}} L$$

$$\mathcal{B} := C^* - \overset{\lim}{\rightarrow} \mathcal{B}_\Lambda \quad (C^* - \text{inductive limit})$$

As a C^*–algebra $\mathcal{B}$ is isomorphic to the (unique) infinite C^*–tensor product $\otimes_{x \in L} \mathcal{B}(\mathcal{H}(x))$. The natural embedding of $\mathcal{B}(x)$ into $\mathcal{B}$ will be denoted

$$j_x : b \in \mathcal{B}(x) \rightarrow j_x(b) = b \otimes 1_{\{x\}^c} \in \mathcal{B} \tag{4}$$

Similarly, for $\Lambda \subseteq L$, we define

$$j_\Lambda := \otimes_{x \in \Lambda} j_x$$

to simplify the notations, in the following we will often identify each $\mathcal{B}_\Lambda$ to the sub–algebra $j_\Lambda(\mathcal{B}_\Lambda)$ of $\mathcal{B}$, through the identification

$$\mathcal{B}_\Lambda \equiv \mathcal{B}_\Lambda \otimes 1_{\Lambda^c} = j_\Lambda(\mathcal{B}_\Lambda)$$

With these notations the elements of the $*$–sub–algebra of $\mathcal{B}_L$ defined by

$$\mathcal{B}_{L,\text{loc}} := \bigcup_{\Lambda \subseteq_{\text{fin}} L} \mathcal{B}_\Lambda \qquad (\text{set theoretical union})$$

will be called local operators (observables if self–adjoint).

4. Definition of QMF

Definition 4.1. A state φ on $\mathcal{B}_L$ is called a generalized quantum Markov state on $\mathcal{B}_L$ if there exist an increasing sequence $\Lambda_n \uparrow L$ (this means: eventually absorbing any finite subset) and, for each Λ_n, a quasi-conditional expectation $E_{\Lambda_n^c}$ with respect to the triplet

$$\mathcal{B}_{\overline{\Lambda_n}^c} \subseteq \mathcal{B}_{\Lambda_n^c} \subseteq \mathcal{B}_L \tag{5}$$

and a state

$$\hat{\varphi}_{\Lambda_n^c} \in \mathcal{S}(\mathcal{B}_{\Lambda_n^c})$$

such that for any $\Lambda_0 \subset\subset \Lambda_n$ one has

$$\varphi|\mathcal{B}_{\Lambda_0} = \hat{\varphi}_{\Lambda_n^c} \circ E_{\Lambda_n^c}|\mathcal{B}_{\Lambda_0} \tag{6}$$

If, in condition (6), one can choose

$$\hat{\varphi}_{\Lambda_n^c} = \varphi|\mathcal{B}_{\Lambda_n^c} \tag{7}$$

then φ is called a quantum Markov state. Finally φ is called a weak Markov state if for all $a \in \mathcal{B}_{L,loc}$ there exists $\Lambda(a) \subseteq_{\text{fin}} L$ such that $\forall n \in \mathbb{N}$ satisfying $\Lambda(a) \subseteq \Lambda_n \subseteq_{\text{fin}} L$ one has:

$$\varphi(a) = \varphi(E_{\Lambda_n^c}(a))$$

Remark. The ergodic argument used in [5] shows that, for quantum Markov states, in Definition (2) above, $E_{\Lambda_n^c}$ can be replaced by an Umegaki conditional expectation from $\mathcal{B}_L$ onto a sub–algebra of $\mathcal{B}_{\Lambda_n^c}$.

Remark. In the case of infinite tensor products (the only one considered here) one has, for any subset, $I \subseteq L$:

$$\mathcal{B}_{I^c} = \mathcal{B}_I' \quad \text{the commutant of } \mathcal{B}_I \tag{8}$$

Recall that a quasi–conditional expectation with respect to the triplet (5) is a CP1 map $E_{\Lambda_n^c} : \mathcal{B}_L \to \mathcal{B}_{\Lambda_n^c}$ satisfying

$$E_{\Lambda_n^c}(a_{\overline{\Lambda_n}^c} a_{\Lambda_n}) = a_{\overline{\Lambda_n}^c} E_{\Lambda_n^c}(a_{\Lambda_n}) \tag{9}$$

Because of (8) this implies that $E_{\Lambda_n^c}(\mathcal{B}_{\Lambda_n}) \subseteq (\mathcal{B}_{\overline{\Lambda}_n^c})' = \mathcal{B}_{(\overline{\Lambda}_n^c)^c} = \mathcal{B}_{\overline{\Lambda}_n}$.

Consequently

$$E_{\Lambda_n^c}(\mathcal{B}_{\Lambda_n}) \subseteq \mathcal{B}_{\Lambda_n^c} \cap \mathcal{B}_{\overline{\Lambda_n}} = \mathcal{B}_{\vec{\partial}\Lambda_n}$$

which is the natural quantum generalization of the multidimensional (discrete) Markov property as originally formulated by Dobrushin [8].

The above argument shows that, whenever (8) holds (e.g. in the case of infinite tensor products) the Markov property

$$E_{\Lambda_n^c}(\mathcal{B}_{\Lambda_n}) \subseteq \mathcal{B}_{\vec{\partial}\Lambda}$$

follows from the basic property (9) of the quasi–conditional expectations. This is not true in general when (8) does not hold (e.g. in the abelian case or in the case of CAR algebras). In all these cases the Markov property should be included in the definition of the various notions of Markov states as an additional requirement.

5. 1–dimensional weak Markov states

In this section we show that there are natural classes of states which are weak Markov states but not Markov states. In this section we will choose $L = \mathbb{N}$ with the usual nearest neighbour relation. Thus $\mathcal{B}_L = \otimes_{\mathbb{N}}\mathcal{B}$ and, if $\Lambda_n = [0, n-1]$ $(n \geq 1)$ then

$$\overline{\Lambda_n} = [0,n] \quad ; \quad \Lambda_n^c = [n, +\infty) \quad ; \quad \overline{\Lambda_n}^c = [n+1, \infty)$$

Lemma 5.1. *Let $\mathcal{E} : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$ be a transition expectation (i.e. a completely positive identity preserving linear (CP1) map), then there exists a unique CP1 map*

$$T : \mathcal{A} \to \mathcal{B}$$

characterized by

$$T(j_0(a_0) \dots j_n(a_n)) = \mathcal{E}(a_0 \otimes \mathcal{E}(a_1 \otimes \dots \otimes \mathcal{E}(a_n \otimes 1) \dots)$$

Proof. Clear.

