

Chapter I

Derivatives and Asset Pricing in a Discrete-Time Setting: Basic Concepts and Strategies

This chapter is organized as follows:

1. Section 2 develops the basic strategies using calls and puts. It presents the main concepts in option strategies.
2. Section 3 illustrates several combined strategies. These strategies can be used in different markets for different underlying assets.
3. Section 4 presents the main framework for asset pricing in a discrete-time context. It develops the mean–variance framework.
4. Section 5 presents the Capital Asset Pricing Model (CAPM) in its simplest version in the lines of Markowitz (1952) and Sharpe (1964). It also develops the general techniques for the derivation of the efficient frontier and presents Merton’s (1987) simple model of capital market equilibrium with incomplete information. Incomplete information refers to the fact that there are some costs in gathering data and transmitting information from one agent to another. All these models are presented in a discrete-time context.
5. Section 6 presents the discrete-time approach for option pricing. It develops the Cox, Ross and Rubinstein (1979) model for the valuation of standard equity options.
6. Section 7 extends the standard discrete-time binomial model of Cox, Ross and Rubinstein (1979) to account for the effects of distributions to the underlying asset.

1. Introduction

This chapter studies the basic concept of options and their uses. It allows the reader to understand the main risks and return patterns

associated with investments in financial markets and in particular in derivative assets. Derivatives correspond to futures, forward contracts, swaps, standard options and more complex options. An option gives the right to its holder to buy (for a call) or sell (for a put) a specified asset at a given strike price for a specified period of time. This is a standard definition of a standard option. Futures or forward contracts have similar definitions as options except that the buyer or the seller of the contract has no option: he has an obligation.

Simple CAPMs allow the reader to understand the main concept of asset pricing in a standard context and in the presence of incomplete information. Simple CAPMs allow the reader to understand the concepts of risk and return in finance. These models can also be applied to the valuation of options with and without incomplete information using the standard analysis in Black and Scholes (1973), Black (1976) and Bellalah (1999, 2000).

Asset pricing theory includes the valuation of a wide range of financial assets and derivative securities.

Modern financial theory is based on some standard assumptions regarding markets and investors. It has had an impact on the development of financial markets.

Over the last three decades, the financial market has lived through a wave of financial innovations and structural changes in the securities industry.

What is a derivative?

A derivative is a generic term to encompass all financial transactions which are not directly traded in the primary physical market. It refers to a financial instrument that helps to manage a given risk. It includes forwards, futures, options, commodity contracts, etc.

What is a forward contract?

A forward contract is the simplest and most basic hedging instrument. It is an agreement between two parties to set the price today for a transaction that will not be completed until a specified date in the future. The only way for the buyer or the seller to cancel the contract at a later date is to enter into a reverse forward contract with the same bank or another institution. However, a reverse contract implies a gain or a loss because the forward rate is likely to change as time passes. Forward rate contracts are flexible and allow for customized hedges since all the terms can be negotiated with the counterparty. However, each side of the contract bears what is called counterparty risk, that is, the risk that the other side defaults on the future

commitments. That is why futures contracts are often preferred to forward contracts.

What is a futures contract?

A future is an exchange-traded contract between a buyer and seller and the clearinghouse of a futures exchange to buy or sell a standard quantity and quality of a commodity at a specified future date and price. The clearinghouse acts as a counterparty in all transactions and is responsible for holding traders' surety bonds to guarantee that transactions are completed.

Like forward contracts, futures contracts are used to lock in the interest rate, exchange rate or commodity price. But, futures contracts are organized in such a way that the counterparty risk of default is always completely eliminated because the clearinghouse steps in between a buyer and a seller, each time a deal is struck in the pit. The clearinghouse adopts the position of the buyer to every seller, and of the seller to every buyer, i.e. the clearinghouse keeps a zero net position. This means that every trader in the futures markets has obligations only to the clearinghouse, and has strong expectations that the clearinghouse will maintain its side of the bargain as well. The credibility of the system is maintained through the requirements of margin and daily settlements. The margin is a deposit in the form of cash, government securities, stock in the clearing corporation or letters of credit issued by an approved bank. The main purpose of the margin is to provide a safeguard to ensure that traders will honor their obligations. It is usually set to the maximum loss a trader can experience in a normal trading day. Daily settlements, called making to market, involve debiting the cash accounts of those whose positions lost money for the day and crediting the cash accounts of those whose positions earned money.

However, the elimination of default risk has a cost. Futures contracts are standardized with respect to quantities and delivery dates and limited to frequently traded financial assets. Therefore, available futures contracts may not correspond perfectly to the risk to be hedged, thereby leaving hedgers with basis risk and correlation risk, which cannot be fully eliminated.

What are standard options?

Options are more flexible than forwards and futures because they protect the buyer against unfavorable outcomes, but allow him to enjoy the benefits associated with favorable outcomes. This price to be paid for this win-win position is called the option premium. A standard or a vanilla option is a security that gives its holder the right to buy or sell the underlying asset

within a specified period of time, at a given price, called the strike price, striking price or exercise price. The right to buy is a call and the right to sell is a put. A call is *in-the-money* when the underlying asset price is higher than the strike price. It is *out-of-the-money*, if the underlying asset price is lower than the strike price. The call is *at-the-money*, if the underlying asset price is equal to the strike price. A put is *in-the-money* when the underlying asset price is lower than the strike price. It is *out-of-the-money*, if the underlying asset price is higher than the strike price. The put is *at-the-money*, if the underlying asset price is equal to the strike price. These definitions apply at maturity and at each instant before expiration. A European style option can be exercised only on the last day of the contract, called the maturity date or the expiration date. An American style option can be exercised at any time during the contract's life.

This chapter deals with the main strategies of derivatives markets and the pricing of assets and options in a discrete-time setting.

Using the definition of a standard or a plain vanilla option, it is evident that the higher the underlying asset price, the greater the call's value. When the underlying asset price is much greater than the strike price, the current option value is nearly equal to the difference between the underlying asset price and the discounted value of the strike price. The discounted value of the strike price is given by the price of a pure discount bond, maturing at the same time as the option, with a face or nominal value equal to the strike price. Hence, if the maturity date is very near, the call's value (put's value) is nearly equal to the difference between the underlying asset price and the strike price or zero.

If the maturity date is very far, then the call's value is nearly equal to that of the underlying asset since the bond's price will be very low. The call's value cannot be negative and cannot exceed the underlying asset price.

In the first part of this chapter, we develop the basic strategies and synthetic option positions: long or short the underlying asset, long a call, long a put and short a put. Then we present some combinations and more elaborated strategies as: long a straddle, short a straddle, long or short a strangle, long a tunnel, short a tunnel, long a call or put bull spread, long a call or a put bear spread, long or short a butterfly, long or short a condor, etc. This first part allows the reader to understand the main strategies and risk–return trade-offs in option markets.

In the second part of this chapter, we introduce the main concepts regarding asset pricing in a discrete-time setting. These models are proposed because they can be used in the pricing of options.

Portfolio theory refers to the work of Markowitz (1952) on portfolio selection. Investors prefer to increase their wealth and to minimize the risk linked with the potential gain. It is not possible to obtain the maximum expected return and the minimum variance. According to Markowitz (1952), “The portfolio with maximum expected return is not necessarily the one with minimum variance. There is a rate at which the investor can gain expected return by taking on variance, or reduce variance by giving up expected return”.

The limitations of the standard portfolio theory and mainly its restrictive assumptions, lead to the extensions of the theory in several direction. The most notable extension is the introduction of information uncertainty and its effects on the pricing of assets. In fact, the acquisition of information and its transmission to other agents are central activities in all areas of finance. Recognition of the different speeds of information diffusion is important in empirical research also. The perfect market model can provide a good description of the financial system in the long run. The analysis in Merton (1987) shows that a reconciling of finance theory with empirical violations of the complete-information, perfect market model need not imply a departure from the standard paradigm. However, as it appears in Merton (1987), “It does, however suggest that researchers be cognizant of the insensitivity of this model to institutional complexities and.... I believe that even a modest recognition of institutional structures and information costs can go a long way toward explaining financial behavior that is otherwise seen anomalous to the standard friction-less-market model”. For these reasons, we present also Merton’s (1987) simple model of capital market equilibrium with incomplete information. This model can be easily applied for the valuation of options and futures contracts as in Bellalah and Jacquillat (1995).

In the third part of this chapter, we deal with the valuation of derivative assets in a discrete-time setting using the binomial approach. The valuation of options in a discrete time setting is more pedagogical than in a continuous time setting. Ironically enough, however, the more complex approach, namely the Black–Scholes (1973) one, was discovered before the simple binomial approach. Even if the discrete-time approach is not always computationally efficient, option valuation with the lattice approach is very flexible. It can handle many situations where no analytical solutions are possible. We are interested in the lattice approach pioneered by Cox, Ross and Rubinstein (1979). These authors proposed a binomial model in a discrete-time setting for the valuation of options.

2. Basic Strategies and Synthetic Positions Using Standard Options

This section develops the main standard option strategies and synthetic option positions.

2.1. Options and Synthetic Positions

Synthetic positions can be constructed by options on spot assets, options on futures contracts and their underlying assets. If we use 0 to denote a horizontal line, -1 for the slope under 0 and 1 for the slope above 0, then it is possible to represent the diagram pay-offs of a long call by $(0, 1)$, a short call by $(0, -1)$, a long put by $(-1, 0)$ and a short put by $(1, 0)$. Adopting this notation for the basic option pay-offs, it is possible to construct all the synthetic positions as well as most elaborated diagram strategies using this representation. We denote by S , (F) : price of the underlying asset, which may be a spot asset, S (or a futures contract F),

K : strike price,

C : call price,

P : put price.


We use the following symbols


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
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
\: -1 .


The results of the basic strategies can be represented as follows:


Long a call: $(0, 1)$: 

Short a call: $(0, -1)$: 

Long a put: $(-1, 0)$: 

Short a put: $(1, 0)$: 

Long the underlying asset: $(1, 1)$ 

Short the underlying asset: $(-1, -1)$ 

The symbols $(-1, 0, 1)$ refer to a downward movement, (-1) , a flat position (0) or an upward movement (1) . The risk-return trade-off of the basic strategies can be represented using the different symbols. Using the above notations, it is possible to construct the risk-reward trade-off of any option

strategy. For example, long a call $(0, 1)$ and short a put $(1, 0)$ are equivalent to long the underlying asset $(1, 1)$. Also, short a call $(0, -1)$ and long a put $(-1, 0)$ are equivalent to a short position in the underlying asset $(-1, -1)$. We give the basic synthetic positions when the options have the same strike prices and maturity dates.

Long a synthetic underlying asset = long a call + short a put
 $(1, 1) = (0, 1) + (1, 0)$

Short a synthetic underlying asset = short a call + long a put
 $(-1, -1) = (0, -1) + (-1, 0)$

Long a synthetic call = long the underlying asset + long a put
 $(0, 1) = (1, 1) + (-1, 0)$

Short a synthetic call = short the underlying asset + short a put
 $(0, -1) = (-1, -1) + (1, 0)$

Long a synthetic put = short the underlying asset + long a call
 $(-1, 0) = (-1, 1) + (0, 1)$

Short a synthetic put = long the underlying asset + short a call
 $(1, 0) = (1, 1) + (0, -1)$

The knowledge of synthetic positions is necessary for market participants since it allows the implementation of hedged positions. When managing an option position, buying a call and a put with the same strike price are two equivalent strategies since when buying a call, the trader or the market maker hedges his transaction by the sale of the underlying asset and when buying a put, he hedges his transaction by purchasing the underlying asset. Buying the call and selling the put are equivalent to a long put with the same strike price. This transaction enables the trader or market maker to make a direct sale of the put since a position in a long call and a short put is equivalent to a long position in the underlying asset.

2.2. Long or Short the Underlying Asset

The risk-return profile for a position which is long or short the underlying asset (for example, a futures contract) shows unlimited profit or loss. If we represent the underlying asset price with a horizontal line and the profit or loss with a vertical line, the pay-off to a long or a short position in the underlying asset can be easily represented. If the asset price rises or falls by one point, the profit or loss will be of the same amount.

2.3. Long a Call

Expectations: The trader expects a rising market and (or) a high volatility until the maturity date.

Definition: Buy a call c with a strike price K .

Specific features: The potential gain is not limited and the potential loss is limited to the option premium.

Buying the call at 1.9 reveals the risk-reward profile given in Figure 1 at expiration.

Table 1: Long a call: $S = 102$, $r = 5\%$, volatility = 20%, $T = 100$ days.

Type	Point	Value
Break-even point	A	$K + c$
Maximal loss		c
Maximal gain		Not limited if $S > K$

Options	$Q: 10$	Long a call: 1.9
	Strike price =	110
	Prime =	1.9
	Cost =	19
	Break-even point =	111.9

Q : quantity

Underlying asset price S	Variation in (%)	Call	Performance in (%)
90.00	-12	-1.9	-100
95.00	-7	-1.9	-100
100.00	-2	-1.9	-100
105.00	3	-1.9	-100
110.00	8	-1.9	-100
115.00	13	3.1	165
120.00	18	8.1	430
125.00	23	13.1	696
130.00	27	18.1	961

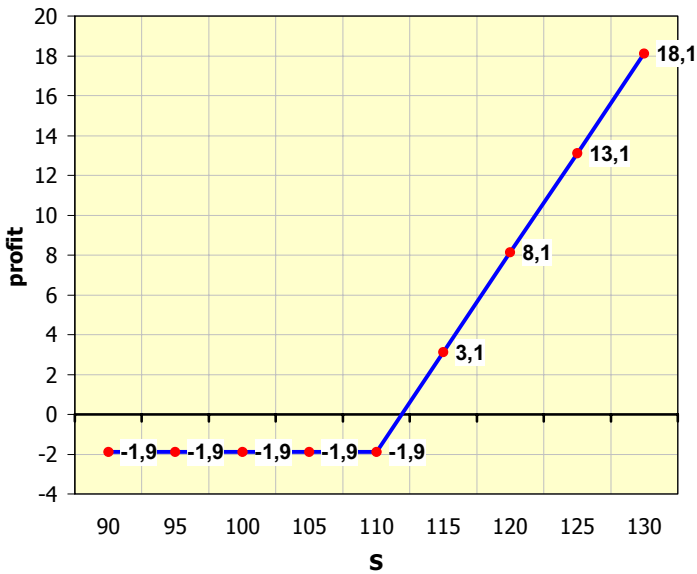


Figure 1: Long a call.

If $S = 111.9$ at maturity; $(110 + 1.9)$, the profit is zero. This is the break-even point of the position. The profit is not limited beyond this level. The maximum loss or performance corresponds to 1.9 or 100%.

In Figure 1, the break-even point is given by the sum of the strike price and the option premium.

2.4. Short a Call

Expectations: The trader expects a falling market and (or) a lower volatility until the maturity date.

Definition: Sell a call c with a strike price K .

Specific features: The potential gain is limited to the perceived premium and the potential loss is not limited. The risk-reward trade-off is inverted when selling calls. The results of the strategy are given in Figure 2.

In Figure 2, the break-even point is given by the sum of the strike price and the option premium.

Table 2: Short a call: $S = 102$, $r = 5\%$, volatility = 20%, $T = 100$ days.

Type	Point	Value
Break-even point	A	$K + c$
Maximal loss		Not limited
Maximal gain		Premium

Options	$Q: 10$	Short a call: 1.9
	Strike price =	110
	Premium =	1.9
	Profit =	19
	Break-even point =	111.9

S	Variation (%)	Call	Performance (%)
90.00	-12	1.9	100
95.00	-7	1.9	100
100.00	-2	1.9	100
105.00	3	1.9	100
110.00	8	1.9	100
115.00	13	-3.1	-165
120.00	18	-8.1	-430
125.00	23	-13.1	-696
130.00	27	-18.1	-961

2.5. Long a Put

Expectations: The trader expects a falling market and (or) a higher volatility until the maturity date.

Definition: Buy a put p with a strike price K .

Specific features: The potential gain is not limited and the potential loss is limited to the option premium.

In Figure 3, the break-even point is given by the algebraic sum of the strike price and the option premium.

