Chapter 1

Integral representations in complex, hypercomplex and Clifford analysis*

Heinrich Begehr

Freie Universität Berlin, Mathematisches Institut
14195 Berlin, Germany
begehr@math.fu-berlin.de

Contents

1.1. Introduction

Integral representations are one of the main tools in analysis. They are useful to determine properties of the functions represented such as smoothness, differentiability, boundary behaviour etc. They serve to reduce boundary value problems etc. for differential equations to integral equations and thus lead to existence and uniqueness proofs. Well–known representation formulas are the Cauchy formula for analytic functions and the Green representation for harmonic functions. Both these formulas are consequences from

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the Gauss divergence theorem where the area integral disappears because homogeneous equations (Cauchy–Riemann and Laplace, respectively) are considered. In the cases of inhomogeneous Cauchy–Riemann equations and the Poisson equation, the area integrals appearing lead to area integral operators of the Pompeiu type. They determine particular solutions to the inhomogeneous equation under consideration.

Now we make the following simple observation. Let ∂ be a linear differential operator and T be its related Pompeiu integral operator. Then ∂T is the identity mapping for a proper function space. For any power ∂^k, $k \in \mathbb{N}$, then the iteration T^k obviously is its right inverse, $\partial^k T^k$ is the identity again. More generally for two such differential operators ∂_1, ∂_2 with right inverses T_1, T_2 then the iteration $T_2 T_1$ is right inverse to $\partial_1 \partial_2$.

On this basis particular solutions for higher order differential operators can be constructed leading also to fundamental solutions. Moreover, these integral operators are useful for determining particular solutions to any higher order differential equation the leading term of which is related to them. In fact, one can solve boundary value problems to these higher order equations if, besides the particular solution for the leading term through the Pompeiu operator, the general solution to the related homogeneous leading term operator equation is taken into consideration.

This sketched procedure can be followed in complex, hypercomplex and Clifford analysis. But the resulting representation formulas of Cauchy-Pompeiu type do not automatically give solutions to related boundary value problems. This, however, is the case whenever these problems are solvable. This phenomenon is known already from the Cauchy formula. Not all functions on the boundary of a domain are boundary values of the analytic functions determined by their Cauchy integrals. In particular solvability conditions have to be observed in the theory of several complex variables where also compatibility conditions for the systems considered are important.

In these lectures the hierarchy of Pompeiu integral operators in the complex case will be presented and some higher order Cauchy–Pompeiu representation formulas given. As an application some orthogonal decompositions of the Hilbert space $L_2(G; \mathbb{C})$, $G \subset \mathbb{C}$ a regular domain, are given. For several complex variables only some results on bidomains are included. As the theory in hypercomplex analysis is analogue to the complex case only some references [2, 9] are given. The situation in Clifford analysis is shortly explained.

1.2. Complex case

Gauss divergence theorem. Let $D \subset \mathbb{R}^2$ be a regular domain, $f, g \in C^1(D;\mathbb{R}) \cap C(\overline{D};\mathbb{R})$. Then

$$\int_D (f_x + g_y)\, dx\, dy = \int_{\partial D} \{f\, dy - g\, dx\} .$$

Complex forms: $z = x + iy$, $\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$,
$w = u + iv \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$. Then

$$\int_D w_{\overline{z}} dx\, dy = \frac{1}{2i}\int_{\partial D} w\, dz\,, \quad \int_D w_z dx\, dy = -\frac{1}{2i}\int_{\partial D} w\, d\overline{z}\,.$$

Cauchy–Pompeiu Representation. Let $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$. Then with $\zeta = \xi + i\eta$

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\,\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\overline{\zeta}}(\zeta)\,\frac{d\xi\, d\eta}{\zeta - z}\,,$$

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\,\frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta)\,\frac{d\xi\, d\eta}{\overline{\zeta - z}}\,.$$

Pompeiu Operator. Let $f \in L_1(D;\mathbb{C})$. Then

$$Tf(z) = -\frac{1}{\pi}\int_D f(\zeta)\,\frac{d\xi\, d\eta}{\zeta - z}\,, \quad \overline{T}f(z) = -\frac{1}{\pi}\int_D f(\zeta)\,\frac{d\xi\, d\eta}{\overline{\zeta - z}}\,.$$

Properties of these operators are developed in [13], see also [1]. Important are

$$\partial_{\overline{z}} Tf = f\,, \quad \partial_z Tf = \Pi f\,, \quad f \in L_1(D;\mathbb{C})\,,$$

where

$$\Pi f(z) = -\frac{1}{\pi}\int_D f(\zeta)\,\frac{d\xi\, d\eta}{(\zeta - z)^2}$$

is a singular integral operator of Calderon–Zygmund type to be taken as a Cauchy principal integral. Here the derivatives are taken in the weak sense.