Corollary 5.1. *For each $n \in \mathbb{N}$ there exists a unique quasi–conditional expectation $E_{[n}$ with respect to the triple $\mathcal{B}_{[n+1} \subseteq \mathcal{B}_{[n} \subseteq \mathcal{B}_L$ such that*

$$E_{[n}(a_{[0,n]}) = j_n(T(a_{[0,n]})) \; ; \quad \forall \, a_{[0,n]} \in \mathcal{B}_{[0,n]}$$

where T is the map defined by Lemma (5.1).

Proof. Using the decompositions

$$\mathcal{B}_L \equiv \mathcal{B}_{[0,n]} \otimes \mathcal{B}_{(n} \; ; \quad \mathcal{B}_{[n} \equiv \mathcal{B}_{\{n\}} \otimes \mathcal{B}_{(n}$$

one defines

$$E_{[n} := (j_n \circ T|_{\mathcal{B}_{[0,n]}}) \otimes id_{\mathcal{B}_{(n}}$$

Proposition 5.1. *Let $\hat{\varphi} \in \mathcal{S}(\mathcal{B}_L)$ be any state with equal marginals, i.e. $\exists \, \varphi_0 \in \mathcal{S}(\mathcal{B})$ such that*

$$\hat{\varphi} \circ j_n = \varphi_0 \; ; \quad \forall n \in \mathbb{N}$$

Then, if $E_{[n}$ and T are defined by Corollary (5.1), one has

$$\lim_{n \to +\infty} \hat{\varphi} \circ E_{[n} = \varphi_0 \circ T =: \varphi \tag{10}$$

in the sense that the limit exists pointwise in $\mathcal{B}_L$ and the identity holds. Moreover φ is a generalized quantum Markov state in the sense of Definition (4.1).

Proof. If $a \in \mathcal{B}_L$ is a local observable, then there exists $n(a) \in \mathbb{N}$ such that, $\forall\, n > n(a)$

$$E_{[n}(a) = j_n(T(a))$$

consequently, $\forall\, n > n(a)$ one has

$$\lim_{N\to\infty} \hat{\varphi}(E_{[N}(a)) = \lim_{N\to\infty} \hat{\varphi}(j_N(T(a))) = \varphi_0(T(a))$$

Since $\mathcal{B}_{L,loc}$ is norm dense in $\mathcal{B}_L$, the limit (10) exists pointwise in $\mathcal{B}_L$.

To prove that it is a generalized quantum Markov state notice that if, in the notations of Definition (4.1), one chooses

$$\hat{\varphi}_{\Lambda_n^c} := \hat{\varphi}|\mathcal{B}_{\Lambda_n^c} \ ; \quad \Lambda_n \subset_{\text{fin}} \mathbb{N}$$

then, with $\Lambda_n = [0,n]$, and $\forall\, a_{[0,n-1]} \in \mathcal{B}_{[0,n-1]}$ one has

$$\hat{\varphi}_{\Lambda_n} E_{[n}((a_{[0,n-1]})) = \hat{\varphi}(j_n(T(a_{[0,n-1]})) = \varphi_0(T(a_{[0,n-1]})) = \varphi(a_{[0,n-1]})$$

Remark. The state φ, constructed in the Proposition above, will not be, in general, a Markov state. In fact

$$\varphi(E_{[n}(a_{[0,n]}a_{(n,n+k]})) = \varphi(E_{[n}(a_{[0,n]})a_{(n,n+k]}) = \varphi(j_n(T(a_{[0,n]}))a_{(n,n+k]}) =$$

$$= \varphi_0(T[j_n(T(a_{[0,n]}))a_{(n,n+k]}])$$

while

$$\varphi(a_{[0,n]}a_{(n,n+k]}) = \varphi_0(T(a_{[0,n]}a_{[n,n+k]}))$$

and in general these two expressions will not be equal. For example, taking $n = 0$, $k = 1$, the former expression is

$$\varphi_0(\mathcal{E}(\mathcal{E}(a_0 \otimes 1) \otimes a_1))$$

and the latter is

$$\varphi_0(\mathcal{E}(a_0 \otimes \mathcal{E}(a_1 \otimes 1)))$$

The following theorem shows that, under simple additional conditions, φ is a weak Markov state.

Theorem 5.1. *In the above notations and assumptions, define the CP1 maps of $\mathcal{B}$ into itself by*

$$P(b) := \mathcal{E}(b \otimes 1) \qquad ; \qquad Q(b) := \mathcal{E}(1 \otimes b) \tag{11}$$

Then φ is a weak Markov state if either one of the following two conditions is satisfied:

$$\varphi_0 \circ Q = \varphi_0 \circ P = \varphi_0 \tag{12}$$

$$\lim_{k\to\infty} Q^k = 1 \otimes \varphi_0 \; ; \quad \varphi_0 \circ P = \varphi_0 \tag{13}$$

Proof. φ is a weak Markov state if and only if $\forall\, a \in \mathcal{B}_{L,loc}\ \exists\, n(a) \in \mathbb{N}$ such that $\forall\, n > n(a)$

$$\varphi(a) = \varphi(E_{[n}(a))$$

Now if $a \in \mathcal{A}$ is a local observable then there exists $n(a) \in \mathbb{N}$ such that

$$E_{[n}(a) = j_n(T(a)) \; ; \quad \forall\, n \geq n(a)$$

Moreover (11) implies that the map T satisfies the following identities:

$$T(j_n(b)) = Q^n P(b) \qquad ; \quad \forall\, b \in \mathcal{B} \qquad , \quad \forall\, n \in \mathbb{N}$$

Consequently, $\forall\, n \geq n(a)$

$$\begin{aligned} \varphi(E_{[n}(a)) &= \varphi(j_n(T(a))) = \lim_k \hat{\varphi}(E_{[k}(j_n(Ta))) \\ &= \lim_k \hat{\varphi}(j_k(T(j_n(T_a))) = \lim_k \varphi_0(Q^k P(T(a))) \end{aligned}$$

If Q and P satisfy (12) then $\forall\, a \in \mathcal{B}_L$ and $\forall\, n \geq n(a)$

$$\varphi(E_{[n}(a)) = \varphi_0(T(a)) = \hat{\varphi}_{[n}(j_n(T(a)) = \varphi(a)$$

Similarly, if Q and P satisfy (13) then

$$\lim_k \varphi_0(Q^k P(T(a))) = \varphi_0(P(T(a))) = \varphi_0(T(a)) = \varphi(a)$$

and this ends the proof.