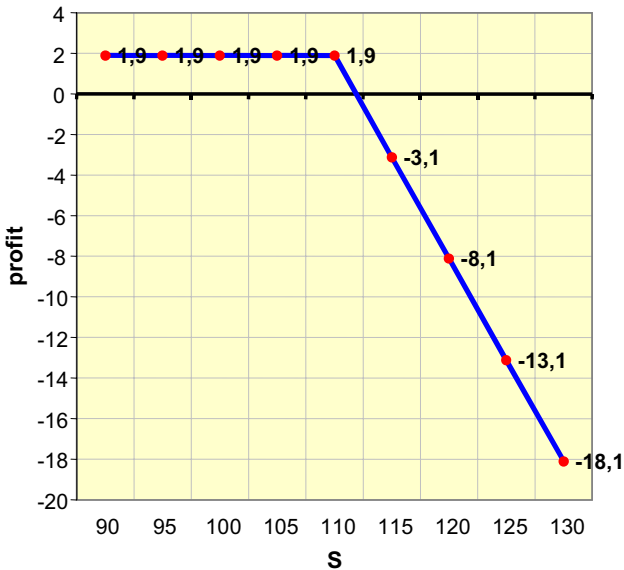


Figure 2: Short a call.

2.6. Short a Put

Expectations: The trader expects a stable and (or) a rising market.

Definition: Sell a put p with a strike price K .

Specific features: The potential gain is limited to the option premium and the potential loss is unlimited.

Figure 3 represents in a certain way the opposite of the risk-reward profile in Figure 2. The profit is limited when the underlying asset price increases and the risk is unlimited when the underlying asset price is decreases.

3. Combined Strategies

This section illustrates several combined strategies involving call and put options. The main features of each strategy are provided.

3.1. Long a Straddle

Expectations: The trader expects a high volatility until the maturity date.

Table 3: Long a put: $S = 102$, $r = 5\%$, volatility = 20%, $T = 100$ days.

Type	Point	Value
Break-even point	A	$K - p$
Maximal loss		Premium
Maximal gain		Not limited

Options	$Q: 10$	Long a put: 2.7
	Strike price =	100
	Prime =	2.7
	Cost =	27
	Break-even point =	97.28

S	Variation (%)	Put	Performance (%)
60.00	-41	37.3	1371
70.00	-31	27.3	1003
80.00	-22	17.3	635
90.00	-12	7.3	268
100.00	-2	-2.7	-100
110.00	8	-2.7	-100
120.00	18	-2.7	-100
130.00	27	-2.7	-100
140.00	37	-2.7	-100

Definition: Buy a call, c and simultaneously buy a put, p on the same underlying for the same maturity date and the same strike price.

Specific features:

- The initial investment is important since the investor buys simultaneously the call and the put.
- The loss is limited to the initial cost (c and p).
- The maximum potential gain is not limited when the market goes up or down.

Buying a straddle needs a simultaneous purchase of call and a put with the same strike price for the same maturity. When the put is worthless, the call is *deep-in-the-money*. When the call is worthless, the put is *in-the-money*.

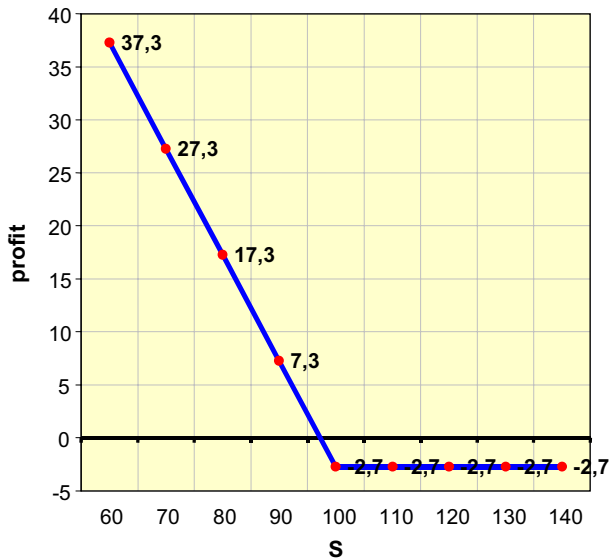


Figure 3: Long a put.

Notes:

- The strike price is chosen according to the trader expectations about the future market direction.
- Simulation
 - Underlying asset $S = 102$
 - Interest rate $r(\%) = 5$
 - Volatility $(\%) = 20$
 - Maturity (in days) = 50

3.2. Short a Straddle

Expectations: The trader expects a low volatility until the maturity date.

Definition: Sell a call, c and simultaneously sell a put, p on the same underlying for the same maturity date and the same strike price.

Specific features:

- The initial revenue is limited to the option premiums.
- The loss is not limited when the market goes up or down.
- The maximum potential gain is limited to the initial premium (c and p).

Table 4: Short a put: $S = 102$, $r = 5\%$, volatility = 20%, $T = 100$ days.

Type	Point	Value
Break-even point	A	$K - p$
Maximal loss		Not limited
Maximal gain		Premium

Options	$Q: 10$	Short a put: 2.7
	Strike price =	100
	Premium =	2.7
	Profit =	27
	Break-even point =	97.28

S	Variation (%)	Put	Performance (%)
80.00	-22	-17.3	-635
85.00	-17	-12.3	-451
90.00	-12	-7.3	-268
95.00	-7	-2.3	-84
100.00	-2	2.7	100
105.00	3	2.7	100
110.00	8	2.7	100
115.00	13	2.7	100
120.00	18	2.7	100

When the underlying asset price is expected to be in a specified interval at maturity, the trader can sell simultaneously a call and a put. The profit is limited to the premium received and the risk may be unlimited.

If the underlying asset is not expected to move much either side, the investor can sell the put and the call. The maximum profit at expiration is obtained when S is in a given interval.

3.3. Long a Strangle

Expectations: The trader expects a high volatility during the options' life.

Definition:

- Buy a call with a strike price K_c .
- Buy a put with a strike price K_p .

Where the $K_p < K_c$.

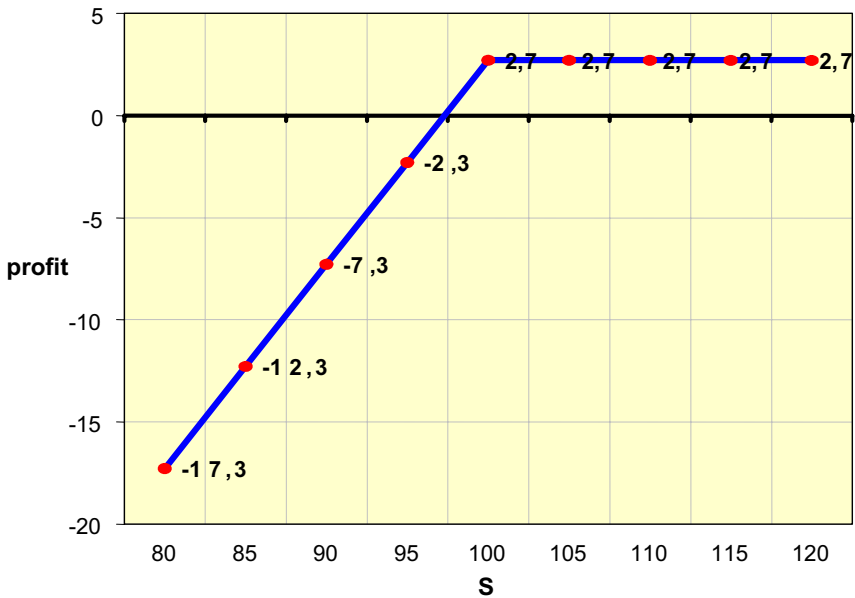


Figure 4: Short a put.

Specific features:

- This strategy costs less than the straddle.
- The maximum loss is limited to the initial cost of $(c + p)$.
- The net result is a profit only when the market movement is important. In this example, the market must increase by 5%, $(107.49 - 102)/102$ or decrease by 9%, $(92.41 - 102)/102$.

Notes:

The trader buys the 105 call and the 95 put. The theoretical prices of these options are, respectively, 2.04 and 0.55, or a total of 2.58. The quantity is 10 and the total cost of the strategy is 25.8.

The two break-even points are computed as follows:

- $105 + (2.04 + 0.55) = \mathbf{107.59}$ or a variation of 5.38%.
- $95 - (2.04 + 0.55) = \mathbf{92.41}$ or a variation of -9.40%.

If the underlying asset price is between the two strike prices at expiration, the maximum loss is reduced to the initial cost 25.8. The net result is a loss if the underlying asset price is between the two break-even points, 92.41

Table 5: Long a straddle.

Type	Point	Value
Break-even point	A	$S = K - (c + p)$
	B	$S = K + (c + p)$
Maximal loss	C	$(c + p)$ if $S = K$
Maximal gain		$K - (c + p)$ if S tends toward 0 Limited if S is beyond the limits

Options	Q	1	1	Strategy
	10	Long a call	Long a put	
Strike price =		100	100	
Prime =		4.5	1.8	6.3
Cost =		45	18	63
Break-even point =		104.50	98.18	5

S	Variation (%)	Call	Put	Straddle	Performance (%)
80.00	-22	-4.5	18.2	13.7	216
85.00	-17	-4.5	13.2	8.7	137
90.00	-12	-4.5	8.2	3.7	58
95.00	-7	-4.5	3.2	-1.3	-21
100.00	-2	-4.5	-1.8	-6.3	-100
105.00	3	0.5	-1.8	-1.3	-21
110.00	8	5.5	-1.8	3.7	58
115.00	13	10.5	-1.8	8.7	137
120.00	18	15.5	-1.8	13.7	216

and 107.59. This loss is less than the initial cost. However, if the underlying asset price is above the break-even points, either side, the trader benefits from the leverage effect. For example, if the underlying asset price is 90 at expiration, or a variation of 12%, the net result is 93%. If the underlying asset goes up by 18% to attain a level of 120, the net profit of 12.4, compared to 2.58, represents a performance of 480%.

Simulation

The parameters used in the simulation are: $S = 102$, $r = 5\%$, volatility = 20%, maturity = 50 days.

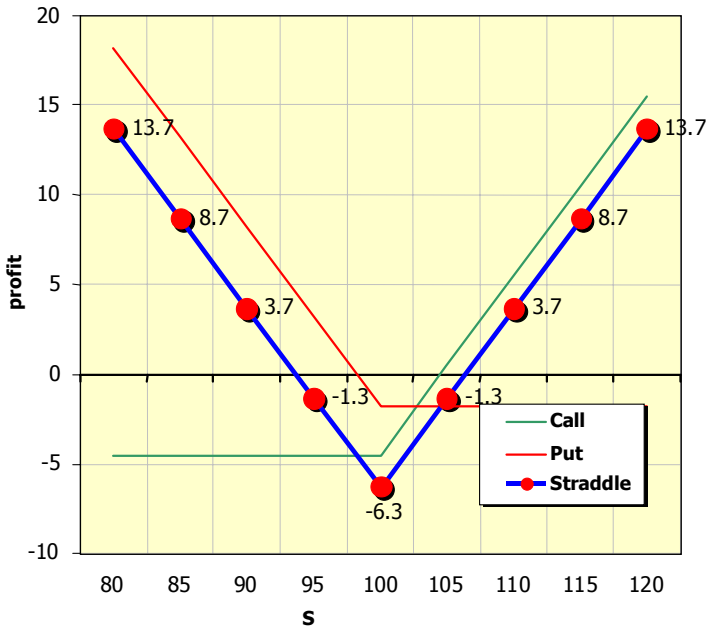


Figure 5: Buying a straddle.

Table 6: Shorting a straddle.

S	Variation (%)	Call	Put	Straddle	Performance (%)
80.00	-22	4.5	-18.2	-13.7	-216
85.00	-17	4.5	-13.2	-8.7	-137
90.00	-12	4.5	-8.2	-3.7	-58
95.00	-7	4.5	-3.2	1.3	21
100.00	-2	4.5	1.8	6.3	100
105.00	3	-0.5	1.8	1.3	21
110.00	8	-5.5	1.8	-3.7	-58
115.00	13	-10.5	1.8	-8.7	-137
120.00	18	-15.5	1.8	-13.7	-216



Figure 6: Short a straddle.

Table 7: Long a strangle.

Type	Point	Value
Break-even point	A	$S = K_p - (c + p)$
	B	$S = K_c + (c + p)$
Maximal loss		$(c + p)$ if $K_p < S < K_c$
Maximal gain	A	$K_p - (c + p)$ if S tends toward 0
	B	Limited if S is higher

	Long a call	Long a put	Strategy
Strike price	105	95	
Premium	2.04	0.55	25.8
Cost	20.4	5.5	
Break-even point	107.59	92.41	

Table 8: Profit (per unit) of a long strangle strategy.

S	Variation (%)	Call	Put	Strangle	Performance (%)
85.00	-17	-2.0	9.5	7.4	287
90.00	-12	-2.0	4.5	2.4	93
95.00	-7	-2.0	-0.5	-2.6	-100
100.00	-2	-2.0	-0.5	-2.6	-100
105.00	3	-2.0	-0.5	-2.6	-100
110.00	8	3.0	-0.5	2.4	93
115.00	13	8.0	-0.5	7.4	287
120.00	18	13.0	-0.5	12.4	480
125.00	23	18.0	-0.5	17.4	674

3.4. Short a Strangle

Expectations: The trader expects a low volatility during the options' life.

Definition: Sell a call with a strike price K_c and sell a put with a strike price K_p where the $K_p < K_c$.

Specific features: The maximum gain is limited to the initial premium of $(c + p)$ and the strategy can show a loss.

The reader can make the specific comments by comparing this strategy with the long strangle.

3.5. Long a Tunnel

Expectations: The trader expects a high volatility during the options' life.

Definition: Buy an *out-of-the money* call and sell an *out-of-the money* put as in Table 10.

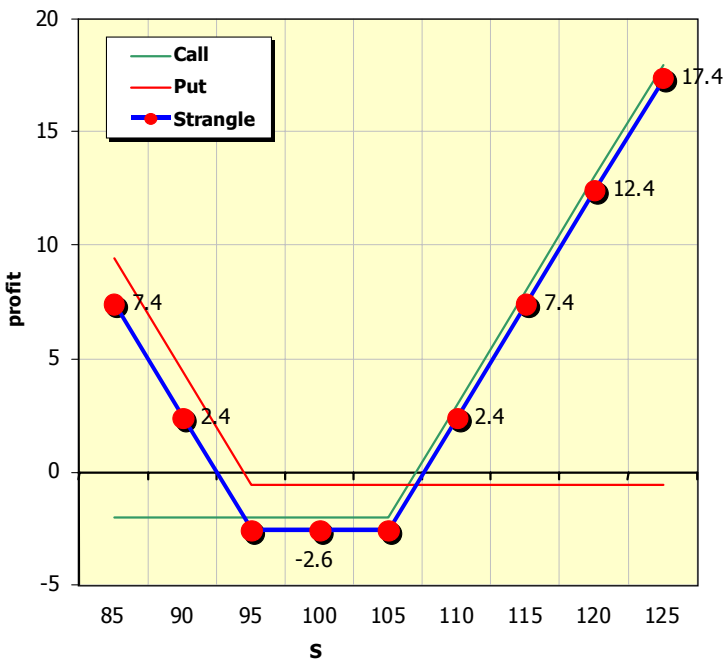


Figure 7: Profit (per unit) of a long strangle strategy.

3.6. Short a Tunnel

This is the opposite of the previous strategy.

3.7. Long a Call Bull Spread

A strategy can be implemented by buying a call with a lower strike price and selling a call with a higher strike price. If the underlying asset price is below the lower strike price at expiration, the maximum loss is limited to the difference between the two option premiums. If the underlying asset price is above the higher strike price at expiration, the lower strike price call is worth the intrinsic value. This strategy shows a limited profit (a loss).

3.8. Long a Put Bull Spread

Expectations: Buying a put spread is equivalent to buying the higher strike price put and selling the lower strike price put. If the underlying asset is

Table 9: Short a strangle.