1.2.1. *Complex first order systems*

Theorem 1.1. *Any solution to $w_{\overline{z}} = f$ in D, $f \in L_1(D;\mathbb{C})$, is representable via $w = \varphi + Tf$ where φ is analytic in D.*

Proof. (1) Obviously $\varphi + Tf$ with $\varphi_{\overline{z}} = 0$ is a solution.
(2) If w is a solution then $(w - Tf)_{\overline{z}} = 0$ i.e. is analytic. □

Generalized Beltrami equation:

$$w_{\overline{z}} + \mu_1 w_z + \mu_2 \overline{w_z} + aw + b\overline{w} = f\,, \quad |\mu_1(z)| + |\mu_2(z)| \leq q_0 < 1\,.$$

Find a particular solution in the form $w = T\rho$! Then ρ must satisfy the singular integral equation

$$\rho + \mu_1 \Pi\rho + \mu_2 \overline{\Pi\rho} + a\,T\rho + b\,\overline{T\rho} = f\,.$$

As $\mu_1 \Pi\rho + \mu_2 \overline{\Pi\rho}$ is contractive and $a\,T\rho + b\,\overline{T\rho}$ is compact this problem is solvable.

1.2.2. *Complex second order equations*

There are two principally different second order elliptic differential operators the main part of which is either the Laplace or the Bitsadze operator. As in the case of the generalized Beltrami equation the solutions to the inhomogeneous Laplace and Bitsadze equations can be used to solve the general equations of second order.

1.2.2.1. *Poisson equation $w_{z\overline{z}} = f$*

The Cauchy–Pompeiu formulas

$$w(z) = \frac{1}{2\pi i} \int_{\partial D} w(\zeta)\, \frac{d\zeta}{\zeta - \tau} - \frac{1}{\pi} \int_D w_{\overline{\widetilde{\zeta}}}(\widetilde{\zeta})\, \frac{d\widetilde{\xi}\, d\widetilde{\eta}}{\widetilde{\zeta} - z}\,,$$

$$w_{\overline{\widetilde{\zeta}}}(\widetilde{\zeta}) = -\frac{1}{2\pi i} \int_{\partial D} w_{\overline{\zeta}}(\zeta)\, \frac{d\overline{\zeta}}{\overline{\zeta - \widetilde{\zeta}}} - \frac{1}{\pi} \int_D w_{\zeta\overline{\zeta}}(\zeta)\, \frac{d\xi\, d\eta}{\overline{\zeta - \widetilde{\zeta}}}$$

lead to

$$w(z) = \frac{1}{2\pi i} \int_{\partial D} \left\{ \frac{w(\zeta)}{\zeta - z}\, d\zeta - \psi(\zeta, z)\, w_{\overline{\zeta}}(\zeta)\, d\overline{\zeta} \right\} - \frac{1}{\pi} \int_D \psi(\zeta, z)\, w_{\zeta\overline{\zeta}}(\zeta)\, d\xi\, d\eta \qquad (1.1)$$

with

$$\psi(\zeta, z) = \frac{1}{\pi} \int_D \frac{1}{\overline{\widetilde{\zeta} - \zeta}}\, \frac{d\widetilde{\xi}\, d\widetilde{\eta}}{\widetilde{\zeta} - z}\,.$$

Applying the Cauchy–Pompeiu formula to $\log|\zeta - z|^2$ in the domain $D_\varepsilon = D\backslash\{z : |z-\zeta| \le \varepsilon\}$ for sufficiently small positive ε gives

$$\log|\zeta - z|^2 = \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \log|\zeta - \widetilde{\zeta}|^2 \frac{d\widetilde{\zeta}}{\widetilde{\zeta} - z} - \frac{1}{\pi}\int_{D_\varepsilon} \frac{1}{\overline{\widetilde{\zeta} - \zeta}}\, \frac{d\widetilde{\xi}\, d\widetilde{\eta}}{\widetilde{\zeta} - z}\,.$$