Lemma 5.2. *Any generalized Markov state on $\mathcal{B}L$ is uniquely determined by any sequence of triplets*

$$\{\Lambda_n, E_{\Lambda_n^c} \ , \ \varphi_{\Lambda_n^c} \ (n \in \mathbb{N}) \ , \ngeq_{\ltimes} \uparrow \mathbb{L}\}$$

through the limit relation

$$\varphi = \lim_{n\to\infty} \varphi_{\Lambda_n^c} \circ E_{\Lambda_n^c} \tag{14}$$

Proof. For all $\Lambda_0 \subset\subset \Lambda_n \subset \Lambda(m)$ one has

$$\varphi_{\Lambda_n^c} \circ E_{\Lambda_n^c} | \mathcal{B}_{\Lambda_0} = \varphi_{\Lambda(m)^c} \circ E_{\Lambda(m)^c} | \mathcal{B}_{\Lambda_0}$$

and therefore the limit (14) stabilizes on $\mathcal{B}_{\Lambda_0}$ as soon as $\Lambda_0 \subset \Lambda_n$.

6. Entangled Markov fields on trees

In this section we prove that, for a very special class of graphs, i.e. the trees, the construction of entangled Markov chains proposed in [1] can be generalized. The simplification coming from considering trees rather than general graphs manifests itself in the fact that the anologue of the basic isometries, used in the construction of [1], in this case commute.

Recall that a tree is a connected graph without loops. This definition implies that any finite subset $\Lambda \subseteq_{fin} L$ enjoys the following fundamental property:

Property (T)

For any $\Lambda \subseteq_{fin} L$ *and for arbitrary* $x \in \vec{\partial}\Lambda$, *there exists a unique point* $y \in \Lambda$ *such that* $x \sim y$.

The fact that Property (T) is the main ingredient used in the proofs of the results below justifies the expectation that our results could be generalized to any graph such that there exists a sequence of $\Lambda_n \subseteq_{fin} L$ such that $\Lambda_n \uparrow L$ and each Λ_n enjoys Property (T) (maybe with the exception of a *small* set of points).

We keep the notations and assumptions of section (2).

Let (L, E) be a graph and let, for each $\{x, y\} \in E$, be given a complex $d \times d$ matrix $(\psi_{xy}(i,j))$ such that the matrix $(|\psi_{xy}(i,j)|^2)$ is bi–stochastic, i.e.

$$\sum_{i=1}^{d} |\psi_{xy}(i,j)|^2 = \sum_{j=1}^{d} |\psi_{xy}(i,j)|^2 = 1$$

$(\psi_{xy}(i,j))$ will be called an *amplitude matrix*: notice that unitarity of the matrices $(\psi_{xy}(x,j))_{i,j}$ is not required. Define the vector

$$\psi_{xy} = \sum_{i,j=1}^{d} \psi_{xy}(i,j) \cdot e_i(x) \otimes e_j(y) \in \mathcal{H}_x \otimes \mathcal{H}_y \tag{15}$$

Lemma 6.1. *In the notation*

$$E_\Lambda := \{\{x,y\} \mid x, y \in \Lambda, x \sim y\}$$

for any $\Lambda \subseteq_{\text{fin}} L$, *define the vector* $\psi_\Lambda \in \mathcal{H}_\Lambda$ *by*

$$\psi_\Lambda := \sum_{\omega_\Lambda} \psi_\Lambda(\omega_\Lambda) e_{\omega_\Lambda} \tag{16}$$

$$\psi_\Lambda(\omega_\Lambda) := \prod_{\{x,y\} \in E_\Lambda} \psi_{xy}(\omega_\Lambda(x), \omega_\Lambda(y)) \tag{17}$$

If Λ enjoys Property T then $\forall x \in \vec{\partial}\Lambda$,

$$\|\psi_{\Lambda\cup\{x\}}\|^2 = \|\psi_\Lambda\|^2$$

Proof. Property T implies that, for arbitrary $x \in \vec{\partial}\Lambda$, there exists a unique point $y \in \Lambda$ such that $x \sim y$. Then

$$\|\psi_{\Lambda\cup\{x\}}\|^2 = \sum_{\omega_\Lambda,\omega_x} |\psi_{\Lambda\cup\{x\}}((\omega_\Lambda,\omega_x))|^2 = \sum_{\omega_{\Lambda\setminus\{y\}},\omega_y,\omega_x} |\psi_{\Lambda\cup\{x\}}((\omega_{\Lambda\setminus\{y\}},\omega_y,\omega_x))|^2$$

$$= \sum_{\omega_{\Lambda\setminus\{y\}},\omega_y} \sum_{\omega_x=1}^{d} |\psi_\Lambda((\omega_{\Lambda\setminus\{y\}},\omega_y))|^2 \cdot |\psi_{xy}(\omega_y,\omega_x)|^2$$

$$= \sum_{\omega_\Lambda} |\psi_\Lambda(\omega_\Lambda)|^2 = \|\psi_\Lambda\|^2$$

which is the thesis.

Proposition 6.1. *Suppose that Λ enjoys Property T and let*

$$\Lambda' \subset\subset \Lambda \subseteq_{\text{fin}} L$$

Then for any $a \in \mathcal{B}_{\Lambda'}$ and $x \in \vec{\partial}\Lambda$ one has:

$$\langle\psi_\Lambda, a\psi_\Lambda\rangle = \langle\psi_{\Lambda\cup\{x\}}, a\psi_{\Lambda\cup\{x\}}\rangle$$