S	Variation (%)	Call	Put	Strangle	Performance (%)
85.00	-17	2.0	-9.5	-7.4	-287
90.00	-12	2.0	-4.5	-2.4	-93
95.00	-7	2.0	0.5	2.6	100
100.00	-2	2.0	0.5	2.6	100
105.00	3	2.0	0.5	2.6	100
110.00	8	-3.0	0.5	-2.4	-93
115.00	13	-8.0	0.5	-7.4	-287
120.00	18	-13.0	0.5	-12.4	-480
125.00	23	-18.0	0.5	-17.4	-674

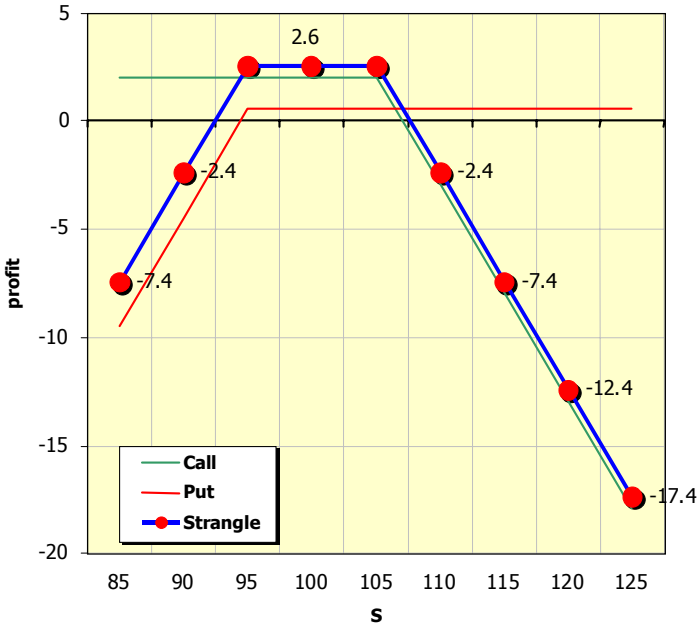


Figure 8: Short a strangle.

around the lower strike price at maturity, the higher strike price put is worth the intrinsic value and the lower strike price is worthless. The maximum profit is given by the difference between the two option premiums. The strategy is done with a debit. The trader can sell the put spread by selling the

Table 10:

Options	10	Long a call	Short a put	
Strike price =		570	550	
Premium =		22.3	15.3	7.0
Cost =		223	153	-70
Break-even point =		592.32	534.68	

Underlying asset S	Variation (%)	Call out-of-the money	Put out-of-the money	Tunnel	Performance (%)
530.00	-5	-22.3	-4.7	-27.0	-386
540.00	-4	-22.3	5.3	-17.0	-243
550.00	-2	-22.3	15.3	-7.0	-100
560.00	0	-22.3	15.3	-7.0	-100
570.00	2	-22.3	15.3	-7.0	-100
580.00	4	-12.3	15.3	3.0	43
590.00	5	-2.3	15.3	13.0	186
600.00	7	7.7	15.3	23.0	329
610.00	9	17.7	15.3	33.0	471

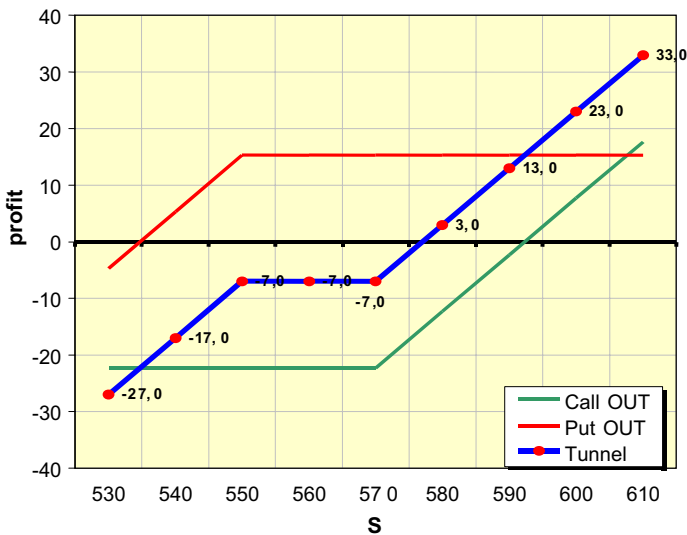


Figure 9: Long a tunnel (buy an out-of-the money call and sell an out-of-the money put).

Table 11:

Options	Q	1	1	
	10	Short a call	Long a put	
Strike price =		570	550	
Premium =		22.3	15.3	7.0
Cost =		223	153	-70
Break-even Point =		592.32	534.68	10

Underlying asset	Variation (%)	Call OUT	Put OUT	Tunnel	Performance (%)
530.00	-5	22.3	4.7	27.0	386
540.00	-4	22.3	-5.3	17.0	243
550.00	-2	22.3	-15.3	7.0	100
560.00	0	22.3	-15.3	7.0	100
570.00	2	22.3	-15.3	7.0	100
580.00	4	12.3	-15.3	-3.0	-43
590.00	5	2.3	-15.3	-13.0	-186
600.00	7	-7.7	-15.3	-23.0	-329
610.00	9	-17.7	-15.3	-33.0	-471

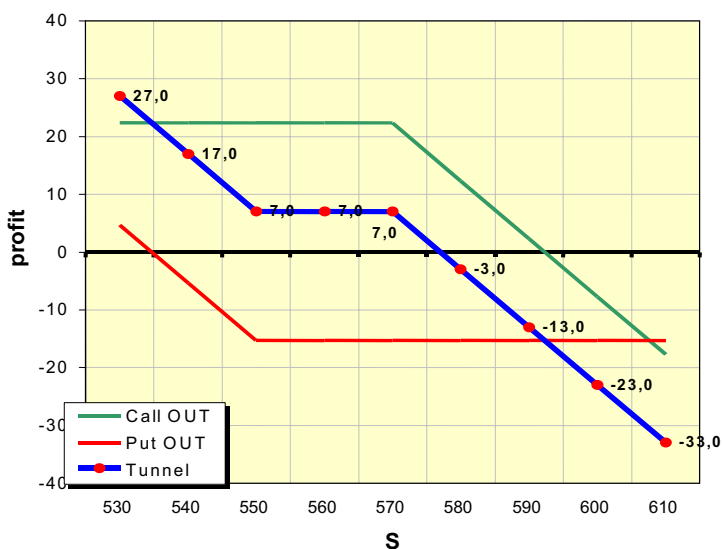

 Figure 10: Short a tunnel (sell an *out-of-the money* call and buy an *out-of-the money* put).

Table 12: Bull spread with calls $S = 102, r = 5\%, \text{volatility} = 20\%, T = 100$ days.

S	Variation (%)	Call IN	Call OUT	Spread	Performance (%)
490.00	-14	-19.9	11.0	-8.9	-100
510.00	-11	-19.9	11.0	-8.9	-100
530.00	-7	-19.9	11.0	-8.9	-100
550.00	-4	-19.9	11.0	-8.9	-100
570.00	0	-19.9	11.0	-8.9	-100
590.00	3	0.1	11.0	11.1	125
610.00	7	20.1	-9.0	11.1	125
630.00	10	40.1	-29.0	11.1	125
650.00	14	60.1	-49.0	11.1	125

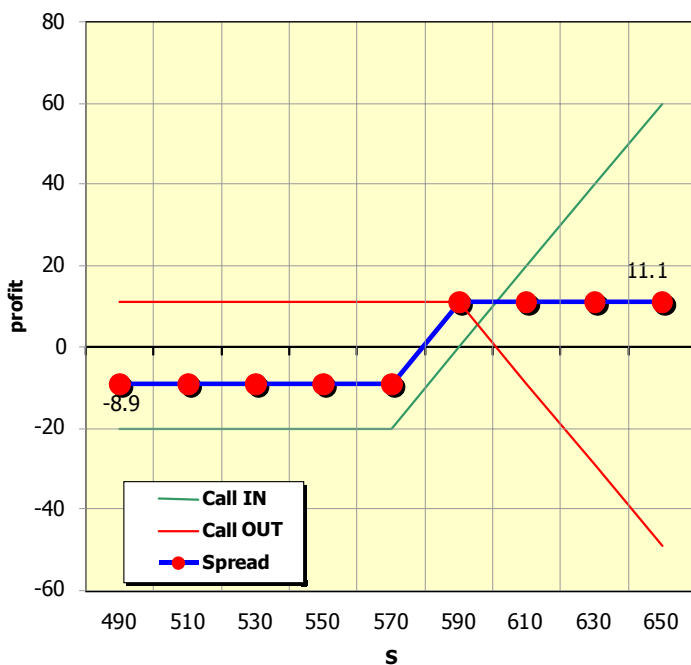


Figure 11: Buying a bull spread with calls.

Table 13: Buying a bull spread with puts.

S	Variation (%)	Put OUT	Put IN	Spread	Performance (%)
490.00	-14	66.0	-75.0	-9.0	82
510.00	-11	46.0	-55.0	-9.0	82
530.00	-7	26.0	-35.0	-9.0	82
550.00	-4	6.0	-15.0	-9.0	82
570.00	0	-14.0	5.0	-9.0	82
590.00	3	-14.0	25.0	11.0	-100
610.00	7	-14.0	25.0	11.0	-100
630.00	10	-14.0	25.0	11.0	-100
650.00	14	-14.0	25.0	11.0	-100

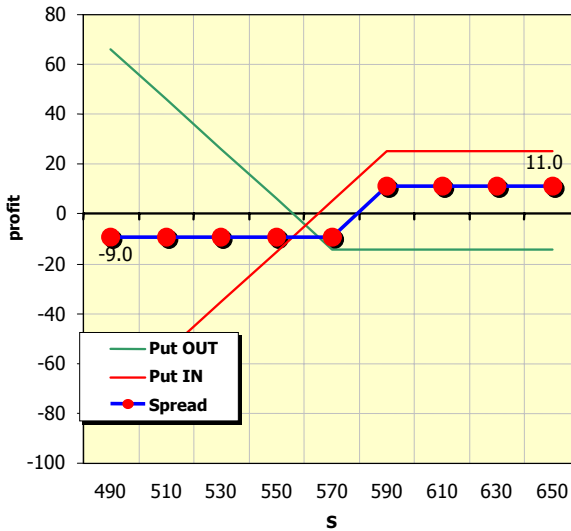


Figure 12: Buying a bull spread with puts.

higher strike price put and buying the lower strike price put. The strategy is done with a credit.

3.9. Short a Call Bear Spread

The strategy is illustrated in Figure 13. The investor buys a call with a strike K_1 and sells a call with a strike K_2 with $K_1 < K_2$.

Table 14: Selling a call bear spread.

S	Variation (%)	Call IN	Call OUT	Spread	Performance (%)
490.00	-14	18.8	-10.3	8.5	100
510.00	-11	18.8	-10.3	8.5	100
530.00	-7	18.8	-10.3	8.5	100
550.00	-4	18.8	-10.3	8.5	100
570.00	0	18.8	-10.3	8.5	100
590.00	4	-1.2	-10.3	-11.5	-134
610.00	7	-21.2	9.7	-11.5	-134
630.00	11	-41.2	29.7	-11.5	-134
650.00	14	-61.2	49.7	-11.5	-134

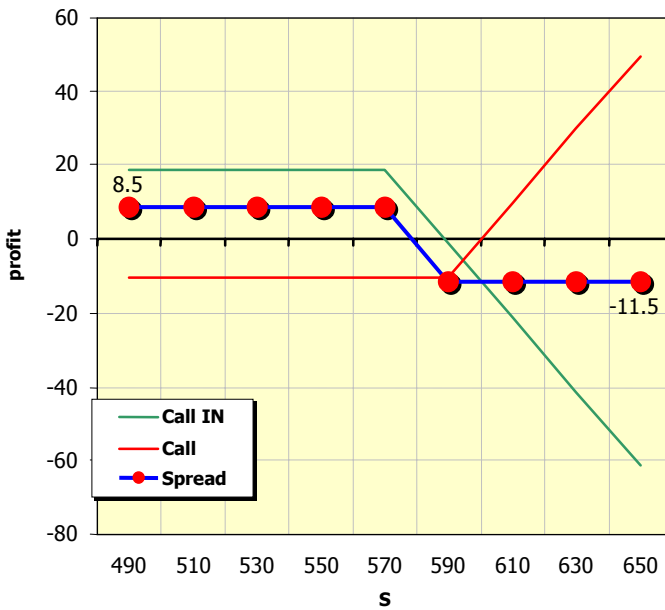


Figure 13: Selling a call bear spread.

3.10. Shorting a Put Bear Spread

Table 15: Selling a put bear spread.

S	Variation (%)	Put OUT	Put IN	Spread	Performance (%)
490.00	-14	-65.1	73.8	8.7	-77
510.00	-11	-45.1	53.8	8.7	-77
530.00	-7	-25.1	33.8	8.7	-77
550.00	-4	-5.1	13.8	8.7	-77
570.00	0	14.9	-6.2	8.7	-77
590.00	4	14.9	-26.2	-11.3	100
610.00	7	14.9	-26.2	-11.3	100
630.00	11	14.9	-26.2	-11.3	100
650.00	14	14.9	-26.2	-11.3	100

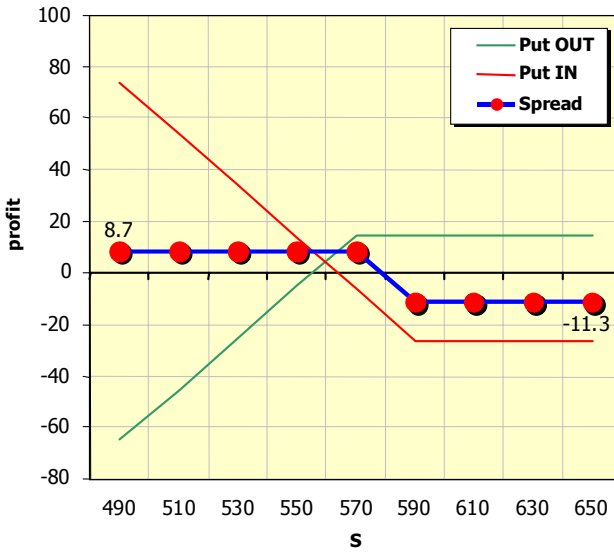


Figure 14: Selling a put bear spread.

3.11. Long a Butterfly

Anticipation: The strategy consists in buying a call with a strike K_1 , selling two calls with strikes K_2 and buying a call with strike K_3 with $K_1 < K_2 < K_3$.

Table 16: Long a butterfly.

S	Variation (%)	Call OUT	Call AT	Call IN	Butterfly	Performance (%)
80.00	-22	-12.7	9.0	-0.7	-4.5	-20
85.00	-17	-12.7	9.0	-0.7	-4.5	-20
90.00	-12	-12.7	9.0	-0.7	-4.5	-20
95.00	-7	-7.7	9.0	-0.7	0.5	2
100.00	-2	-2.7	9.0	-0.7	5.5	25
105.00	3	2.3	-1.0	-0.7	0.5	2
110.00	8	7.3	-11.0	-0.7	-4.5	-20
115.00	13	12.3	-21.0	4.3	-4.5	-20
120.00	18	17.3	-31.0	9.3	-4.5	-20

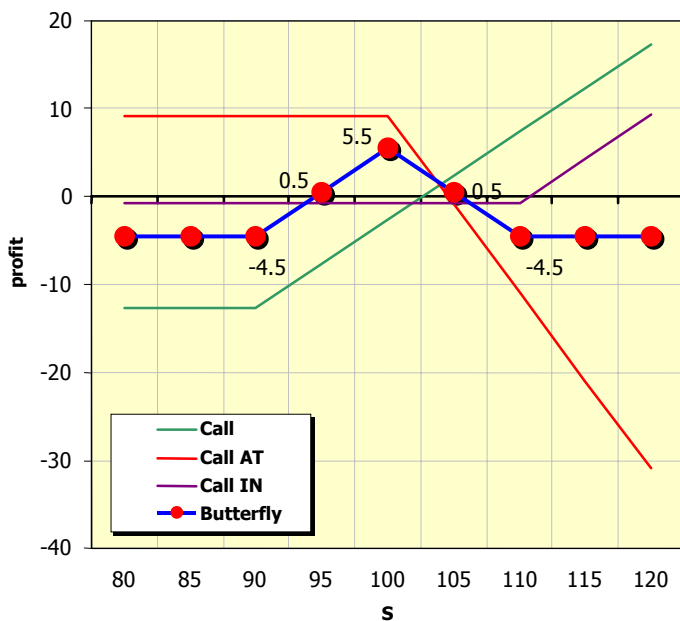


Figure 15: Long a butterfly.

Table 17: Short a butterfly.

S	Variation (%)	Call OUT	Call AT	Call IN	Butterfly	Performance (%)
80.00	-22	12.7	-9.0	0.7	4.5	20
85.00	-17	12.7	-9.0	0.7	4.5	20
90.00	-12	12.7	-9.0	0.7	4.5	20
95.00	-7	7.7	-9.0	0.7	-0.5	-2
100.00	-2	2.7	-9.0	0.7	-5.5	-25
105.00	3	-2.3	1.0	0.7	-0.5	-2
110.00	8	-7.3	11.0	0.7	4.5	20
115.00	13	-12.3	21.0	-4.3	4.5	20
120.00	18	-17.3	31.0	-9.3	4.5	20

3.12. Short a Butterfly

Anticipation: The strategy consists in selling two calls with a strike K_1 and a strike K_3 and buying the two calls with a strike K_2 .

4. Asset Pricing in a Discrete-Time Setting: The Mean–Variance Framework

Asset pricing in a discrete time setting is often analyzed with respect to simple models of capital market equilibrium. These models are based on the concepts of risk and return. They can also be applied for the valuation of derivatives.