As for $\zeta \ne z$

$$\frac{1}{2\pi i}\int_{|\widetilde{\zeta}-\zeta|=\varepsilon} \log|\zeta - \widetilde{\zeta}|^2 \frac{d\widetilde{\zeta}}{\widetilde{\zeta} - z} = \frac{\varepsilon\log\varepsilon}{\pi}\int_0^{2\pi} \frac{e^{it}}{\varepsilon e^{it} + \zeta - z}\, dt$$

and

$$\frac{1}{\pi}\int_{|\widetilde{\zeta}-\zeta|<\varepsilon} \frac{1}{\overline{\widetilde{\zeta} - \zeta}}\, \frac{d\widetilde{\xi}\, d\widetilde{\eta}}{\widetilde{\zeta} - z} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\varepsilon} e^{it}\frac{dt\, dr}{re^{it} + \zeta - z}$$

tend to zero with ε then

$$\log|\zeta - z|^2 = \widetilde{\psi}(\zeta, z) - \psi(\zeta, z) \tag{1.2}$$

with

$$\widetilde{\psi}(\zeta, z) = \frac{1}{2\pi i}\int_{\partial D} \log|\widetilde{\zeta} - \zeta|^2 \frac{d\widetilde{\zeta}}{\widetilde{\zeta} - z}\,.$$

Because for $z, \zeta \in D$

$$\begin{aligned}\partial_\zeta \widetilde{\psi}(\zeta, z) &= -\frac{1}{2\pi i}\int_{\partial D} \frac{d\widetilde{\zeta}}{(\widetilde{\zeta} - \zeta)(\widetilde{\zeta} - z)} \\ &= -\frac{1}{2\pi i}\int_{\partial D} \left(\frac{1}{\widetilde{\zeta} - \zeta} - \frac{1}{\widetilde{\zeta} - z}\right)\frac{d\widetilde{\zeta}}{\zeta - z} = 0\end{aligned}$$

from the Gauss theorem

$$\frac{1}{2\pi i}\int_{\partial D} \widetilde{\psi}(\zeta, z)\, w_{\overline{\zeta}}(\zeta)\, d\overline{\zeta} + \frac{1}{\pi}\int_D \widetilde{\psi}(\zeta, z)\, w_{\zeta\overline{\zeta}}(\zeta)\, d\xi\, d\eta = 0\,.$$

Adding this to the right–hand side of (1.1) and observing (1.2) show

$$\begin{aligned}w(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{w(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i}\int_{\partial D} \log|\zeta - z|^2 w_{\overline{\zeta}}(\zeta)\, d\overline{\zeta} \\ + \frac{1}{\pi}\int_D \log|\zeta - z|^2 w_{\zeta\overline{\zeta}}(\zeta)\, d\xi\, d\eta\,.\end{aligned} \tag{1.3}$$

As is well known $2/\pi \log|\zeta - z|$ is the fundamental solution to the Laplacian $\partial_z\partial_{\overline{z}}$. The representation (1.3) has the form

$$w = \varphi + \overline{\psi} + T_{1,1}f\,, \qquad f = w_{z\overline{z}}\,,$$

with analytic functions φ and ψ.

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1.2.2.2. *Bitsadze equation* $w_{\overline{z}\,\overline{z}} = f$

Similarly to the preceding subsection the Cauchy–Pompeiu formulas

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\,\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\overline{\widetilde{\zeta}}}(\widetilde{\zeta})\,\frac{d\widetilde{\xi}\,d\widetilde{\eta}}{\widetilde{\zeta} - z}\,,$$

$$w_{\overline{\widetilde{\zeta}}}(\widetilde{\zeta}) = \frac{1}{2\pi i}\int_{\partial D} w_{\overline{\zeta}}(\zeta)\,\frac{d\zeta}{\zeta - \widetilde{\zeta}} - \frac{1}{\pi}\int_D w_{\overline{\zeta}\,\overline{\zeta}}(\zeta)\,\frac{d\xi\,d\eta}{\zeta - \widetilde{\zeta}}$$