Proof. Because of Property T, given $x \in \vec{\partial}\Lambda$, there exists a unique point $y \in \Lambda$ such that $x \sim y$. Then we have

$$\begin{aligned}
&\langle\psi_{\Lambda\cup\{x\}}, a\psi_{\Lambda\cup\{x\}}\rangle = \\
&= \sum_{\omega_{\Lambda'},\omega'_{\Lambda'}} \sum_{\omega_{\Lambda\setminus\{\Lambda'\cup\{y\}\}}} \sum_{\omega_x,\omega_y} \psi_{\Lambda\cup\{x\}}((\omega_{\Lambda'},\omega_{\Lambda\setminus\{\Lambda'\cup\{y\}\}},\omega_x,\omega_y))^* \\
&\quad \cdot a_{\omega_{\Lambda'}\omega'_{\Lambda'}} \psi_{\Lambda\cup\{x\}}((\omega'_{\Lambda'},\omega_{\Lambda\setminus\{\Lambda'\cup\{y\}\}},\omega_x,\omega_y)) \\
&= \sum_{\omega_{\Lambda'},\omega'_{\Lambda'}} \sum_{\omega_{\Lambda\setminus\{\Lambda'\cup\{y\}\}}} \sum_{\omega_x,\omega_y} \psi_\Lambda((\omega_{\Lambda'},\omega_{\Lambda\setminus\{\Lambda'\cup\{y\}\}},\omega_y))^* \\
&\quad a_{\omega_{\Lambda'}\omega'_{\Lambda'}} \psi_\Lambda((\omega'_{\Lambda'},\omega_{\Lambda\setminus\{\Lambda'\cup\{y\}\}},\omega_y)) |\psi_{xy}(\omega_x,\omega_y)|^2 \\
&= \sum_{\omega_{\Lambda'},\omega'_{\Lambda'}} \sum_{\omega_{\Lambda\setminus\Lambda'}} \psi_\Lambda((\omega_{\Lambda'},\omega_{\Lambda\setminus\Lambda'}))^* a_{\omega_{\Lambda'}\omega'_{\Lambda'}} \psi_\Lambda((\omega'_{\Lambda'},\omega_{\Lambda\setminus\Lambda'})) \\
&= \langle\psi_\Lambda, a\psi_\Lambda\rangle
\end{aligned}$$

The trouble with Property T is that, if Λ has Property T and $x \in \vec{\partial}\Lambda$, unfortunately it is not true that also $\Lambda \cup x$ has Property T. However trees have a very special property given by the following Lemma.

Lemma 6.2. *In a tree every finite subset $\Lambda \subseteq L$ enjoys Property T.*

Proof. Let $\Lambda \subseteq L$ be a finite subset and let $x \in \vec{\partial}\Lambda$. If there exist $y, z \in \Lambda$ such that $y \sim x$, $z \sim x$, then since a tree is connected, there is a path between y and z and this would give a loop. Against the definition of tree.

Corollary 6.1. *If (L, E) is a tree, and the vector ψ_Λ is defined by (16), (17), then, for any $\Lambda \subseteq_{\text{fin}} L$ of cardinality ≥ 2, one has:*

$$\|\psi_\Lambda\|^2 = d \tag{18}$$

and the limit

$$\varphi(a) = \frac{1}{d} \lim_{\Lambda \uparrow L} \langle \psi_\Lambda, a\psi_\Lambda \rangle$$

exists for any a in the local algebra $\mathcal{B}$ and defines a state φ on $\mathcal{B}$.

Proof. The first statement follows by induction from Proposition (6.1) and Lemma (6.2) because, if $\Lambda = \{x, y\}$, then we get

$$\|\psi_{xy}\|^2 = \sum_{i,j} |\psi_{xy}(i,j)|^2 = d$$

The second statement follows from the first one and Proposition (6.1)

Proposition 6.2. *For $\Lambda \subseteq_{\text{fin}} L$, $x \in \vec{\partial}\Lambda$ and $z \in \Lambda$, with $z \sim x$, define $V_{(z|x)} : \mathcal{H}_z \to \mathcal{H}_z \otimes \mathcal{H}_x$ by*

$$V_{(z|x)} e_{i_z} = \sum_{i_x} \psi_{xz}(i_x, i_z) e_{i_x} \otimes e_{i_z} \tag{19}$$

and extend it naturally to a map $\mathcal{H}_{\Lambda_0} \to \mathcal{H}_{\Lambda_0}$ for any Λ_0 containing $\overline{\Lambda}$. Then $\forall x, y \in \vec{\partial}\Lambda$, $z \in \Lambda$ with $x \sim z$, $y \sim z$, $V_{(z|x)}$ and $V_{(z|y)}$ are isometries satisfying:

$$V_{(z|x)}\psi_\Lambda = \psi_{\Lambda \cup \{x\}}$$

$$V_{(z|x)}V_{(z|y)} = V_{(z|y)}V_{(z|x)}$$

Proof. We have

$$\begin{aligned}\langle V_{(z|x)}e_{i_z}, V_{(z|x)}e_{j_z} \rangle &= \delta_{i_z, j_z} \sum_{i_x, j_x} \langle \psi_{xz}(i_x, i_z) e_{i_x}, \psi_{xz}(j_x, i_z) e_{j_x} \rangle \\ &= \delta_{i_z, j_z} \sum_{i_x} |\psi_{xz}(i_z, i_x)|^2 = \delta_{i_z, j_z} = \langle e_{i_z}, e_{j_z} \rangle\end{aligned}$$

Therefore any $V_{(z|x)}$ is an isometry. Next, we get

$$\begin{aligned}
V_{(z|x)}\psi_\Lambda &= V_{(z|x)}\Big(\sum_{\omega_{\Lambda\setminus\{z\}},i_z}\psi_\Lambda((\omega_{\Lambda\setminus\{z\}},i_z))e_{\omega_{\Lambda\setminus\{z\}}}\otimes e_{i_z}\Big)\\
&= \sum_{\omega_{\Lambda\setminus\{z\}},i_z}\psi_\Lambda((\omega_{\Lambda\setminus\{z\}},i_z))\Big(\sum_{i_x}\psi_{xz}(i_x,i_z)e_{\omega_{\Lambda\setminus\{z\}}}\otimes e_{i_x}\otimes e_{i_z}\Big)\\
&= \sum_{\omega_{\Lambda\setminus\{z\}},i_x,i_z}\psi_{\Lambda\cup\{x\}}((\omega_{\Lambda\setminus\{z\}},i_x,i_z))e_{\omega_{\Lambda\setminus\{z\}}}\otimes e_{i_x}\otimes e_{i_z}\\
&= \psi_{\Lambda\cup\{x\}}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
V_{(z|x)}V_{(z|y)}e_{i_z} &= V_{(z|x)}\Big(\sum_{i_y}\psi_{yz}(i_y,i_z)e_{i_y}\otimes e_{i_z}\Big)\\
&= \sum_{i_x,i_y}\psi_{xz}(i_x,i_z)\psi_{yz}(i_y,i_z)e_{i_x}\otimes e_{i_y}\otimes e_{i_z}\\
&= V_{(z|y)}\Big(\sum_{i_x}\psi_{xz}(i_x,i_z)e_{i_x}\otimes e_{i_z}\Big)\\
&= V_{(z|y)}V_{(z|x)}e_{i_z}.
\end{aligned}$$