4.1. Risk and Return: Some Definitions

The mean–variance framework refers to a risk–return trade-off. Table 18 shows an uncertain return and its corresponding probability.

The sum of the probabilities equals one.

$$\begin{aligned} \sum_{i=1}^m P_i &= P_1 + P_2 + P_3 + P_4 + P_5 + P_6 \\ &= \frac{1}{12} + \frac{2}{12} + \frac{4}{12} + \frac{3}{12} + \frac{1}{12} + \frac{1}{12} = 1. \end{aligned}$$

The expected return from the investment can be computed as a weighted average of each uncertain return by its corresponding probability.

Table 18: The probable results of the investment according to the state of nature.

Rate of return	Probability P_i	Probability in %, P_i
$R_1 = 6\%$	$P_1 = 1/12$	8.33
$R_2 = 8\%$	$P_2 = 2/12$	16.66
$R_3 = 10\%$	$P_3 = 4/12$	33.33
$R_4 = 12\%$	$P_4 = 3/12$	25
$R_5 = 14\%$	$P_5 = 1/12$	8.33
$R_6 = 16\%$	$P_6 = 1/12$	8.33
Total	$\sum_{i=1} P_i = 1$	$\sum_{i=1} P_i = 100\%$

$$E(R) = \sum_{i=1}^n P_i R_i = 10.67\%.$$

The variance is computed as the deviations around the mean:

$$\sigma^2 = \sum_{i=1}^n P_i [[R_i - E(R)]^2].$$

Table 19 shows the procedure for the computation of the variance of returns in this context.

Table 19: Computing the variance.

P_i	R_i	$E(R)$	$(R_i - E(R))^2$	$P_i \times (R_i - E(R))^2$
1/12	6.00	10.67	21.78	1.81
2/12	8.00	10.67	7.11	1.19
4/12	10.00	10.67	0.44	0.15
3/12	12.00	10.67	1.78	0.44
1/12	14.00	10.67	11.11	0.93
1/12	16.00	10.67	28.44	2.37

Hence $\sigma^2 = 6.89$ (%).

Risk is often defined as the deviations of the expected return with respect to the mean return. The following example illustrates the procedure for the computation of risk and return. Portfolio selection consists in the selection of a portfolio with respect to the mean variance framework. The investor prefers a higher expected return for a given variance or a lower variance

for a given expected return. The computation of the variance and expected return for a portfolio with two or N assets is simple.

4.2. Portfolio Selection

Portfolio theory is based on the concepts of risk and return. Consider a portfolio with a proportion X_A of asset A and X_B of asset B, with:

$$E_A < E_B \sigma_A < \sigma_B, X_A + X_B = 1.$$

The expected return is

$$E_P = X_A E_A + X_B E_B. \quad (1)$$

The variance is

$$\sigma_p^2 = X_A^2 \sigma_A^2 + X_B^2 \sigma_B^2 + 2X_A X_B \rho_{AB} \sigma_A \sigma_B \quad (2)$$

where ρ_{AB} is the correlation coefficient between A and B. When the two assets are perfectly correlated, $\rho_{AB} = 1$, and the variance of the portfolio is:

$$\sigma_p^2 = X_A^2 \sigma_A^2 + X_B^2 \sigma_B^2 + 2X_A X_B \sigma_A \sigma_B$$

or:

$$\sigma_p^2 = (X_A \sigma_A + X_B \sigma_B)^2.$$

The standard deviation is:

$$\sigma_p = X_A \sigma_A + X_B \sigma_B.$$

Using this system for different values of ρ_{AB} in the interval $(-1, 1)$, it is possible to generate point by point all the curves reflecting the relationships between the pairs (E_p, σ_p) of a portfolio. Consider the pairs (E_p, σ_p) . There is a rate at which the investor can gain expected return by taking on variance, or reduce variance by giving up expected return. Investors choose the assets to be included in their portfolios using the mean variance framework. Figure 16 shows that for each level of risk, it is possible to determine the portfolio with the highest expected return. The points corresponding to this situation allow the definition of the efficient frontier.

For the case of a portfolio with two assets A and B where A is the risk-less asset, the expected return and the variance:

$$\begin{aligned} E_P &= X_A E_A + X_B E_B \\ \sigma_p^2 &= X_A^2 \sigma_A^2 + X_B^2 \sigma_B^2 + 2X_A X_B \rho_{AB} \sigma_A \sigma_B \end{aligned}$$

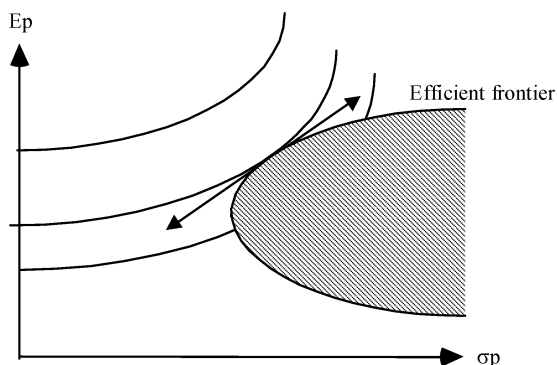


Figure 16: Portfolio choice.

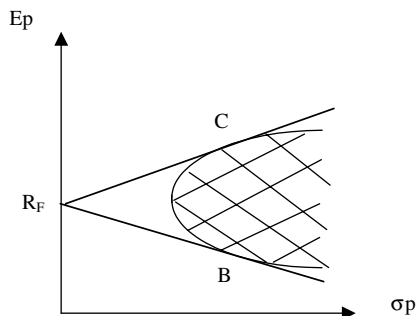


Figure 17: Introduction of a risk-free asset.

become

$$E_P = X_A E_A + X_B E_B$$

$$\sigma_p^2 = X_B^2 \sigma_B^2.$$

When a riskless asset is used, it is possible to determine the “best” combination between the riskless asset and point C on the efficient frontier.

Point C dominates all the other portfolios on the efficient frontier. It is often referred to as the market portfolio, M.

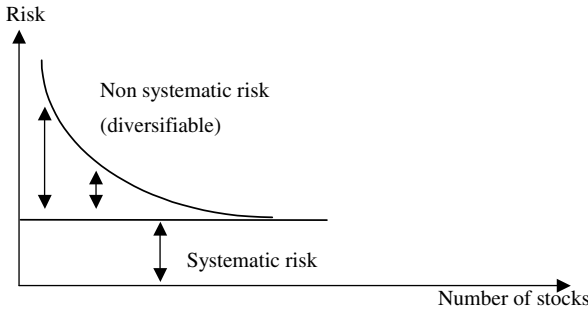


Figure 18: Diversification principal.

4.3. Systematic Risk and Diversification: An Introduction

The diversification principle is based on a relationship between risk and the number of stocks to be included in a portfolio. The idea is illustrated in Figure 18.

When investors hold the market portfolio, the contribution of each asset to the risk of a portfolio can be easily determined as in Table 20.

Table 20: Variance–covariance matrix of the market portfolio.

Stock	1	2	⋮	N
1	$X_1^2 \sigma_1^2$	$X_1 X_2 \text{Cov}(R_1, R_2)$	⋮	$X_1 X_N \text{Cov}(R_1, R_N)$
2	$X_2 X_1 \text{Cov}(R_2, R_1)$	$X_2^2 \sigma_2^2$		$X_2 X_N \text{Cov}(R_2, R_N)$
3	$X_3 X_1 \text{Cov}(R_3, R_1)$	$X_3 X_2 \text{Cov}(R_3, R_2)$	⋮	$X_3 X_N \text{Cov}(R_3, R_N)$
⋮	⋮	⋮	⋮	⋮
N	$X_N X_1 \text{Cov}(R_N, R_1)$	$X_N X_2 \text{Cov}(R_N, R_2)$	⋮	$X_N^2 \sigma_N^2$

For example, the contribution of asset 3 in line 3 shows its covariance with the other assets. Line 3 can be written as:

$$\begin{aligned} & X_3 X_1 \text{cov}(R_1, R_3) + X_3 X_2 \text{cov}(R_3, R_2) + X_3^2 \text{cov}(R_3, R_3) \\ & + X_3 X_4 \text{cov}(R_3, R_4) + \cdots + X_3 X_N \text{cov}(R_3, R_N) \\ & = X_3 [X_1 \text{cov}(R_3, R_1) + X_2 \text{cov}(R_3, R_3) \\ & + X_4 \text{cov}(R_3, R_4) + \cdots + X_N \text{cov}(R_3, R_N)]. \end{aligned}$$

This is the contribution of asset 3 to the global risk of the portfolio weighted by the fraction of this asset in the value of the portfolio.

The contribution of an asset to the risk of the market portfolio is measured by its covariance $\text{cov}(R_i, R_M)$ with the market portfolio and its beta. The beta is given by: $\beta_i = [\text{Cov}(R_i, R_M)]/[\sigma^2(R_M)]$ or $\beta_i = \rho_{iM}(\sigma_i/\sigma_M)$.

5. Asset Pricing Models in Discrete Time: The Capital Asset Pricing Model, CAPM and the CAPMI of Merton (1987)

At equilibrium, there is a simple relationship between the expected return and risk, given by the beta of an asset.

5.1. The Capital Asset Pricing Model, CAPM

When short sales are allowed and the investor can borrow and lend at the riskless rate, the composition of the optimal portfolio requires the maximization of the slope between a given portfolio and the risk free rate, or when the partial derivative with respect to the assets in the portfolio is set to zero, we obtain the following system of simultaneous equations:

$$(\bar{R}_k - R_F) = x_1 \sigma_{1k} + x_2 \sigma_{2k} + \Lambda + x_k \sigma_k^2 + x_{N-1} \sigma_{N-1,k} + x_N \sigma_{N,k}.$$

Since investors have homogeneous expectations regarding the optimal portfolio, the right-hand side of this equality can be written as:

$$(\bar{R}_k - R_F) = \gamma \text{cov}(R_k, R_m). \quad (3)$$

We can check that the following quantity

$$(\bar{R}_k - R_F) = \gamma(x_1 \sigma_{1k} + x_2 \sigma_{2k} + \Lambda + x_k \sigma_k^2 + x_{N-1} \sigma_{N-1,k} + x_N \sigma_{N,k}) \quad (4)$$

is equivalent to:

$$(\bar{R}_m - R_F) = \gamma \text{cov}(R_m, R_m).$$

Recall that the return on the market portfolio is given by:

$$\sum_{i=1}^N X_{*i} R_i$$

where X_{*i} correspond to the weights invested in the market portfolio. The covariance between the return an asset k and the market portfolio can be written as:

$$\text{cov}(R_k, R_m) = E\{(R_k - \bar{R}_k)(R_m - \bar{R}_m)\}$$

or:

$$\text{cov}(R_k, R_m) = E\left\{(R_k - \bar{R}_k) \left(\sum_{i=1}^N X_{*i} R_i - \sum_{i=1}^N X_{*i} \bar{R}_i \right)\right\}.$$

Factoring by the quantity $\sum_{i=1}^N X_{*i}$ gives:

$$\text{cov}(R_k, R_m) = E\left\{(R_k - \bar{R}_k) \left(\sum_{i=1}^N X_{*i} (R_i - \bar{R}_i) \right)\right\}.$$

Developing the terms within the expectation operator gives:

$$\begin{aligned} \text{cov}(R_k, R_m) = E\{ & (R_k - \bar{R}_k) X_{*1} (R_1 - \bar{R}_1) + (R_k - \bar{R}_k) X_{*2} (R_2 - \bar{R}_2) \\ & + K + (R_k - \bar{R}_k) X_{*k} (R_k - \bar{R}_k) \\ & + K + (R_k - \bar{R}_k) X_{*N} (R_N - \bar{R}_N)\}. \end{aligned}$$

Factoring by X_{*i} and applying the expectation operator provides:

$$\begin{aligned} \text{cov}(R_k, R_m) = & X_{*1} E\{(R_k - \bar{R}_k)(R_1 - \bar{R}_1)\} \\ & + X_{*2} E\{(R_k - \bar{R}_k)(R_2 - \bar{R}_2)\} \\ & + X_{*3} E\{(R_k - \bar{R}_k)(R_3 - \bar{R}_3)\} \\ & + \cdots + X_{*k} E\{(R_k - \bar{R}_k)(R_k - \bar{R}_k)\} \\ & + \cdots + X_{*N} E\{(R_k - \bar{R}_k)(R_N - \bar{R}_N)\}. \end{aligned}$$

When these terms are compared with the right-hand side of Eq. (4), we see that they are the same. Hence, Eq. (4) can be written as: $\gamma \text{cov}(R_k, R_m) = \bar{R}_k - R_F$.

The equality is verified for each asset and portfolio and in particular for the market portfolio: $(\bar{R}_m - R_F) = \gamma \text{cov}(R_m, R_m)$ or: $(\bar{R}_m - R_F) = \gamma \sigma_m^2$. Hence, $\gamma = (\bar{R}_m - R_F) / \sigma_m^2$. Replacing this value of γ in Eq. (3), we obtain:

$$\bar{R}_k = R_F + \left[\frac{(\bar{R}_m - R_F)}{\sigma_m^2} \right] \text{cov}[R_k, R_m].$$

This is the standard version of the capital asset pricing model.

5.2. The Efficient Frontier when Investors can Borrow and Lend in the Presence of Short Selling Restrictions

In this case, we face the same problem, except the fact that the weights must be positive. The maximization problem becomes: $F = (\bar{R}_p - R_F) / \sigma_p$ under the constraint: $\sum_{i=1}^N x_i = 1, x_i \geq 0$. Since the variance has terms in x^2 and products $x_i x_j$, the solution can be found using the Kuhn–Tucker conditions.

5.3. The Efficient Frontier when Investors are not Allowed to Borrow and Lend at the Risk Free Rate in the Presence of Short Selling Restrictions

In this case, we face the following maximization problem:

$$F = \sum x_i^2 \sigma_i^2 + \sum_{i=1}^N \sum_{\substack{j=1 \\ i \neq j}}^N x_i x_j \sigma_{ij}$$

under the constraint:

$$\sum_{i=1}^N x_i = 1, \quad \sum_{i=1}^N x_i \bar{R}_i = \bar{R}_p, \quad x_i \geq 0$$

for $i = 1, \dots, N$.

The efficient frontier can be determined by varying R_p between the return on the minimum variance portfolio and the return on the portfolio with the maximum variance.

5.4. Capital Market Equilibrium with Incomplete Information

Merton's (1987) model is based on the standard assumptions of frictionless markets, no transaction costs and no taxes, and borrowing and short selling without restrictions. There are n firms in the economy and N investors. Investors pay information costs λ before they include assets in their portfolios. It is important to regard information costs as: the cost of gathering and processing data, and the cost of information transmission from one party to another. In the literature of the principal agent and signalling models, the cost of transmitting information can be considerable. Investors pay information costs before they can process detailed information released from time to time about the firm. Information comes from the firm, stock market advisory services, brokerage houses, professional portfolio managers, etc. The background of information costs fits well with the theory of "generic" or "neglected" stocks which are not followed by large numbers of professional analysts. The relationship between the equilibrium market value V_k on firm k if all investors were informed about firm k and its value in the context of incomplete information is given by:

$$V_k^* = V_k \left[1 + \frac{\lambda_k}{R} \right], \quad \text{hence } V_k^* = V_k / \left[1 + \frac{\lambda_k}{R} \right].$$

The term λ_k reflects information costs on the asset k . It has dimensions of incremental expected rate of return and R refers to one plus the riskless rate. This equation shows that "the effect of incomplete information on equilibrium price is similar to applying an additional discount rate". When information is complete, the model reduces to the standard capital asset pricing model of Sharpe (1964). In fact, if we define in a standard fashion: $\beta_k = \text{cov}(\tilde{R}_k, \tilde{R}_M) / [\text{var}(\tilde{R}_M)]$, then the equilibrium expected return on security k can be written as

$$\bar{R}_k = R + \beta_k(\bar{R}_M - R - \lambda_m) + \lambda_k.$$

6. Option Pricing in a Discrete-Time Setting: The Cox, Ross and Rubinstein Model for Equity Options

Cox, Ross and Rubinstein (1979) propose the first discrete-time model for the pricing of stock options. Rendleman and Barter (1980) develop a similar model for the pricing of interest rate sensitive instruments.

6.1. The Monoperiodic Model

To illustrate the foundations of the binomial model, consider the following data:

- Underlying asset price: $S = 40$,
- Strike price: K or $E = 40$,
- Riskless interest rate: $r = 10\%$ or $R = 1 + r = 1.1$,
- Time of maturity: 1 year.