imply

$$w(z) = \frac{1}{2\pi i}\int_{\partial D}\left\{\frac{w(\zeta)}{\zeta - z} + \psi(\zeta, z)\,w_{\overline{\zeta}}(\zeta)\right\}d\zeta - \frac{1}{\pi}\int_D \psi(\zeta, z)\,w_{\overline{\zeta}\,\overline{\zeta}}(\zeta)\,d\xi\,d\eta \tag{1.4}$$

with

$$\psi(\zeta, z) = \frac{1}{\pi}\int_D \frac{1}{\overline{\widetilde{\zeta} - \zeta}}\;\frac{d\widetilde{\xi}\,d\widetilde{\eta}}{\widetilde{\zeta} - z}\,.$$

Applying the Cauchy–Pompeiu formula to $\frac{\overline{\zeta - z}}{\zeta - z}$ in the domain $D_\varepsilon = D\backslash\{z : |z - \zeta| \le \varepsilon\}$ for sufficiently small positive ε gives

$$\frac{\overline{\zeta - z}}{\zeta - z} = \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \frac{\overline{\zeta - \widetilde{\zeta}}}{\zeta - \widetilde{\zeta}}\;\frac{d\widetilde{\zeta}}{\widetilde{\zeta} - z} + \frac{1}{\pi}\int_{D_\varepsilon}\frac{1}{\zeta - \widetilde{\zeta}}\;\frac{d\widetilde{\xi}\,d\widetilde{\eta}}{\widetilde{\zeta} - z}\,.$$

Observing that when ε tends to zero

$$\frac{1}{2\pi i}\int_{|\widetilde{\zeta} - \zeta| = \varepsilon} \frac{\overline{\widetilde{\zeta} - \zeta}}{\widetilde{\zeta} - \zeta}\;\frac{d\widetilde{\zeta}}{\widetilde{\zeta} - z} = \frac{\varepsilon}{2\pi}\int_0^{2\pi} e^{-it}\frac{dt}{\varepsilon e^{it} + \zeta - z}$$

and

$$\frac{1}{\pi}\int_{|\widetilde{\zeta} - \zeta| < \varepsilon} \frac{1}{\widetilde{\zeta} - \zeta}\;\frac{d\widetilde{\zeta}\,d\widetilde{\eta}}{\widetilde{\zeta} - z} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\varepsilon} e^{-it}\frac{dt\,dr}{re^{it} + \zeta - z}$$

tend to zero then

$$\frac{\overline{\zeta - z}}{\zeta - z} = \widetilde{\psi}(\zeta, z) - \psi(\zeta, z) \tag{1.5}$$

with

$$\widetilde{\psi}(\zeta, z) = \frac{1}{2\pi i}\int_{\partial D} \frac{\overline{\widetilde{\zeta} - \zeta}}{\widetilde{\zeta} - \zeta}\;\frac{d\widetilde{\zeta}}{\widetilde{\zeta} - z}\,.$$

As for $z, \zeta \in D$

$$\partial_{\overline{\zeta}}\widetilde{\psi}(\zeta, z) = -\frac{1}{2\pi i}\int_{\partial D}\frac{1}{\widetilde{\zeta} - \zeta}\;\frac{d\widetilde{\zeta}}{\widetilde{\zeta} - z} = 0$$

from the Gauss theorem

$$\frac{1}{2\pi i}\int_{\partial D}\widetilde{\psi}(\zeta,z)\,w_{\overline{\zeta}}(\zeta)\,d\zeta-\frac{1}{\pi}\int_{D}\widetilde{\psi}(\zeta,z)\,w_{\overline{\zeta}\,\overline{\zeta}}(\zeta)\,d\xi\,d\eta=0\,.$$

Subtracting this from the right–hand side of (1.4) and observing (1.5) lead to

$$w(z)=\frac{1}{2\pi i}\int_{\partial D}\frac{w(\zeta)}{\zeta-z}\,d\zeta-\frac{1}{2\pi i}\int_{\partial D}\frac{\overline{\zeta-z}}{\zeta-z}\,w_{\overline{\zeta}}(\zeta)\,d\zeta \qquad (1.6)$$
$$+\frac{1}{\pi}\int_{D}\frac{\overline{\zeta-z}}{\zeta-z}\,w_{\overline{\zeta}\,\overline{\zeta}}(\zeta)\,d\xi\,d\eta\,.$$

The kernel $(\overline{\zeta-z})/[(\zeta-z)\pi]$ is the fundamental kernel to the Bitsadze operator $\partial_{\overline{z}}^2$. Representation (1.6) is of the form

$$w=\varphi+\overline{z}\psi+T_{0,2}f\,,\qquad f=w_{\overline{z}\,\overline{z}}\,,$$

with analytic functions φ und ψ.