Proposition 6.3. *Define the transition expectation* $E_{(z|x)} : \mathcal{B}_x\otimes\mathcal{B}_z\to\mathcal{B}_z$ *by*

$$E_{(z|x)}(a_x\otimes a_z) = V^*_{(z|x)}(a_x\otimes a_z)V_{(z|x)}$$

and extend it to $\mathcal{B}$ *in the usual way. Let* x_o *be any (initial) point in* L *and denote*

$$\varphi_0 = \frac{1}{d}\langle\sum_{i_{x_0}=1}^{d} e_{i_{x_o}},\cdot\sum_{j_{x_0}=1}^{d} e_{j_{x_o}}\rangle$$

the $e(x_o)$*–maximally entangled state on* $\mathcal{B}_{x_0}$*. Define inductively* $L_0=\{x_0\}$ *and*

$$L_n = \bar{L}_{n-1}$$

$$E_{L_n} := \prod\{E_{(x|y)}\ :\ x\in L_n\ ,\ y\in\vec{\partial}L_n\ ,\ x\sim y\}$$

where the product is well-defined because, due to Proposition 6.2, the factors commute. Then for any $\Lambda\subseteq L_n$ *and any* $a_\Lambda\in\mathcal{B}_\Lambda$*:*

$$\varphi(a_\Lambda) = \varphi_0\circ E_{L_0}\circ\cdots\circ E_{L_n}(a_\Lambda)$$

and φ *is a QMF.*

Proof. From Proposition 6.2, we have

$$
\begin{aligned}
&\varphi_0 \circ E_{L_0} \circ \cdots \circ E_{L_n}(a_\Lambda) \\
&= \frac{1}{d}\langle \sum_{i_{x_0}=1}^{d} \prod_{x\in L_1} V_{(x_0|x)} e_{i_{x_0}}, E_{L_1} \circ \cdots \circ E_{L_n}(a_\Lambda) \prod_{x\in L_1} V_{(x_0|x)} \sum_{j_{x_0}=1}^{d} e_{j_{x_o}} \rangle \\
&= \frac{1}{d}\langle \psi_{L_1}, E_{L_1} \circ \cdots \circ E_{L_n}(a_\Lambda)\psi_{L_1} \rangle \\
&\vdots \\
&= \frac{1}{d}\langle \psi_{L_{n+1}}, a_\Lambda \psi_{L_{n+1}} \rangle \\
&= \varphi(a_\Lambda)
\end{aligned}
$$

Now, we prove that φ is a QMF. For each $\Lambda \subseteq_{\text{fin}} L$ and $\omega_{\vec{\partial}\Lambda} \in \Omega_{\vec{\partial}\Lambda}$, we define

$$\psi_{\omega_{\vec{\partial}\Lambda}} = \sum_{\omega_\Lambda} \psi(\omega_\Lambda, \omega_{\vec{\partial}\Lambda}) \cdot e_{\omega_\Lambda} \otimes e_{\omega_{\vec{\partial}\Lambda}}$$

and $V_{\vec{\partial}\Lambda,\bar{\Lambda}} : \mathcal{H}_{\vec{\partial}\Lambda} \to \mathcal{H}_{\bar{\Lambda}}$ by

$$V_{\vec{\partial}\Lambda,\bar{\Lambda}}(e_{\omega_{\vec{\partial}\Lambda}}) = \|\psi_{\omega_{\vec{\partial}\Lambda}}\|^{-1} \psi_{\omega_{\vec{\partial}\Lambda}}$$

Then $V_{\vec{\partial}\Lambda,\bar{\Lambda}}$ is well defined because $\|\psi_{\omega_{\vec{\partial}\Lambda}}\| \neq 0$ for each $\omega_{\vec{\partial}\Lambda}$ (otherwise $\psi(\omega_\Lambda, \omega_{\vec{\partial}\Lambda}) = 0$ for each ω_Λ, contradicting (18) with Λ replaced by $\overline{\Lambda}$). Moreover, since the $\psi_{\omega_{\vec{\partial}\Lambda}}$ are mutually orthogonal, $V_{\vec{\partial}\Lambda,\bar{\Lambda}}$ is an isometry. If we put

$$\psi_{\vec{\partial}\Lambda} = \sum_{\omega_{\vec{\partial}\Lambda}} \|\psi_{\omega_{\vec{\partial}\Lambda}}\| e_{\omega_{\vec{\partial}\Lambda}}$$

then we have

$$V_{\vec{\partial}\Lambda,\bar{\Lambda}}(\psi_{\vec{\partial}\Lambda}) = \sum_{\omega_{\vec{\partial}\Lambda}, \omega_\Lambda} \psi(\omega_\Lambda, \omega_{\vec{\partial}\Lambda}) e_{\omega_\Lambda} \otimes e_{\omega_{\vec{\partial}\Lambda}} = \psi_{\bar{\Lambda}}$$

Since V_Λ is an isometry, we get $\|\psi_{\vec{\partial}\Lambda}\|^2 = d$. Denoting

$$\varphi_{\vec{\partial}\Lambda} := d^{-1}\langle \psi_{\vec{\partial}\Lambda}, \cdot \psi_{\vec{\partial}\Lambda} \rangle$$

$$\mathcal{E}_{\Lambda^c}(a_{\bar{\Lambda}}) := V^*_{\vec{\partial}\Lambda,\bar{\Lambda}} a_{\bar{\Lambda}} V_{\vec{\partial}\Lambda,\bar{\Lambda}} \qquad , \qquad E_{\Lambda^c} := \mathcal{E}_{\Lambda^c} \otimes \mathrm{id}_{\mathcal{B}_{\bar{\Lambda}^c}}$$

for each $a_{\bar{\Lambda}} \in \mathcal{B}_{\bar{\Lambda}}$, we see that

$$\varphi_{\vec{\partial}\Lambda} E_{\Lambda^c}(a_\Lambda) = d^{-1}\langle \psi_{\vec{\partial}\Lambda}, V^*_{\vec{\partial}\Lambda,\bar{\Lambda}} a_\Lambda V_{\vec{\partial}\Lambda,\bar{\Lambda}} \rangle = d^{-1}\langle \psi_{\bar{\Lambda}}, a_\Lambda \psi_{\bar{\Lambda}} \rangle = \varphi(a_\Lambda)$$

for each $a_\Lambda \in \mathcal{B}_\Lambda$. Hence φ is a QMF.