At the end of the year, the underlying stock can increase by 20%, from 40 to (40×1.2) , or 48, as it can decrease by the same amount from 40 to (40×0.8) , or 32 as in Figure 19.

The dynamics of the option is nearly similar to that of the underlying asset. The call option price at the maturity date is given by the greater of zero and the intrinsic value. As in Figure 20, the option price can go up to C_u or down to C_d .

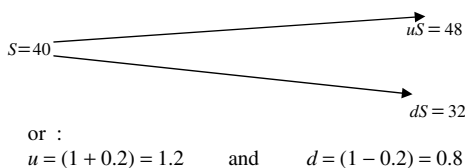


Figure 19: One period binomial model.

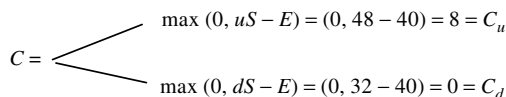


Figure 20:



Figure 21: Dynamics of the hedge portfolio.

The strike price is often denoted by K or E . It is possible to construct an initial hedge portfolio using the underlying asset S and a certain number H of options as $(S - HC)$. If this portfolio hedges the investor against risk, it must lead to the same result at the maturity date as in Figure 21.

We can compute the number H as follows:

$$H = \frac{S(u - d)}{(C_u - C_d)} = \frac{40(1.2 - 0.8)}{(8 - 0)} = 16/8 = 2.$$

When the stock price increases, the value of the hedge portfolio is:

$$uS - HC_u = 1.2(40) - 2(8) = 48 - 16 = 32.$$

When the stock price decreases, the value of the hedge portfolio is:

$$dS - HC_d = 0.8(40) - 2(0) = 32.$$

What is the option price at time 0 in this simple binomial model?

Since the initial portfolio value is $(S - HC)$, its final value must be multiplied by the riskless rate since it is a hedge portfolio. The value of a hedged portfolio at the maturity date becomes $R(S - HC)$. In order to avoid risk-less arbitrage, we must have:

$$R(S - HC) = (uS - HC_u)$$

which gives $C = (S(R - u) + HC_u)/HR$. Since the value of H is given by $H = (S(u - d))/(C_u - C_d)$. The call price is given by

$$C = \left[C_u \frac{(R - d)}{(u - d)} + C_d \frac{(u - R)}{(u - d)} \right] / R.$$

This is the option price in a mono-periodic binomial model.

Example. Using the following data:

$C_u = 8$, $C_d = 0$, $u = 1.2$, $d = 0.8$, $R = 1.1$, the option price is:

$$C = \left[8 \frac{(1.1 - 0.8)}{(1.2 - 0.8)} + 0 \frac{(1.2 - 1.1)}{(1.2 - 0.8)} \right] / 1.1 = (6 + 0)/1.1 = 5.4545.$$

The call price can also be written as:

$$C = [pC_u + (1 - p)C_d]/R$$

with $p = (R - d)/(u - d)$, $(1 - p) = (u - R)/(u - d)$ where p refers to the probability associated to an increase in the underlying asset price.

6.2. The Multiperiodic Model

This simple mono-periodic model can be repeated N times to construct the multi-periodic binomial option pricing model. Time to maturity T is divided into N intervals of length Δt where the underlying asset price increases from S to uS or decreases from S down to dS with a probability p and $(1 - p)$. In a risk-neutral world, the expected value of S is $Se^{r\Delta t}$. The expected value can also be calculated as follows:

$$pSu + (1 - p)Sd.$$

The equality between the two expected values gives:

$$Se^{r\Delta t} = pSu + (1 - p)Sd.$$

Simplifying by S gives:

$$e^{r\Delta t} = pu + (1 - p)d. \quad (5)$$

The variance of S over the same time interval $\sqrt{\Delta t}$ is $\sigma^2 S^2 \Delta t$ since the variance of a random variable X is given by $E(X^2) - E(X)^2$.

Calculating the variance and simplifying gives:

$$\sigma^2 \Delta t = pu^2 + (1 - p)d^2 - [pu + (1 - p)d]^2. \quad (6)$$

Using Eqs. (5) and (6), and $u = 1/d$, it is possible to show that the following relationships are verified:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = \frac{1}{u}, \quad m = e^{r\Delta t}, \quad p = \frac{m - d}{u - d}. \quad (7)$$

At each node, the underlying asset value can be written as $Su^j d^{i-j}$ for j varying from 0 to i . The first index i correspond to the period and the second index j indicate the position. For example, when the option's maturity date is in one period, $i = 1$ and j varies from 0 to i , i.e., 0 to 1.

Using 0 for the lowest position at each period, when the underlying asset value decreases, we have: $Su^0 d^{1-0} = Sd$. When it increases, we have: $Su^1 d^{1-1} = Su$.

The value of a European or an American option at each pair (i, j) is denoted by $F_{i,j}$. The option price at time 0 can be computed by a recursive starting from the maturity date T . The option price is given by its expected future value discounted to the present at the appropriate riskless rate.

At maturity, the payoff from a European call is:

$$F_{N,j} = \max[0, Su^j d^{N-j} - K].$$

The payoff from a European put:

$$F_{N,j} = \max[0, K - Su^j d^{N-j}].$$

The option value at each node can be computed using the two immediate successive nodes. The expected value must be discounted using the riskless rate as follows:

$$F_{i,j} = e^{-r\Delta t} \cdot [pF_{i+1,j+1} + (1-p)F_{i+1,j}]$$

for $0 \leq i \leq M-1$ and $0 \leq j \leq i$.

Since the value of an American call option must be at least equal to its intrinsic value, the following condition must be satisfied:

$$F_{i,j} = \max[Su^j d^{i-j} - K, e^{-r\Delta t}(pF_{i+1,j+1} + (1-p)F_{i+1,j})].$$

The value of an American put option must satisfy the following condition:

$$F_{i,j} = \max[K - Su^j d^{i-j}, e^{-r\Delta t}(pF_{i+1,j+1} + (1-p)F_{i+1,j})].$$

This model appears in CRR (1979), Cox and Rubinstein (1985), Hull (2000), etc.

6.3. Examples

6.3.1. Examples with Two Periods

Consider the following data for the pricing of a European call:

$$S = 100, \quad K = 100, \quad T = 1 \text{ year}, \quad N = 2, \quad \sigma = 0.2, \quad r = 0.1.$$

In a first step, the values of the model parameters must be computed:

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.2\sqrt{\frac{1}{2}}} = 1.1519, \quad d = \frac{1}{u} = e^{-0.2\sqrt{\frac{1}{2}}} = 0.8681,$$

$$m = e^{r\Delta t} = e^{0.1\sqrt{\frac{1}{2}}} = 1.0732, \quad p = \frac{m-d}{u-d} = 0.7227.$$

$$\begin{array}{rcl}
 & & Suu = 132.68 \\
 S = 100 & Su = 115.19 & Sud = 100 \\
 & Sd = 86.81 & Sdd = 75.36
 \end{array}$$

Figure 22: Dynamics of the underlying asset price.

$$\begin{array}{rcl}
 & & S_{2,2} = 132.68 \\
 S_{0,0} = 100 & S_{1,1} = 115.19 & S_{2,1} = 100 \\
 & S_{1,0} = 86.81 & S_{2,0} = 75.36
 \end{array}$$

Figure 23: Dynamics of the underlying asset price.

$$\begin{array}{rcl}
 & & C_{2,2} = 132.68 - 100 = 32.68 \\
 C_{0,0} = ? & C_{1,1} = ? & C_{2,1} = 100 - 100 = 0 \\
 & C_{1,0} = ? & C_{2,0} = 75.36 - 100 = 0
 \end{array}$$

Figure 24: Dynamics of the option price.

For an initial value of $S = 100$, the two possible values in the next period are:

$$Su = 100(1.1519) = 115.19, \quad Sd = 100(0.8681) = 86.81.$$

These two values of the underlying asset price lead to three possible values as:

$$\begin{aligned}
 Suu &= 115.19(1.1519) = 132.68, & Sud &= 115.19(0.8681) = 100, \\
 Sdd &= 86.81(0.8681) = 75.36.
 \end{aligned}$$

Using the index representation, (i, j) , we have Figure 23.

In the above representation, $S_{0,0}$ refers to the initial time 0 and the lowest position 0. $S_{1,0}$ corresponds to period 1 and the lowest position 0. $S_{1,1}$ refers to period 1 and the first position after 0, i.e., 1.

The dynamics of the option price are given in Figure 24.

The option value can be computed by simple application of the following formula:

$$F_{i,j} = e^{-r\Delta t} \cdot [pF_{i+1,j+1} + (1 - p)F_{i+1,j}].$$

The option value at node (1, 1), i.e., $C_{1,1}$ is given by:

$$C_{1,1} = e^{-r\Delta t} \cdot [pC_{2,2} + (1 - p)C_{2,1}]$$

or:

$$C_{1,1} = e^{-0.1(1/2)} \cdot [0.7227(32.68) + (1 - 0.7227)0] = 22.465.$$

The option value at node (1, 0), i.e., $C_{1,0}$ is:

$$C_{1,0} = e^{-r\Delta t} \cdot [pC_{2,1} + (1 - p)C_{2,0}]$$

or:

$$C_{1,0} = e^{-0.1(1/2)} \cdot [0.7227(22.465) + (1 - 0.7227)0] = 0.$$

Using the possible values in one period, what is the option price at time 0 ?

Using the same formula, $C_{0,0}$ is given by:

$$C_{0,0} = e^{-r\Delta t} \cdot [pC_{1,1} + (1 - p)C_{1,0}]$$

or :

$$C_{0,0} = e^{-0.1(1/2)} \cdot [0.7227(0) + (1 - 0.7227)0] = 15.443.$$

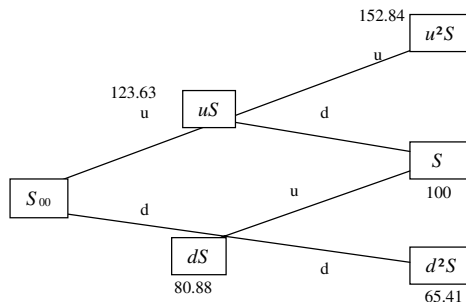
The option value at time 0 is 15.443.

6.3.2. Other Applications of the Binomial Model of Cox, Ross and Rubinstein for Two Periods

Consider the following data for the valuation of European and American call and put options:

$$S = 100, \quad K = 100, \quad r = 5\%, \quad \sigma = 30\%, \quad N = 2, \quad T = 1 \text{ year.}$$

European call prices: Using the above date, the dynamics of the underlying asset are given by:



In this case, option values are given by:

$$C_{2,2} = 52.84, \quad C_{1,1} = 0, \quad C_{2,0} = 0,$$

$$C_{1,1} = \frac{0.5064 * 52.84 + 0 * (1 - 0.5064)}{1.025} = 26.105, \quad C_{1,0} = 0$$

$$C_{0,0} = \frac{0.5064 * 26.105 + 0}{1.025} = 12.897.$$

Hence, the option price is 12.897.

European put prices: Option prices are given by:

$$P_{2,2} = 0, \quad P_{2,1} = 0, \quad P_{2,0} = 100 - 65.41 = 34.59,$$

$$P_{1,1} = \frac{0.5064 * 0 + 0}{1.025} = 0,$$

$$P_{1,0} = \frac{0.5064 * 0 + 0.4936 * 34.59}{1.025} = 16.657,$$

$$P_{0,0} = \frac{0 + 0.4936 * 16.657}{1.025} = 8.021.$$

Hence, the option price is 8.021. We can check that the put–call parity theorem is verified: $P = C + Ke^{-rt} - S = 12.897 + 100e^{-0.05*1} - 100 = 8.02$.

American call prices: Option prices are computed as:

$$C_{1,1} = \max\left(\frac{0.5064 * 52.84 + 0.4936 * 0}{1.025}; 23.63\right) = 26.105$$

$$C_{1,0} = \max\left(\frac{0.5064 * 0 + 0}{1.025}; 0\right) = 0$$

$$C_{0,0} = \frac{0.5064 * 26.105}{1.025} = 12.88.$$

American put prices: Option prices are computed as:

$$P_{1,1} = \max\left(\frac{0.5064 * 0 + 0.4936 * 0}{1.025}; 100 - 80.88\right) = 0$$

$$P_{1,0} = \max \left(\frac{0.5064 * 0 + 0.4936 * 34.59}{1.025}; 100 - 80.88 \right) = 19.12$$

$$P_{0,0} = \frac{0.5064 * 0 + 0.4936 * 19.12}{1.025} = 9.2.$$

6.3.3. Applications of the Binomial Model of Cox, Ross and Rubinstein for Three Periods

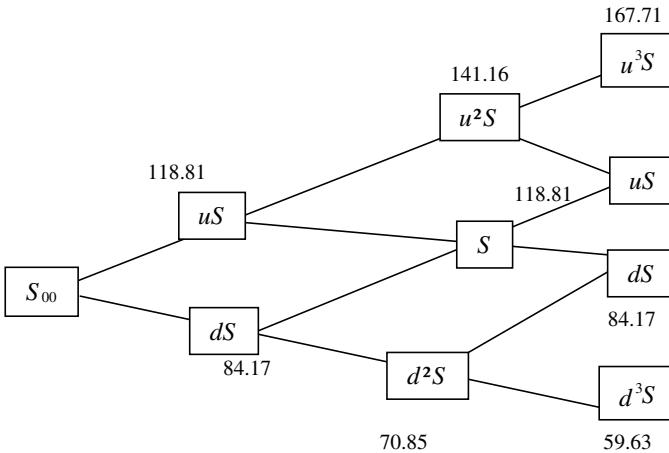
Consider the following data for the valuation of European and American options: $S = 100$, $K = 100$, $r = 5\%$, $\sigma = 30\%$, $N = 3$, $T = 1$ year.

European call prices: Using the above data, we have

$$u = e^{\sigma\sqrt{\Delta t}} \quad \text{with } N\Delta t = T \Leftrightarrow \Delta t = \frac{1}{3} = 0.33,$$

$$u = e^{0.30\sqrt{0.33}} = 1.1881, \quad d = \frac{1}{u} = \frac{1}{1.1881} = 0.8417,$$

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05*0.33} - 0.8417}{1.1881 - 0.8417} = 0.5050.$$



The option pay-off at maturity is:

$$C_{3,3} = 167.71 - 100 = 67.71, \quad C_{3,2} = 118.81 - 100 = 18.81,$$

$$C_{3,1} = 84.17 - 100 = 0, \quad C_{3,0} = 59.63 - 100 = 0.$$

Before maturity, option prices are computed as:

$$C_{2,2} = \frac{0.5050 * 67.71 + 0.495 * 18.81}{1.017} = 42.78,$$

$$C_{2,1} = \frac{0.5050 * 18.81 + 0.495 * 0}{1.017} = 9.34,$$

$$C_{2,0} = \frac{0.5050 * 0 + 0.495 * 0}{1.017} = 0,$$

$$C_{1,1} = \frac{0.5050 * 42.78 + 0.495 * 9.34}{1.017} = 25.79,$$

$$C_{1,0} = \frac{0.5050 * 9.34 + 0.495 * 0}{1.017} = 4.64,$$

$$C_{0,0} = \frac{0.5050 * 25.79 + 0.495 * 4.64}{1.017} = 15.06.$$

The option price is $C_{0,0} = 15.06$.