7. Maximally Entangled Markov fields on general graphs

In order to extend the construction of the previous section to more general graphs, we need the condition that, for each $\Lambda \subseteq \vec{\partial} x$,

$$\sum_{i_x} \prod_{y \in \Lambda, y \sim x} |\psi_{xy}(i_x, i_y)|^2$$

is constant, i.e. independent of the choice of the i_y's. This is not true in general.

In this section we prove that, for a very special class of bi-stochastic amplitude matrices, which could be called the maximally entangled ones, the construction of the previous section can be carried over to general graphs.

Markov chains proposed in [1] can be generalized. The simplification coming from considering trees rather than general graphs manifests itself in the fact that the analogue of the basic isometries, used in the construction of [1], in this case commute.

We want to extend the states introduced in Section 6 to general cases. But if $x, y, z \in L$ are connected one to another, then in general one can only say that

$$\sum_{i_x, i_y, i_z} |\psi_{xy}(i_x, i_y)\psi_{yz}(i_y, i_z)\psi_{zx}(i_z, i_x)|^2 \leq d$$

Hence, we need more assumptions. Now, we assume

$$\psi_{xy}(i_x, i_y) = \frac{1}{\sqrt{d}} e^{i\theta_{xy}(i_x, i_y)} \tag{20}$$

where $\theta_{xy}(i_x, i_y) \in \mathbb{R}$.

For any $\Lambda \subseteq_{\text{fin}} L$, let v_Λ be the number of vertices in Λ and ϵ_Λ be the number of edges in Λ. Then α_Λ, defined by:

$$\alpha_\Lambda := v_\Lambda - \epsilon_\Lambda \tag{21}$$

is a numerical invariant of the graph.

Lemma 7.1. *For $\Lambda \subseteq_{\text{fin}} L$ let be defined by (17) and (16). Then*

$$\|\psi_\Lambda\|^2 = d^{\alpha_\Lambda}$$

Proof. For each $\omega_\Lambda \in \Omega_\Lambda$ (20) and (17) imply that

$$|\psi_\Lambda(\omega_\Lambda)|^2 = d^{-\epsilon_\Lambda}$$

Since the number of configurations is d^{v_Λ}, we obtain

$$\|\psi_\Lambda\|^2 = \sum_{\omega_\Lambda} |\psi_\Lambda(\omega_\Lambda)|^2 = d^{-\epsilon_\Lambda} d^{v_\Lambda} = d^{\alpha_\Lambda}$$

Proposition 7.1. *Let $\Lambda' \subset\subset \Lambda \subseteq_{\text{fin}} L$. Then for any $a \in \mathcal{B}_{\Lambda'}$ and $x \in \vec{\partial}\Lambda$*

$$d^{-\alpha_\Lambda}\langle\psi_\Lambda, a\psi_\Lambda\rangle = d^{-\alpha_{\Lambda\cup\{x\}}}\langle\psi_{\Lambda\cup\{x\}}, a\psi_{\Lambda\cup\{x\}}\rangle$$

Proof. Denoting

$$\partial x = \vec{\partial} x \cap \Lambda$$

we find

$$\begin{aligned}
&d^{-\alpha_{\Lambda\cup\{x\}}}\langle\psi_{\Lambda\cup\{x\}}, a\psi_{\Lambda\cup\{x\}}\rangle = \\
&= d^{-\alpha_{\Lambda\cup\{x\}}} \sum_{\omega_{\Lambda'},\omega'_{\Lambda'}} \sum_{\omega_{\Lambda\setminus\{\Lambda'\cup\partial x\}}} \sum_{\omega_{\partial x}} \sum_{i_x} \psi_{\Lambda\cup\{x\}}((\omega_{\Lambda'}, \omega_{\Lambda\setminus\{\Lambda'\cup\partial x\}}, \omega_{\partial x}, i_x))^* \\
&\quad \cdot a_{\omega_{\Lambda'}\omega'_{\Lambda'}} \psi_{\Lambda\cup\{x\}}((\omega'_{\Lambda'}, \omega_{\Lambda\setminus\{\Lambda'\cup\partial x\}}, \omega_{\partial x}, i_x)) \\
&= d^{-\alpha_{\Lambda\cup\{x\}}} \sum_{\omega_{\Lambda'},\omega'_{\Lambda'}} \sum_{\omega_{\Lambda\setminus\{\Lambda'\cup\partial x\}}} \sum_{\omega_{\partial x}} \sum_{i_x} \psi_{\Lambda}((\omega_{\Lambda'}, \omega_{\Lambda\setminus\{\Lambda'\cup\partial x\}}, \omega_{\partial x}))^* \\
&\quad \cdot a_{\omega_{\Lambda'}\omega'_{\Lambda'}} \psi_{\Lambda}(\omega'_{\Lambda'}, \omega_{\Lambda\setminus\{\Lambda'\cup\partial x\}}, \omega_{\partial x})) \prod_{y\in\partial x} |\psi_{xy}(i_x, \omega_{\partial x}(y))|^2 \\
&= d^{-\alpha_{\Lambda\cup\{x\}}} d^{\epsilon_\Lambda - \epsilon_{\Lambda\cup\{x\}}+1} \sum_{\omega_{\Lambda'},\omega'_{\Lambda'}} \sum_{\omega_{\Lambda\setminus\Lambda'}} \psi_\Lambda((\omega_{\Lambda'}, \omega_{\Lambda\setminus\Lambda'}))^* \\
&\quad \cdot a_{\omega_{\Lambda'}\omega'_{\Lambda'}} \psi_\Lambda((\omega'_{\Lambda'}, \omega_{\Lambda\setminus\Lambda'})) \\
&= d^{-\alpha_\Lambda}\langle\psi_\Lambda, a\psi_\Lambda\rangle
\end{aligned}$$

Now, we can define the state φ on $\mathcal{B}$ by

$$\varphi(a) = \lim_{\Lambda\uparrow L} d^{-\alpha_\Lambda}\langle\psi_\Lambda, a\psi_\Lambda\rangle$$

for any a in the local algebra. In fact, this is well-defined from Proposition (7.1). From the definition of φ, we have

$$\varphi(E_{\omega_\Lambda}) = d^{-v_\Lambda}$$

For any $\Lambda \subseteq_{\text{fin}} L$, we define the operator $V_{\vec{\partial}\Lambda,\bar{\Lambda}} : \mathcal{H}_{\vec{\partial}\Lambda} \to \mathcal{H}_{\bar{\Lambda}}$ by

$$V_{\vec{\partial}\Lambda,\bar{\Lambda}} e_{\omega_{\vec{\partial}\Lambda}} = d^{\frac{\beta_\Lambda}{2}} \sum_{\omega_\Lambda} \psi_\Lambda(\omega_\Lambda, \omega_{\vec{\partial}\Lambda}) e_{\omega_\Lambda} \otimes e_{\omega_{\vec{\partial}\Lambda}}$$

where $\beta_\Lambda = |\Lambda_E| - v_\Lambda$ and

$$\psi_\Lambda(\omega_\Lambda, \omega_{\vec{\partial}\Lambda}) = \prod_{\{x,y\}\in\Lambda_E} \psi_{xy}((\omega_\Lambda, \omega_{\vec{\partial}\Lambda})(x), (\omega_\Lambda, \omega_{\vec{\partial}\Lambda})(y))$$

Lemma 7.2. *For any* $\Lambda \subseteq_{\text{fin}} L$*, the operator* $V_{\vec{\partial}\Lambda,\bar{\Lambda}}$ *is isometry. Moreover, for sufficiently large* $\Lambda' \subseteq_{\text{fin}} L$*, we obtain*

$$V_{\vec{\partial}\Lambda,\bar{\Lambda}} d^{-\frac{\alpha_{\Lambda'\setminus\Lambda}}{2}} \psi_{\Lambda'\setminus\Lambda} = d^{-\frac{\alpha_{\Lambda'}}{2}} \psi_{\Lambda'}$$

Proof. For each orthonormal basis $e_{\omega_{\vec{\partial}\Lambda}}, e_{\omega'_{\vec{\partial}\Lambda}} \in \mathcal{H}_{\vec{\partial}\Lambda}$, we have

$$\begin{aligned}
&\langle V_{\vec{\partial}\Lambda} e_{\omega_{\vec{\partial}\Lambda}}, V_{\vec{\partial}\Lambda} e_{\omega'_{\vec{\partial}\Lambda}} \rangle \\
&= \langle d^{\frac{\beta_\Lambda}{2}} \sum_{\omega_\Lambda} \psi_\Lambda(\omega_\Lambda, \omega_{\vec{\partial}\Lambda}) e_{\omega_\Lambda} \otimes e_{\omega_{\vec{\partial}\Lambda}}, d^{\frac{\beta_\Lambda}{2}} \sum_{\omega'_\Lambda} \psi_\Lambda(\omega'_\Lambda, \omega'_{\vec{\partial}\Lambda}) e_{\omega'_\Lambda} \otimes e_{\omega'_{\vec{\partial}\Lambda}} \rangle \\
&= \delta_{\omega_{\vec{\partial}\Lambda}, \omega'_{\vec{\partial}\Lambda}} d^{\beta_\Lambda} \langle \sum_{\omega_\Lambda} \psi_\Lambda(\omega_\Lambda, \omega_{\vec{\partial}\Lambda}) e_{\omega_\Lambda}, \sum_{\omega'_\Lambda} \psi_\Lambda(\omega'_\Lambda, \omega_{\vec{\partial}\Lambda}) e_{\omega'_\Lambda} \rangle \\
&= \delta_{\omega_{\vec{\partial}\Lambda}, \omega'_{\vec{\partial}\Lambda}} d^{\beta_\Lambda} \sum_{\omega_\Lambda} |\psi_\Lambda(\omega_\Lambda, \omega_{\vec{\partial}\Lambda})|^2 \\
&= \delta_{\omega_{\vec{\partial}\Lambda}, \omega'_{\vec{\partial}\Lambda}} d^{\beta_\Lambda} d^{v_\Lambda} d^{-|\Lambda_E|} = \delta_{\omega_{\vec{\partial}\Lambda}, \omega'_{\vec{\partial}\Lambda}}
\end{aligned}$$

Therefore, $V_{\vec{\partial}\Lambda,\bar{\Lambda}}$ is isometry. Furthermore, $\Lambda_E \cup (\Lambda'\setminus\Lambda)_e = E'_\Lambda$. Now, we have

$$\begin{aligned}
&V_{\vec{\partial}\Lambda,\bar{\Lambda}} d^{-\frac{\alpha_{\Lambda'\setminus\Lambda}}{2}} \psi_{\Lambda'\setminus\Lambda} \\
&= V_{\vec{\partial}\Lambda,\bar{\Lambda}} d^{-\frac{\alpha_{\Lambda'\setminus\Lambda}}{2}} \sum_{\omega_{\Lambda'\setminus\Lambda}} \psi_{\Lambda'\setminus\Lambda}(\omega_{\Lambda'\setminus\Lambda}) e_{\omega_{\Lambda'\setminus\Lambda}} \\
&= d^{-\frac{\alpha_{\Lambda'\setminus\Lambda}}{2}} d^{\frac{\beta_\Lambda}{2}} \sum_{\omega_{\Lambda'\setminus\bar{\Lambda}}, \omega_\Lambda, \omega_{\vec{\partial}\Lambda}} \psi_{\Lambda'\setminus\Lambda}((\omega_{\Lambda'\setminus\bar{\Lambda}}, \omega_{\vec{\partial}\Lambda})) \psi_\Lambda(\omega_\Lambda, \omega_{\vec{\partial}\Lambda}) \\
&\quad \cdot e_{\omega_{\Lambda'\setminus\bar{\Lambda}}} \otimes e_{\omega_{\vec{\partial}\Lambda}} \otimes e_{\omega_\Lambda} \\
&= d^{-\frac{\alpha_{\Lambda'}}{2}} \sum_{\omega_{\Lambda'}} \psi(\omega_{\Lambda'}) e_{\omega_{\Lambda'}} = d^{-\frac{\alpha_{\Lambda'}}{2}} \psi_{\Lambda'}
\end{aligned}$$