European put prices: The option pay-off at maturity is given by:

$$P_{3,3} = 100 - 167.71 = 0,$$

$$P_{3,2} = 100 - 118.81 = 0,$$

$$P_{3,1} = 100 - 84.17 = 15.83,$$

$$P_{3,0} = 100 - 59.63 = 40.37.$$

Before maturity, option prices are given by:

$$P_{2,2} = \frac{0.5050 * 0 + 0.495 * 0}{1.017} = 0,$$

$$P_{2,1} = \frac{0.5050 * 0 + 0.495 * 15.83}{1.017} = 7.70,$$

$$P_{2,0} = \frac{0.5050 * 15.83 + 0.495 * 40.37}{1.017} = 27.51,$$

$$P_{1,1} = \frac{0.5050 * 0 + 0.495 * 7.70}{1.017} = 3.75,$$

$$P_{1,0} = \frac{0.5050 * 7.70 + 0.495 * 27.51}{1.017} = 17.21,$$

$$P_{0,0} = \frac{0.5050 * 3.75 + 0.495 * 17.21}{1.017} = 10.24.$$

We can check the put call parity relationship in this context:

$$P = C + Ke^{-rt} - S = 15.06 + 100e^{-0.05*1} - 100 = 10.2.$$

American call prices: American call option prices are computed as follows:

$$C_{2,2} = \max\left(\frac{0.5050 * 27.71 + 0.495 * 18.81}{1.017}; 141.16 - 100\right) = 42.78$$

$$C_{2,1} = \max\left(\frac{0.5050 * 18.81 + 0.495 * 0}{1.017}; 100 - 100\right) = 9.34$$

$$C_{2,0} = \max\left(\frac{0.5050 * 0 + 0.495 * 0}{1.017}; 0\right) = 0$$

$$C_{1,1} = \max\left(\frac{0.5050 * 42.78 + 0.495 * 9.34}{1.017}; 118.8 - 100\right) = 25.79$$

$$C_{1,0} = \max\left(\frac{0.5050 * 9.34 + 0.495 * 0}{1.017}; 84.17 - 100\right) = 4.64$$

$$C_{0,0} = \frac{0.5050 * 25.79 + 0.495 * 4.64}{1.017} = 15.06.$$

American put prices: American put prices are computed as follows:

$$P_{2,2} = \max\left(\frac{0.5050 * 0 + 0.495 * 0}{1.017}; 100 - 141.16\right) = 0$$

$$P_{2,1} = \max\left(\frac{0.5050 * 0 + 0.495 * 15.83}{1.017}; 100 - 100\right) = 7.70$$

$$P_{2,0} = \max\left(\frac{0.5050 * 15.83 + 0.495 * 40.37}{1.017}; 100 - 70.85\right) = 29.15$$

$$P_{1,1} = \max\left(\frac{0.5050 * 0 + 0.495 * 7.70}{1.017}; 100 - 118.8\right) = 3.75$$

$$P_{1,0} = \max\left(\frac{0.5050 * 7.70 + 0.495 * 29.15}{1.017}; 100 - 84.17\right) = 18.01$$

$$P_{0,0} = \frac{0.5050 * 3.75 + 0.495 * 18.01}{1.017} = 10.63.$$

6.3.4. Applications of the Binomial Model of Cox, Ross and Rubinstein for Four Periods

Consider the following data for the valuation of European and American call and put options:

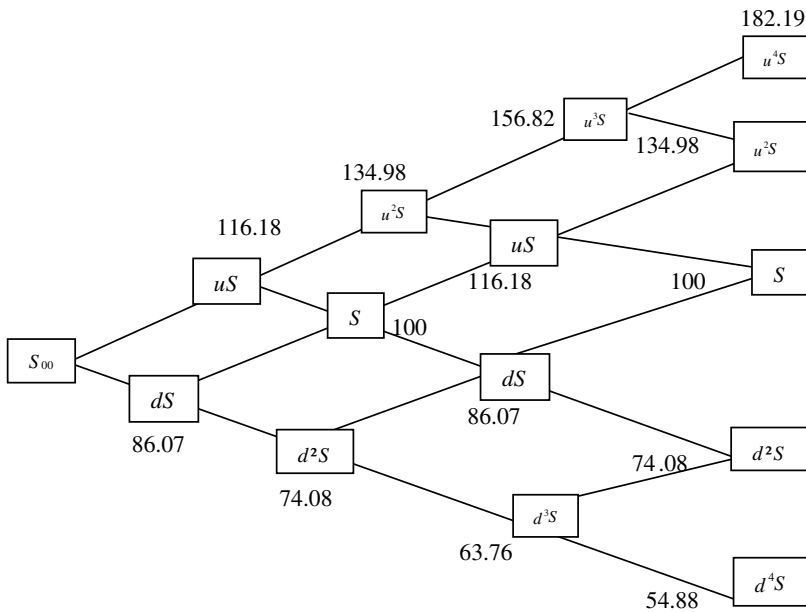
$$S = 100, \quad E = 100, \quad r = 5\%, \quad \sigma = 30\%, \quad T = 1 \text{ year.}$$

$$N = 4, \quad N * \Delta t = T \Leftrightarrow \Delta t = \frac{1}{4} = 0.25 \quad u = e^{\sigma\sqrt{\Delta t}} = e^{0.30*\sqrt{0.25}} = 1.1618$$

$$d = \frac{1}{u} = \frac{1}{1.1618} = 0.8607,$$

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05*0.25} - 0.8607}{1.1618 - 0.8607} = 0.5044.$$

European call prices: Using the above data, the dynamics of the underlying asset are given by:



Option prices at maturity are given by:

$$C_{4,4} = 182.19 - 100 = 82.19, \quad C_{4,3} = 134.98 - 100 = 34.98,$$

$$C_{4,2} = 100 - 100 = 0, \quad C_{4,1} = 74.08 - 100 = 0,$$

$$C_{4,0} = 54.88 - 100 = 0.$$

Before maturity, option prices are given by:

$$C_{3,3} = \frac{0.5044 * 82.19 + 0.4956 * 34.98}{1.013} = 58.04,$$

$$C_{3,2} = \frac{0.5044 * 34.98 + 0.4956 * 0}{1.013} = 17.42,$$

$$C_{3,1} = \frac{0.5044 * 0 + 0.4956 * 0}{1.013} = 0,$$

$$C_{3,0} = \frac{0.5044 * 0 + 0.4956 * 0}{1.013} = 0,$$

$$C_{2,2} = \frac{0.5044 * 58.04 + 0.4956 * 17.42}{1.013} = 37.42,$$

$$C_{2,1} = \frac{0.5044 * 17.42 + 0.4956 * 0}{1.013} = 8.67,$$

$$C_{2,0} = \frac{0.5044 * 0 + 0.4956 * 0}{1.013} = 0,$$

$$C_{1,1} = \frac{0.5044 * 37.42 + 0.4956 * 8.67}{1.013} = 22.87,$$

$$C_{1,0} = \frac{0.5044 * 8.67 + 0.4956 * 0}{1.013} = 4.32,$$

$$C_{0,0} = \frac{0.5044 * 22.87 + 0.4956 * 4.32}{1.013} = 13.50.$$

The European call price is 13.50.

European put prices: At maturity, the option pay-off is given by:

$$P_{4,4} = 100 - 182.19 = 0, \quad P_{4,3} = 100 - 134.98 = 0,$$

$$P_{4,2} = 100 - 100 = 0, \quad P_{4,1} = 100 - 74.08 = 25.92,$$

$$P_{4,0} = 100 - 54.88 = 45.12, \quad P_{3,3} = \frac{0.5044 * 0 + 0.4956 * 0}{1.013} = 0,$$

$$P_{3,2} = \frac{0.5044 * 0 + 0}{1.013} = 0,$$

$$P_{3,1} = \frac{0.5044 * 0 + 0.4956 * 25.92}{1.013} = 12.68,$$

$$P_{3,0} = \frac{0.5044 * 25.92 + 0.4956 * 45.12}{1.013} = 34.98.$$

Option prices are given by:

$$P_{2,2} = \frac{0.5044 * 0 + 0.4956 * 0}{1.013} = 0,$$

$$P_{2,1} = \frac{0.5044 * 0 + 0.4956 * 12.68}{1.013} = 6.2,$$

$$P_{2,0} = \frac{0.5044 * 12.68 + 0.4956 * 34.98}{1.013} = 23.43,$$

$$P_{1,1} = \frac{0.5044 * 0 + 0.4956 * 6.2}{1.013} = 3.03,$$

$$P_{1,0} = \frac{0.5044 * 6.2 + 0.4956 * 23.43}{1.013} = 14.55,$$

$$P_{0,0} = \frac{0.5044 * 22.87 + 0.4956 * 4.32}{1.013} = 8.63.$$

We can check that the put–call relationship is verified in this context:

$$P = C + Ke^{-rt} - S = 13.50 + 100e^{-0.05*1} - 100 = 8.62.$$

American call prices: American call option prices are computed as follows:

$$C_{3,3} = \max\left(\frac{0.5044 * 82.19 + 0.4956 * 34.98}{1.013}; 156.82 - 100\right) = 58.04,$$

$$C_{3,2} = \max\left(\frac{0.5044 * 34.98 + 0.4956 * 0}{1.013}; 116.18 - 100\right) = 17.42,$$

$$C_{3,1} = \max(0; 0) = 0, \quad C_{3,0} = \max(0; 0) = 0.$$

$$C_{2,2} = \max(37.42; 134.98 - 100) = 37.42,$$

$$C_{2,1} = \max(8.67; 0) = 8.67, \quad C_{2,0} = \max(0; 0) = 0,$$

$$C_{1,1} = \max(22.87; 16.18) = 22.87,$$

$$C_{1,0} = \max(4.32; 0) = 4.32,$$

$$C_{0,0} = \frac{0.5044 * 22.87 + 0.4956 * 4.32}{1.013} = 13.50.$$

American put prices: American put prices are computed as follows:

$$P_{3,3} = \max(0; 100 - 156.82) = 0,$$

$$P_{3,2} = \max(0; 100 - 116.18) = 0,$$

$$P_{3,1} = \max\left(\frac{0.5044 * 0 + 0.4956 * 25.92}{1.013}; 100 - 86.07\right) = 13.93,$$

$$P_{3,0} = \max\left(\frac{0.5044 * 25.92 + 0.4956 * 45.12}{1.013}; 100 - 63.76\right) = 36.24,$$

$$P_{2,2} = \max\left(\frac{0.5044 * 0 + 0.4956 * 0}{1.013}; 0\right) = 0,$$

$$P_{2,1} = \max\left(\frac{0.5044 * 0 + 0.4956 * 13.93}{1.013}; 0\right) = 6.81,$$

$$P_{2,0} = \max\left(\frac{0.5044 * 13.93 + 0.4956 * 36.24}{1.013}; 25.92\right) = 25.92,$$

$$P_{1,1} = \max\left(\frac{0.5044 * 0 + 0.4956 * 6.81}{1.013}; 0\right) = 3.33,$$

$$P_{1,0} = \max\left(\frac{0.5044 * 6.81 + 0.4956 * 25.92}{1.013}; 13.93\right) = 16.07,$$

$$P_{0,0} = \frac{0.5044 * 3.33 + 0.4956 * 16.07}{1.013} = 9.52.$$

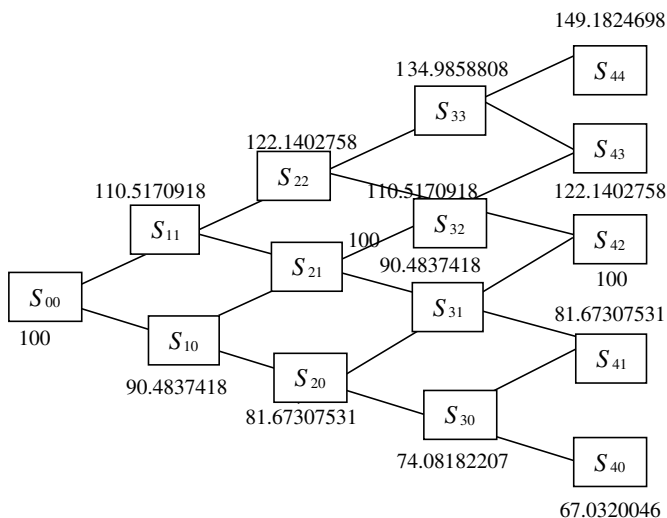
6.3.5. Other Applications of the Cox, Ross and Rubinstein for Four Periods

Consider the valuation of European and American call and put options in the following context: $S = 100$, $K = 100$, $\sigma = 20\%$, $r = 14\%$, $N = 4$, $T = 1$. Since $T = 1$, we have

$$\begin{aligned} \Rightarrow \Delta T &= \frac{T}{N} = \frac{1}{4} = 0.25, & u &= e^{0.2\sqrt{1/4}} = 1.105170916, \\ d &= \frac{1}{u} = 0.904837418, & p &= e^{r\Delta T} - d = 0.6528228716, \\ 1 - p &= 0.3471771284, & m &= e^{r\Delta T} = 1.035619709. \end{aligned}$$

The price dynamics of the underlying asset are given in the following figure:

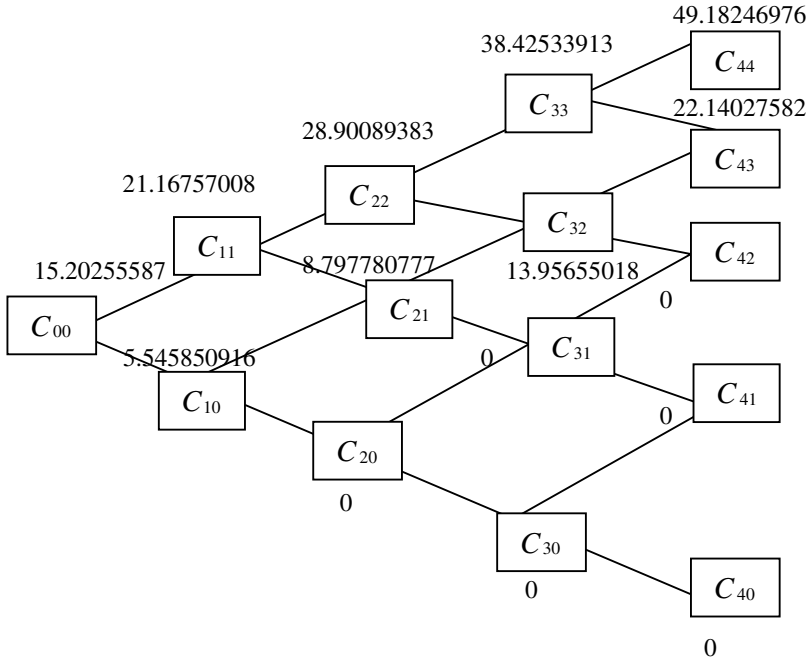
Dynamics of the underlying asset



Using the initial underlying asset value $S_{00} = 100$, we can generate all the other values using: $S_{ij} = S \cdot u^j \cdot d^{i-j}$ with $i \in \{0, 1, 2, 3, 4\}$ and $j \in \{0, 1, 2, 3, 4\}$. For example, $S_{10} = 100 * 1.105170918^0 * 0.904837418^1 = 904837418$.

The European call price can be computed as follows:

Computation of the European call price



At maturity, option prices are computed as:

$$C_{4j} = \text{Max}\{0; S_{ij} - K\} \text{ with } j \in \{0, 1, 2, 3, 4\}, \quad C_{40} = 0.$$

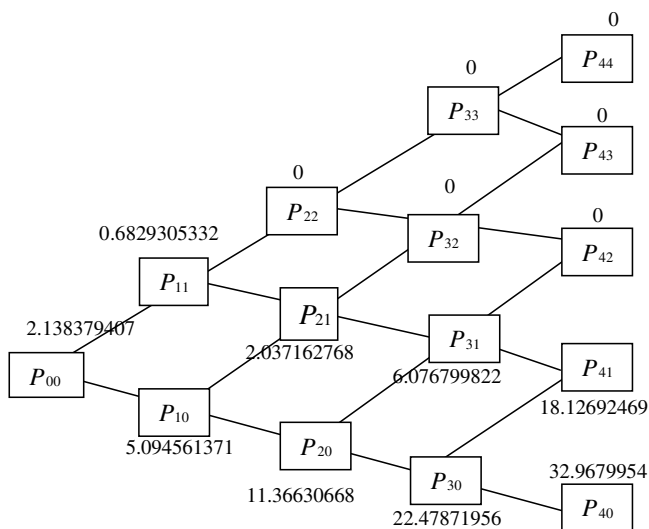
The other values are computed as:

$$C_{i,j} = \frac{C_{i+1,j+1} * p + C_{i+1,j} * (1 - p)}{m} \text{ for } i < 4,$$

where p is the probability.

European put prices:

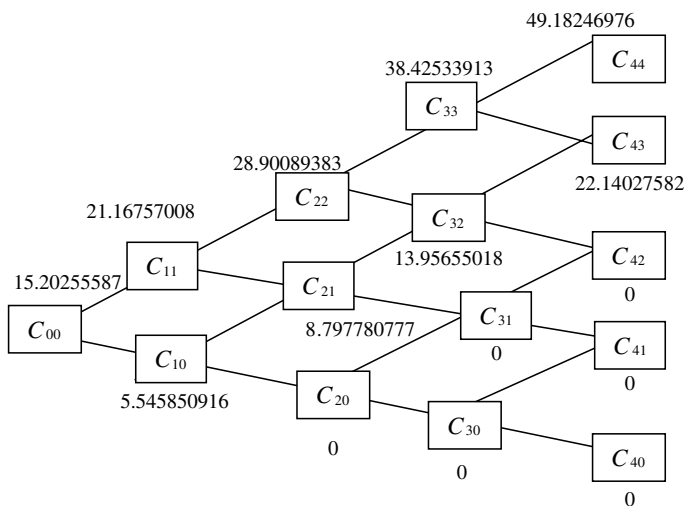
The same underlying asset prices can be used in the computation of the European put price.