For each $\Lambda \subseteq_{\text{fin}} L$, we define the transition maps $\mathcal{E}_{\Lambda^c} : \mathcal{B}_{\bar{\Lambda}} \to \mathcal{B}_{\vec{\partial}\Lambda}$ by

$$\mathcal{E}_{\Lambda^c}(a_{\bar{\Lambda}}) = V^*_{\vec{\partial}\Lambda,\bar{\Lambda}} a_{\bar{\Lambda}} V_{\vec{\partial}\Lambda,\bar{\Lambda}}$$

for any $a_{\bar{\Lambda}} \in \mathcal{B}_{\bar{\Lambda}}$, and quasi-conditional expectations $E_{\Lambda^c} : \mathcal{B} \to \mathcal{B}$ by

$$E_{\Lambda^c}(a_{\bar{\Lambda}} \otimes a_{\bar{\Lambda}^c}) = I_{\mathcal{B}_\Lambda} \otimes \mathcal{E}_{\Lambda^c}(a_{\bar{\Lambda}}) \otimes a_{\bar{\Lambda}^c}$$

for any $a_{\bar{\Lambda}} \in \mathcal{B}_{\bar{\Lambda}}$ and $a_{\bar{\Lambda}^c} \in \mathcal{B}_{\bar{\Lambda}^c}$. By definition, we have

$$\mathcal{E}_{\Lambda^c}(e_{\omega_\Lambda, \omega'_\Lambda}) = \sum_{\omega_{\vec{\partial}\Lambda}} d^{\beta_\Lambda} \psi_\Lambda(\omega_\Lambda, \omega_{\vec{\partial}\Lambda})^* \psi_\Lambda(\omega'_\Lambda, \omega_{\vec{\partial}\Lambda}) E_{\omega_{\vec{\partial}\Lambda}}$$

In particular,

$$\mathcal{E}_{\Lambda^c}(E_{\omega_\Lambda}) = d^{-v_\Lambda} \cdot 1$$

Lemma 7.3. *For any* $\Lambda_0 \subset\subset \Lambda \subseteq_{\text{fin}} L$ *and* $a_{\Lambda_0} \in \mathcal{B}_{\Lambda_0}$*, we have*

$$E_{\Lambda^c}(a_{\Lambda_0}) = \varphi(a_{\Lambda_0})$$

In particular the family $\{E_{\Lambda^c}\}$ *is weakly projective and* φ *is the unique weakly invariant state.*

Proof. From the above calculation, we obtain

$$\begin{aligned}
&E_{\Lambda^c}(e_{\omega_{\Lambda_0},\omega'_{\Lambda_0}}) \\
&= \sum_{\omega_{\Lambda\setminus\Lambda_0},\omega_{\vec{\partial}\Lambda}} d^{\beta_\Lambda}\psi_\Lambda((\omega_{\Lambda\setminus\Lambda_0},\omega_{\Lambda_0}),\omega_{\vec{\partial}\Lambda})^*\psi_\Lambda((\omega_{\Lambda\setminus\Lambda_0},\omega'_{\Lambda_0}),\omega_{\vec{\partial}\Lambda})E_{\omega_{\vec{\partial}\Lambda}} \\
&= \sum_{\omega_{\vec{\partial}\Lambda},\omega_{\vec{\partial}\Lambda_0}} d^{|\Lambda_{0E}|-v_{\bar{\Lambda}_0}}\psi_{\Lambda_0}(\omega_{\Lambda_0},\omega_{\vec{\partial}\Lambda_0})^*\psi_{\Lambda_0}(\omega'_{\Lambda_0},\omega_{\vec{\partial}\Lambda_0})E_{\omega_{\vec{\partial}\Lambda}} \\
&= \sum_{\omega_{\vec{\partial}\Lambda_0}} d^{|\Lambda_{0E}|-v_{\bar{\Lambda}_0}}\psi_{\Lambda_0}(\omega_{\Lambda_0},\omega_{\vec{\partial}\Lambda_0})^*\psi_{\Lambda_0}(\omega'_{\Lambda_0},\omega_{\vec{\partial}\Lambda_0})\cdot I
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\varphi(e_{\omega_{\Lambda_0},\omega'_{\Lambda_0}}) &= d^{-\alpha_{\bar{\Lambda}_0}}\langle\psi_{\bar{\Lambda}_0}, e_{\omega_{\Lambda_0},\omega'_{\Lambda_0}}\psi_{\bar{\Lambda}_0}\rangle \\
&= d^{-\alpha_{\bar{\Lambda}_0}}\sum_{\omega_{\vec{\partial}\Lambda_0}}\psi_{\bar{\Lambda}_0}((\omega_{\vec{\partial}\Lambda_0},\omega_{\Lambda_0}))^*\psi_{\bar{\Lambda}_0}((\omega_{\vec{\partial}\Lambda_0},\omega'_{\Lambda_0})) \\
&= d^{|\Lambda_{0E}|-v_{\bar{\Lambda}_0}}\sum_{\omega_{\vec{\partial}\Lambda_0}}\psi_{\Lambda_0}(\omega_{\Lambda_0},\omega_{\vec{\partial}\Lambda_0})^*\psi_{\Lambda_0}(\omega'_{\Lambda_0},\omega_{\vec{\partial}\Lambda_0})
\end{aligned}$$

Lemma 7.4. *The family* $\{E_{\Lambda^c}\}$ *is not projective. In particular,* φ *is not a Markov state.*

Proof. For arbitrary $e_{\omega_{\bar{\Lambda}},\omega'_{\bar{\Lambda}}} \in \mathcal{B}_{\bar{\Lambda}}$, we get

$$\mathcal{E}_{\Lambda^c}(e_{\omega_{\bar{\Lambda}},\omega'_{\bar{\Lambda}}}) = d^{\beta_\Lambda}\psi_\Lambda(\omega_\Lambda,\omega_{\vec{\partial}\Lambda})^*\psi_\Lambda(\omega'_\Lambda,\omega'_{\vec{\partial}\Lambda})e_{\omega_{\vec{\partial}\Lambda},\omega'_{\vec{\partial}\Lambda}}$$

where $\omega_{\bar{\Lambda}} = (\omega_\Lambda,\omega_{\vec{\partial}\Lambda})$ and $\omega'_{\bar{\Lambda}} = (\omega'_\Lambda,\omega'_{\vec{\partial}\Lambda})$. From the proof of Lemma (7.3) we have that

$$E_{\Lambda'^c}E_{\Lambda^c} \neq E_{\Lambda'^c}$$

for $\Lambda \subset\subset \Lambda' \subseteq_{\text{fin}} L$.

7.1. *Interpretation*

In many models used in statistical mechanics, the vertices $x \in L$ are identified to particles, the Hilbert space $\mathcal{H}(x)$ to their state space, the basis

$(e_j(x))$ to the eigenvectors of some non degenerate observable $A(x)$ and the index set $S(x)$ to the eigenvalues of this observable, say

$$S(x) = \{1, \ldots, d_{\mathcal{H}}\} \equiv \{a_1(x), \ldots, a_{d_{\mathcal{H}}}(x)\}$$

With these identifications the section ω_Λ is identified to the event or configuration:

$$\omega_\Lambda \equiv \{[A(x) = a(x)\omega_\Lambda(x)] \; ; \quad \forall x \in \Lambda\}$$

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