At maturity, European put prices are given by: $P_{4j} = \text{Max}\{0; K - S_{4j}\}$ with $j \in \{0, 1, 2, 3, 4\}$. Before maturity, put prices are computed as:

$$P_{ij} = \frac{P_{i+1,j+1} * p + P_{i+1,j} * (1 - p)}{m} \quad \text{for } i < 4.$$

American call prices:



The following condition must be satisfied at each node in the computation of the American call price:

$$C_{ij} = \text{Max} \left\{ S_{ij} - E; \frac{C_{i+1,j+1} * p + C_{i+1,j} * (1 - p)}{m} \right\} \quad \text{for } i < 4.$$

Option values are computed as follows:

$$C_{33} = \text{Max}\{13498 - 100; 38.42539913\} = 38.42539913,$$

$$C_{32} = \text{Max}\{110.5170918 - 100; 13.95655018\},$$

$$C_{31} = \text{Max}\{90.483741 - 100; 0\},$$

$$C_{30} = \text{Max}\{74.08182207 - 100; 0\},$$

$$C_{22} = \text{Max}\{122.1402758 - 100; 28.90089383\},$$

$$C_{21} = \text{Max}\{100 - 100; 8.797780777\},$$

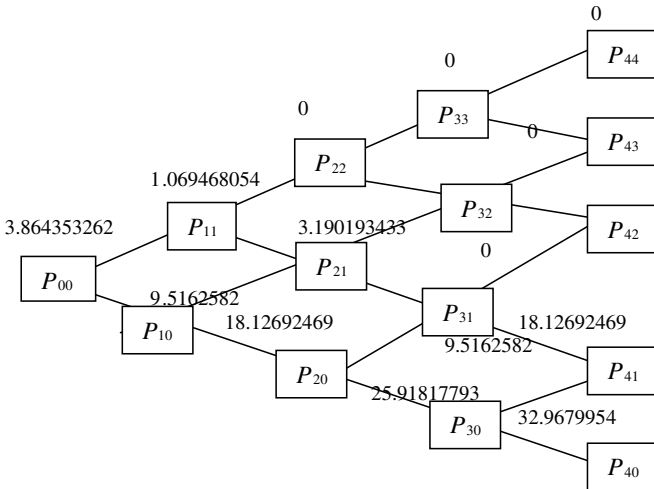
$$C_{20} = \text{Max}\{81.87307531 - 100; 0\},$$

$$C_{11} = \text{Max}\{110.5170918 - 100; 21.16757008\},$$

$$C_{00} = 15.20255587.$$

The American call price is 15.20255587.

American put prices: The same procedure is used for the computation of option prices.



The following condition must be satisfied before maturity:

$$P_{ij} = \text{Max} \left\{ K - S_{ij}; \frac{P_{i+1,j+1} * p + C_{i+1,j} * (1 - p)}{m} \right\} \quad \text{for } i < 4,$$

or:

$$P_{33} = \text{Max}\{100 - 134.98, 58.808 - 100; 0\},$$

$$P_{32} = \text{Max}\{100 - 110.5170918; 0\},$$

$$P_{31} = \text{Max}\{100 - 90.4837418; 6.0767\},$$

$$P_{30} = \text{Max}\{100 - 74.08182207; 22.47\},$$

$$P_{22} = \text{Max}\{100 - 122.14; 0\},$$

$$P_{21} = \text{Max}\{100 - 100; 0\},$$

$$P_{20} = \text{Max} \left\{ 100 - 81.87307531; \frac{9.5162582 * p + 25.918177930(1 - p)}{m} \right\} = 14.68746632,$$

$$P_{11} = \text{Max} \left\{ 100 - 110.5170918; \frac{3.190193433(1 - P)}{m} \right\} = 1.069468054,$$

$$P_{10} = \text{Max} \left\{ 100 - 90.4837418; \frac{1.069468054 * P + 9.5162582(1 - P)}{m} \right\} \\ = 3.864353262.$$

Hence, the option price is 3.86435.

6.3.6. Examples with Five Periods

Example: Consider the valuation of European and American options in the following context:

Underlying asset, $S = 100$, strike price $K = 100$, interest rate = 0.1, volatility = 0.4, $T = 5$ months, $N = 5$.

In this case, we have : $p = 0.5073$, $d = 0.8909$, $u = 1.1224$.

Dynamics of the underlying asset price					178.1312
				158.7055	
			141.3982		141.3982
		125.9784		125.9784	
	112.2401		112.2401		112.2401
100		100.0000		100.0000	
	89.0947		89.0947		89.0947
		79.3787		79.3787	
			70.7222		70.7222
				63.0098	
					56.1384
<hr/>					
The valuation of European put options					
					0.0000
				0.0000	
			0.0000		0.0000
		1.2720		0.0000	
	4.2282		2.6033		0.0000
8.6380		7.3442		5.3282	
	13.3256		12.3506		10.9053
		19.7110		19.7914	
			27.6249		29.2778
				36.1603	
					43.8616
<hr/>					
The valuation of American put options					
					0.0000
				0.0000	
			0.0000		0.0000
		1.2720		0.0000	
	4.3250		2.6033		0.0000
8.9769		7.5423		5.3282	
	13.9195		12.7561		10.9053
		20.7226		20.6213	
			29.2778		29.2778
				36.9902	
					43.8616

The valuation of European call options					
					78.1312
				59.5354	
			43.0511		41.3982
		29.7194		26.8082	
	19.7467		16.4963		12.2401
12.7191		9.8132		6.1581	
	5.6987		3.0982		0.0000
		1.5587		0.0000	
			0.0000		0.0000
				0.0000	
					0.0000

The valuation of American call options					
					78.1312
				59.5354	
			43.0511		41.3982
		29.7194		26.8082	
	19.7467		16.4963		12.2401
12.7191		9.8132		6.1581	
	5.6987		3.0982		0.0000
		1.5587		0.0000	
			0.0000		0.0000
				0.0000	
					0.0000

7. The Binomial Model and the Distributions to the Underlying Assets

We denote by:

$C(\cdot)$, $c(\cdot)$: prices of American and European call options,

$P(\cdot)$, $p(\cdot)$: prices of American and European put options.

7.1. The Model

The model is a simple extension of the basic lattice approach which has the flexibility to account for the magnitude and the timing of dividends, and

other cash payments. The basic lattice approach suggested by CRR (1979) considers the situation where there is only one state variable: the price of a non-dividend paying stock. The time to maturity of the option is divided into N equal intervals of length Δt during which the stock price moves from its initial value S to one of two new values Su and Sd with probabilities p and $(1 - p)$. When $u = 1/d$, it can be shown that: $p = (a - d)/(u - d)$, $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$, $a = e^{r\Delta t}$.

The nature of the lattice of stock prices is completely specified and the nodes correspond to $Su^j d^{i-j}$ for $j = 0, 1, \dots, i$. The option is evaluated by starting at time T and working backward. We denote by $F_{i,j}$, the option value at time $t + i\Delta t$ when the stock price is $Su^j d^{i-j}$. At time $t + i\Delta t$, the option holder can choose to exercise the option and receives the amount by which K (or S) exceeds the current stock price (or K) or wait. The American call is given by:

$$F_{i,j} = \max[Su^j d^{i-j} - K, e^{-r\Delta t}(pF_{i+1,j+1} + (1 - p)F_{i+1,j})].$$

The American put is given by:

$$F_{i,j} = \max[K - Su^j d^{i-j}, e^{-r\Delta t}(pF_{i+1,j+1} + (1 - p)F_{i+1,j})].$$

The extension of the lattice approach to the valuation of American options on stocks paying a known cash income is as follows. When there is only one cash income at date, τ , between $k\Delta t$ and $(k + 1)\Delta t$, it is possible to design trees where the number of nodes at time Δt is always $(i + 1)$. The analysis which parallels that in Hull (2000) and Briys *et al.* (1998) can be simplified by assuming that the implicit spot stock price has two components: a part which is stochastic and a part which is the present value of all future cash payments during the option's life. When there is just one ex-cash income date τ , during the option's life and $k\Delta t \leq \tau \leq (k + 1)\Delta t$, then at time x , the value of the stochastic component S is given by:

$$\begin{aligned} S^*(x) &= S(x), & \text{when } x > \tau \\ S^*(x) &= S(x) - (D_i + R_i)e^{-r(\tau-x)}, & \text{when } x \leq \tau. \end{aligned}$$

Assume σ^* is the constant volatility of S^* . Using the parameters p , u , and d , at time $t + i\Delta t$, the nodes on the tree define the stock prices:

$$\begin{aligned} \text{If } i\Delta t < \tau: & S^*(t)u^j d^{i-j} + (D_i + R_i)e^{-r(\tau-i\Delta t)}, & j = 0, 1, \dots, i. \\ \text{If } i\Delta t \geq \tau: & S^*(t)u^j d^{i-j}, & j = 0, 1, \dots, i. \end{aligned}$$

7.2. Simulations for a Small Number of Periods in the Presence of Dividends

Example 1. Applications of the Cox, Ross and Rubinstein model for five periods.

Consider the following data for the valuation of European and American call and put options: $S = 110$, $K = 115$, $r = 10\%$, $\sigma = 40\%$, $N = 5$, $t = 5$ months, $\Delta t = 1$ month. Date of dividend: 105 days, Dividend amount: $D = 10$.

The European call price: Using the above data, the dynamics of the underlying asset are computed using:

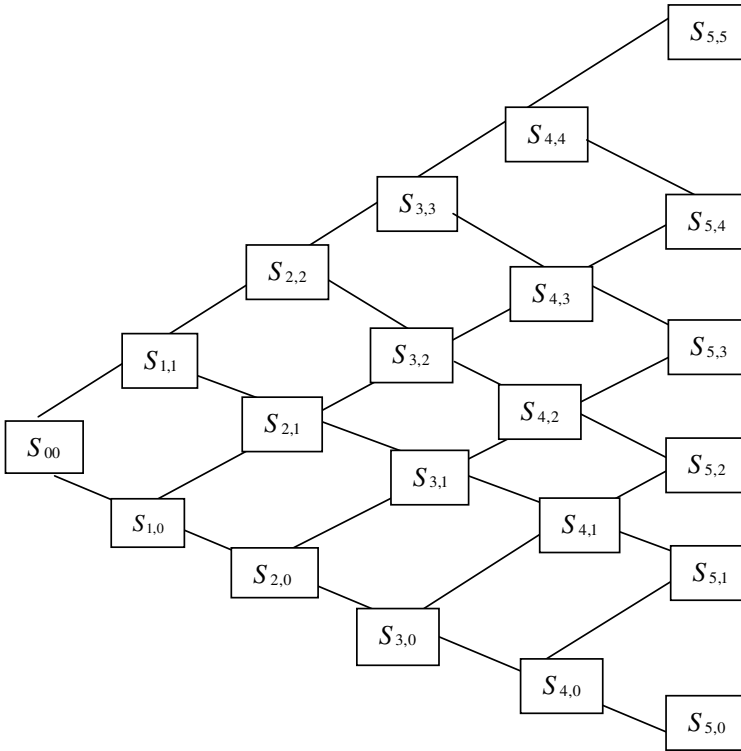
$$\begin{aligned}
 u &= e^{\sigma\sqrt{\Delta t}}, \quad u = e^{0.4\sqrt{1/2}}, \quad \text{or } u = 1.1224, \quad d = \frac{1}{u} = 0.8909, \\
 p &= \frac{e^{r\Delta t} - d}{u - d} = 0.5073, \\
 S^* &= S - D = 110 - 10 = 100, \\
 S_{0,0} &= S^* u^0 d^0 + De^{-r(105/365)} = 109.7164, \\
 S_{1,1} &= S^* u^1 d^0 + De^{-r((105/365)-(1/12))} = 100(1.1224), \\
 &\quad + 10De^{-r((105/365)-(1/12))} = 122.0377, \\
 S_{1,0} &= S^* u^0 d^1 + De^{-r((105/365)-(1/12))} = 98.8925, \\
 S_{2,2} &= S^* u^2 d^0 + De^{-r((105/365)-(2/12))} = 135.8579, \\
 S_{2,1} &= S^* u^1 d^1 + De^{-r((105/365)-(2/12))} = 109.8797, \\
 S_{2,0} &= S^* u^0 d^2 + De^{-r((105/365)-(2/12))} = 89.2586, \\
 S_{3,3} &= S^* u^3 d^0 + De^{-r((105/365)-(3/12))} = 151.3603, \\
 S_{3,2} &= S^* u^2 d^1 + De^{-r((105/365)-(3/12))} = 122.2024, \\
 S_{3,1} &= S^* u^1 d^2 + De^{-r((105/365)-(3/12))} = 99.0572, \\
 S_{3,0} &= S^* u^0 d^3 + De^{-r((105/365)-(3/12))} = 80.6848, \\
 S_{4,4} &= S^* u^4 d^0 = 158.7050, \\
 S_{4,3} &= S^* u^3 d^1 = 125.782, \\
 S_{4,2} &= S^* u^2 d^2 = 100.00, \\
 S_{4,1} &= S^* u^1 d^3 = 79.3788, \\
 S_{4,0} &= S^* u^0 d^4 = 63.0100, \\
 S_{5,5} &= S^* u^5 d^0 = 178.1305, \\
 S_{5,4} &= S^* u^4 d^1 = 141.3979, \\
 S_{5,3} &= S^* u^3 d^2 = 112.2400,
 \end{aligned}$$

$$S_{5,2} = S^* u^2 d^3 = 89.0948,$$

$$S_{5,1} = S^* u^1 d^4 = 70.7224,$$

$$S_{5,0} = S^* u^0 d^5 = 56.1386.$$

The dynamics of the underlying asset are represented in the following way:



The option's maturity value is computed as:

$$C_{5,5} = \max[0; S_{5,5} - K] = 63.1305,$$

$$C_{5,4} = \max[0; S_{5,4} - K] = 26.3979,$$

$$C_{5,3} = \max[0; S_{5,3} - K] = 0,$$

$$C_{5,2} = \max[0; S_{5,2} - K] = 0,$$

$$C_{5,1} = \max[0; S_{5,1} - K] = 0,$$

$$C_{5,0} = \max[0; S_{5,0} - K] = 0.$$

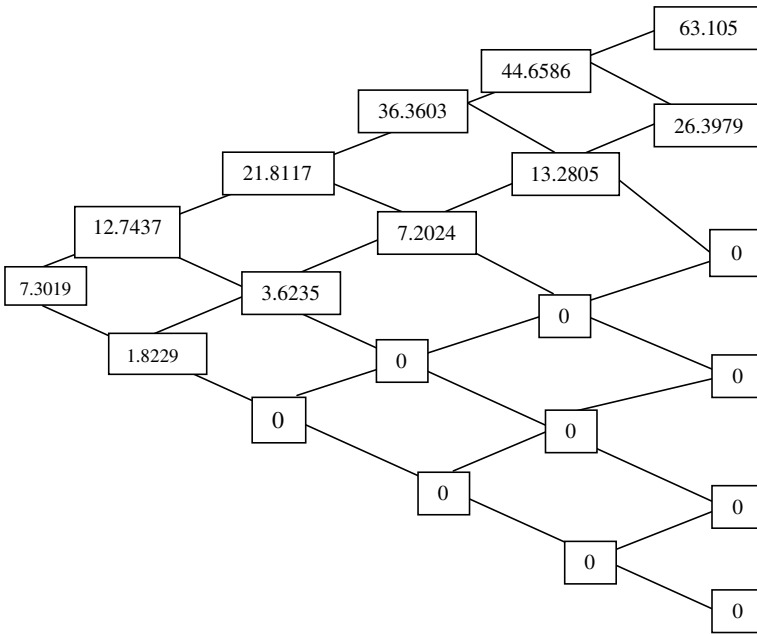
The American call option price is computed as:

$$\begin{aligned} C_{4,4} &= \max \left[\frac{p \times C_{5,5} + q \times C_{5,4}}{e^{r\Delta t}}; IV \right] = \max[44.6586; S_{4,4} - K] \\ &= \max[44.6586; 43.7050] = 44.6586 \end{aligned}$$

where IV stands for the option intrinsic value.

$$\begin{aligned} C_{4,3} &= \max \left[\frac{p \times C_{5,4} + q \times C_{5,3}}{e^{r\Delta t}}; S_{4,3} - K \right] \\ &= \max[13.2805; 10.9782] = 13.2805, \\ C_{4,2} &= \max \left[\frac{p \times C_{5,3} + q \times C_{5,2}}{e^{r\Delta t}}; \max[0; S_{4,2} - K] \right] = 0, \\ C_{4,1} &= \max \left[\frac{p \times C_{5,2} + q \times C_{5,1}}{e^{r\Delta t}}; \max[0; S_{4,1} - E] \right] = 0, \\ C_{4,0} &= \max \left[\frac{p \times C_{5,1} + q \times C_{5,0}}{e^{r\Delta t}}; \max[0; S_{4,0} - K] \right] = 0, \\ C_{3,3} &= \max \left[\frac{p \times C_{4,4} + q \times C_{4,3}}{e^{r\Delta t}}; S_{3,3} - K \right] \\ &= \max[28.9563; 36.3603] = 36.3603, \\ C_{3,2} &= \max \left[\frac{p \times C_{4,3} + q \times C_{4,2}}{e^{r\Delta t}}; S_{3,2} - K \right] \\ &= \max[6.6813; 7.2024] = 7.2024, \\ C_{3,1} &= \max \left[\frac{p \times C_{4,2} + q \times C_{4,1}}{e^{r\Delta t}}; S_{3,1} - K \right] = 0, \\ C_{3,0} &= \max \left[\frac{p \times C_{4,1} + q \times C_{4,0}}{e^{r\Delta t}}; S_{3,0} - K \right] = 0. \end{aligned}$$

American call option prices are computed as follows:



$$C_{2,2} = \max \left[\frac{p \times C_{3,3} + q \times C_{3,2}}{e^{r\Delta t}}; S_{2,2} - K \right]$$

$$= \max[21.8117; 20.8579] = 21.8117,$$

$$C_{2,1} = \max \left[\frac{p \times C_{3,2} + q \times C_{3,1}}{e^{r\Delta t}}; \max[0; S_{2,1} - K] \right]$$

$$= \max[3.6235; 0] = 3.6235,$$

$$C_{2,0} = \max \left[\frac{p \times C_{3,1} + q \times C_{3,0}}{e^{r\Delta t}}; \max[0; S_{2,0} - K] \right] = 0,$$

$$C_{1,1} = \max \left[\frac{p \times C_{2,2} + q \times C_{2,1}}{e^{r\Delta t}}; \max[0; S_{1,1} - K] \right]$$

$$= \max[12.7437; 7.0377] = 12.7437,$$

$$C_{1,0} = \max \left[\frac{p \times C_{2,1} + q \times C_{2,0}}{e^{r\Delta t}}; \max[0; S_{1,0} - K] \right]$$

$$\begin{aligned}
&= \max[1.8229; 0] = 1.8229, \\
C_{0,0} &= \max \left[\frac{p \times C_{1,1} + q \times C_{1,0}}{e^{r\Delta t}}; \max[0; S_{0,0} - K] \right] \\
&= \max[7.3019; 0] = 7.3019.
\end{aligned}$$

The American put price:

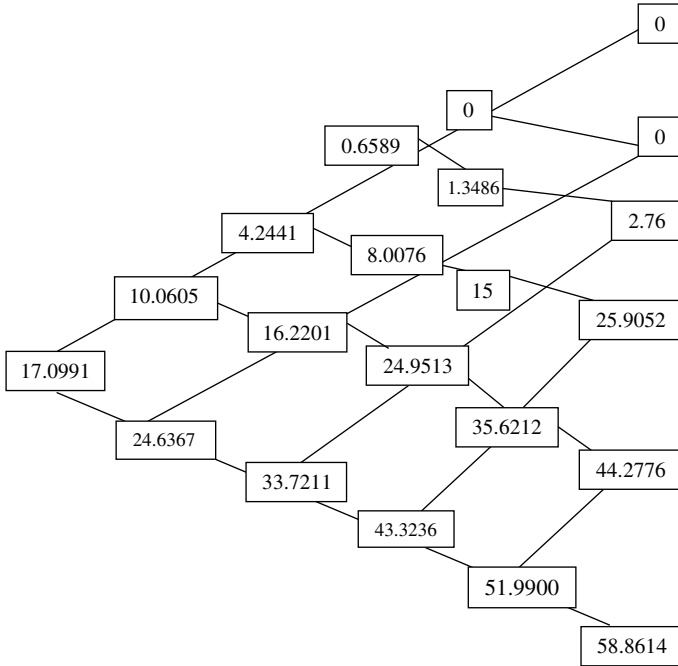
Using the above data, the American put price is computed as follows at different nodes:

$$\begin{aligned}
P_{5,5} &= \max[0; K - S_{5,5}] = 0, \\
P_{5,4} &= \max[0; K - S_{5,4}] = 0, \\
P_{5,3} &= \max[0; K - S_{5,3}] = 2.76, \\
P_{5,2} &= \max[0; K - S_{5,2}] = 25.9052, \\
P_{5,1} &= \max[0; K - S_{5,1}] = 44.2776, \\
P_{5,0} &= \max[0; K - S_{5,0}] = 58.8614.
\end{aligned}$$

Before maturity, the option price is computed as:

$$\begin{aligned}
P_{4,4} &= \max \left[\frac{p \times P_{5,5} + q \times P_{5,4}}{e^{r\Delta t}}; \max[0; K - S_{4,4}] \right] = 0, \\
P_{4,3} &= \max \left[\frac{p \times P_{5,4} + q \times P_{5,3}}{e^{r\Delta t}}; \max[0; K - S_{4,3}] \right] \\
&= \max[1.3486; 0] = 1.3486, \\
P_{4,2} &= \max \left[\frac{p \times P_{5,3} + q \times P_{5,2}}{e^{r\Delta t}}; \max[0; K - S_{4,2}] \right] \\
&= \max[14.0461; 0] = 15, \\
P_{4,1} &= \max \left[\frac{p \times P_{5,2} + q \times P_{5,1}}{e^{r\Delta t}}; \max[0; K - S_{4,1}] \right] \\
&= \max[34.6672; 35.6212] = 35.6212, \\
P_{4,0} &= \max \left[\frac{p \times P_{5,1} + q \times P_{5,0}}{e^{r\Delta t}}; \max[0; K - S_{4,0}] \right] \\
&= \max[51.0360; 51.9900] = 51.9900.
\end{aligned}$$

Option prices are reported in the following figure:



$$P_{3,3} = \max \left[\frac{p \times P_{4,4} + q \times P_{4,3}}{e^{r\Delta t}}; \max[0; K - S_{3,3}] \right]$$

$$= \max[0.6589; 0] = 0.6589,$$

$$P_{3,2} = \max \left[\frac{p \times P_{4,3} + q \times P_{4,2}}{e^{r\Delta t}}; \max[0; K - S_{3,2}] \right]$$

$$= \max[8.0076; 0] = 8.0076,$$

$$P_{3,1} = \max \left[\frac{p \times P_{4,2} + q \times P_{4,1}}{e^{r\Delta t}}; \max[0; K - S_{3,1}] \right]$$

$$= \max[24.9513; 15.9428] = 24.9513,$$

$$P_{3,0} = \max \left[\frac{p \times P_{4,1} + q \times P_{4,0}}{e^{r\Delta t}}; \max[0; K - S_{3,0}] \right]$$

$$= \max[43.3236; 34.3152] = 43.3236,$$

$$P_{2,2} = \max \left[\frac{p \times P_{3,3} + q \times P_{3,2}}{e^{r\Delta t}}; \max[0; K - S_{2,2}] \right]$$

$$= \max[4.2441; 0] = 4.2441,$$

$$P_{2,1} = \max \left[\frac{p \times P_{3,2} + q \times P_{3,1}}{e^{r\Delta t}}; \max[0; K - S_{2,1}] \right]$$

$$= \max[16.2201; 5.1203] = 16.2201,$$

$$P_{2,0} = \max \left[\frac{p \times P_{3,1} + q \times P_{3,0}}{e^{r\Delta t}}; \max[0; K - S_{2,0}] \right]$$

$$= \max[33.7211; 25.7414] = 33.7211,$$

$$P_{1,1} = \max \left[\frac{p \times P_{2,2} + q \times P_{2,1}}{e^{r\Delta t}}; \max[0; K - S_{1,1}] \right]$$

$$= \max[10.0605; 0] = 10.0605,$$

$$P_{1,0} = \max \left[\frac{p \times P_{2,1} + q \times P_{2,0}}{e^{r\Delta t}}; \max[0; K - S_{1,0}] \right]$$

$$= \max[24.6367; 16.1075] = 24.6367,$$

$$P_{0,0} = \max \left[\frac{p \times P_{1,1} + q \times P_{1,0}}{e^{r\Delta t}}; \max[0; K - S_{0,0}] \right]$$

$$= \max[17.0991; 5.2836] = 17.0991.$$

Example 2. Consider the valuation of European and American options in the following context:

Underlying asset, $S = 100$, strike price $K = 100$, interest rate = 0.1, volatility = 0.4, $T = 5$ months, $N = 5$, dividend = 10, dividend date = 105. In this case, we have : $p = 0.5073$, $d = 0.8909$, and $u = 1.1224$.

Dynamics of the underlying asset for five periods					178.1312
				158.7055	141.3982
		135.8581	151.3606	125.9784	112.2401
110	122.0378	109.8797	122.2025	100.0000	89.0947
	98.8925	89.2584	99.0571	79.3787	70.7222
			80.6846	63.0098	56.1384

The valuation of European put options					0.0000
				0.0000	0.0000
		1.2720	0.0000	0.0000	0.0000
	4.2282	7.3442	2.6033	5.3282	10.9053
8.6380	13.3256	19.7110	12.3506	19.7914	29.2778
			27.6249	36.1603	43.8616

The valuation of American put options					0.0000
				0.0000	0.0000
		1.2720	0.0000	0.0000	0.0000
	4.3250	7.5423	2.6033	5.3282	10.9053
8.8801	13.7214	20.3171	12.7561	20.6213	29.2778
			28.4479	36.9902	43.8616

The valuation of European call options

				59.5354	78.1312
			43.0511		41.3982
		29.7194		26.8082	
	19.7467		16.4963		12.2401
12.7191		9.8132		6.1581	
	5.6987		3.0982		0.0000
		1.5587		0.0000	
			0.0000		0.0000
				0.0000	
					0.0000

The valuation of American call options

				59.5354	78.1312
			51.3606		41.3982
		36.6880		26.8082	
	24.6554		22.2025		12.2401
15.8944		12.6840		6.1581	
	7.1430		3.0982		0.0000
		1.5587		0.0000	
			0.0000		0.0000
				0.0000	
					0.0000

Example 3. Consider the valuation of European and American options in the following context:

Underlying asset, $S = 80$, strike price $K = 100$, interest rate = 0.1, volatility = 0.4, $T = 5$ months, $N = 5$, dividend = 10, dividend date = 105. In this case, we have: $p = 0.5073$, $d = 0.8909$, $u = 1.1224$.

Dynamics of the underlying for five periods					142.5050
				126.9644	113.1186
		110.6624	123.0810	100.7827	89.7921
90	99.5898	89.8797	99.7545	80.0000	71.2758
	81.0735	73.3827	81.2382	63.5030	56.5778
			66.5402	50.4078	44.9107
Valuation of European put options					0.0000
				0.0000	0.0000
		7.0283	2.4369	4.9875	10.2079
19.3423	12.9178	19.2016	11.8756	19.1701	28.7242
	26.2863	34.0280	27.0714	35.6672	43.4222
			41.7694	48.7623	55.0893
Valuation of American put options					0.0000
				0.0000	0.0000
		7.2265	2.4369	4.9875	10.2079
19.8853	13.3136	19.8077	12.2811	20.0000	28.7242
	26.9900	34.8442	27.8943	36.4970	43.4222
			42.5923	49.5922	55.0893

 Valuation of European call options

				27.7943	42.5050
			17.2083		13.1186
		10.2801		6.6001	
	5.9882		3.3206		0.0000
3.4234		1.6706		0.0000	
	0.8405		0.0000		0.0000
		0.0000		0.0000	
			0.0000		0.0000
				0.0000	
					0.0000

 Valuation of American call options

				27.7943	42.5050
			23.0810		13.1186
		13.2347		6.6001	
	7.4747		3.3206		0.0000
4.1713		1.6706		0.0000	
	0.8405		0.0000		0.0000
		0.0000		0.0000	
			0.0000		0.0000
				0.0000	
					0.0000

Summary

This chapter develops the main concepts regarding the pricing of assets and derivatives in a simple discrete-time context. The analysis in a discrete-time setting is more intuitive and provides the foundations about the convergence to continuous-time models. This chapter provides the reader with the main concepts and tools used in this book regarding risk, return, uncertainty, incomplete information and asset pricing in an intuitive manner.

The most well known strategies in portfolio management involve combinations of options. They include vertical spreads, calendar spreads, diagonal spreads, ratio spreads, volatility spreads and synthetic contracts.

A vertical spread involves the purchase of an option and the sale of another with the same time to maturity and a different strike price. When the strategy produces a cash-out flow, we say that the investor is long the spread. When the strategy generates a cash-inflow, the investor is said short the spread. The strategy can be implemented by calls or puts using different strike prices.

A vertical bull spread is implemented when an *at-the-money* option is bought and an *out-of-the-money* option is sold.

A calendar spread strategy represents a position where the investor is long an option with a longer term and short an option with a short term for the same strike price.

A diagonal spread involves the purchase of an option with a longer term and the sale of another with a short term where both options have different strike prices.

A bullish diagonal spread is implemented when the purchased option is at parity and the short option is *out-of-the-money*.

The option price depends on the underlying asset price S , the strike price K , the interest rate r , the time to maturity, T , the volatility σ and dividend payouts. The option maturity corresponds to the number of days until expiration. It is often given in a fraction of a year or in days. The dividends must be known or estimated before using an option pricing model.

Asset pricing and option pricing are based on the concepts of risk and return. Portfolio theory provides the basis for asset valuation. Sharpe (1963), among others, shows how an investor should divide wealth between risky assets and a riskless asset. He shows that “the proportionate composition of the non-cash assets is independent of their aggregate share of the investment balance. This fact makes it possible to describe the investor’s decisions as if there was a single non-cash asset, a composite formed by combining the multitude of actual non-cash assets in fixed proportions”.

Sharpe (1963) develops a simplified model for portfolio analysis by observing that stocks are likely to co-move with the market. The contributions of Markowitz and Sharpe were honored by the first Nobel Prize in financial economics in 1991.

The analysis in Merton’s (1987) provides a simple model of capital market equilibrium with incomplete information. He shows that a reconciling of finance theory with empirical violations of the complete-information, perfect market model need not imply a departure from the standard paradigm.

As Merton (1987) asserts “Financial markets dominated by rational agents may nevertheless produce anomalous behavior relative to the perfect

market model. Institutional complexities and information costs may cause considerable variations in the time scales over which different types of anomalies are expected to be eliminated in the market place". These models can be applied for the valuation of derivative assets. We present also a discrete-time approach for the valuation of options. The analysis is based on the standard model of Cox, Ross and Rubinstein (1979). This binomial model can be used for the valuation of options on different underlying assets.

Questions

1. Define the strategy of buying (selling) calls.
2. Define the strategy of buying (selling) puts.
3. Define the strategy of buying (selling) spreads.
4. Define the strategy of buying (selling) combinations.
5. What are the determinants of an option price?
6. Define the specific features of long (or short) a straddle.
7. Define the specific features of long (or short) a strangle.
8. Define the specific features of long (or short) a tunnel.
9. Define the specific features of long (or short) spreads.
10. Define the specific features of long (or short) butterfly.
11. What is risk?
12. How can we compute expected return?
13. How is the efficient frontier derived?
14. How is the Capital Asset Pricing Model derived?
15. Describe Merton's (1987) simple model of capital market equilibrium with incomplete information.
16. Describe the Cox, Ross and Rubinstein model for equity options for one period.
17. Describe the Cox, Ross and Rubinstein model for equity options for two periods (N periods).
18. What are the valuation parameters in the lattice approach for stock prices?
19. How is an option priced in the lattice approach for stock prices?
20. What modifications are necessary to the standard lattice approach to apply it to American options?
21. What are the effects of cash distributions on the stock price?
22. What are the main definitions of risk and return?
23. What is portfolio selection and how can it be implemented?

24. What is portfolio diversification in a standard context?
25. How is the standard CAPM derived? Can it be applied to the pricing of assets and options?
26. Does information affect the pricing of assets?
27. How is Merton's (1987) simple model of capital market equilibrium with incomplete information obtained?
28. What is the definition of information?
29. How can options be priced?
30. What are the specific features of a general binomial model?
31. What are the effects of dividends on option pricing?
32. Analyze the simulation results provided in the tables of this chapter. What are your main comments and remarks regarding results of the simulations?

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