1.2.2.3. *General complex second order equations*

A general second order equation with leading term $\partial_{\overline{z}}^2$ has the form

$$w_{\overline{z}\,\overline{z}}+\mu_1 w_{z\overline{z}}+\mu_2\overline{w_{z\overline{z}}}+a_1 w_{\overline{z}}+a_2\overline{w_{\overline{z}}}+b_1 w_z+b_2\overline{w_z}+c_1 w+c_2\overline{w}=d$$

where $|\mu_1(z)|+|\mu_2(z)|\le q_0<1$. Setting $w_{\overline{z}\,\overline{z}}=f$ so that $w=\varphi+\overline{z}\psi+T_{0,2}f$ leads to a singular integral equation for f. With proper integral operators $T_{\mu,\nu}$, see the following section, it is, similarly to the generalized Beltrami equation,

$$f+\mu_1 T_{-1,1}f+\mu_2 T_{1,-1}\overline{f}+a_1 T_{0,1}f+a_2 T_{1,0}\overline{f}+b_1 T_{-1,2}f+b_2 T_{2,-1}\overline{f}$$
$$+c_1 T_{0,2}f+c_2 T_{2,0}\overline{f}=d-\mu_1\psi'-\mu_2\overline{\psi'}-a_1\psi-a_2\overline{\psi}-b_1(\varphi'+\overline{z}\psi')$$
$$-b_2(\overline{\varphi'}+z\overline{\psi'})-c_1(\varphi+\overline{z}\psi)-c_2(\overline{\varphi}+z\overline{\psi})\,.$$

Here $T_{-1,1}$ and $T_{1,-1}$ are singular integral operators while the other ones are just weakly singular and give a compact operator.

1.2.3. *Complex higher order equations*

Continuing in the way indicated in the preceding subsections the prototyp $\partial_z^m\partial_{\overline{z}}^n w=f$ can be treated in regular domains $D\subset\mathbb{C}$.

Definition 1.1. Let for $m, n \in \mathbb{Z}$, $0 \le m+n$, $1 \le m^2+n^2$

$$K_{m,n}(z) = \begin{cases} \dfrac{(-m)!(-1)^m}{(n-1)!\,\pi}\, z^{m-1}\overline{z}^{n-1}, & m \le 0, \\ \dfrac{(-n)!\,(-1)^n}{(m-1)!\,\pi}\, z^{m-1}\overline{z}^{n-1}, & n \le 0, \\ \dfrac{z^{m-1}\overline{z}^{n-1}}{(m-1)!\,(n-1)!\,\pi}\Big[\log|z|^2 - \sum\limits_{\mu=1}^{m-1}\dfrac{1}{\mu} - \sum\limits_{\nu=1}^{n-1}\dfrac{1}{\nu}\Big], & 1 \le m, n. \end{cases}$$

These kernel functions determine fundamental solutions to $\partial_z^m \partial_{\overline{z}}^n$ for $0 \le m, n$, $0 < m^2+n^2$, see [8]. Their essential properties are

$$K_{m,n} = \partial_z K_{m+1,n} = \partial_{\overline{z}} K_{m,n+1}$$

and

$$\int_{|z|<R} |K_{m,n}(z)|\, dx\, dy < +\infty \quad \text{for} \quad 0 < m+n\,,\ 0 < R\,.$$

Definition 1.2. For $D \subset \mathbb{C}$ a domain, $f \in L_1(D;\mathbb{C})$ and $m, n \in \mathbb{Z}$ with $0 \le m+n$

$$T_{m,n}f(z) = \int_D K_{m,n}(z-\zeta)\, f(\zeta)\, d\xi\, d\eta \quad \text{if} \quad 1 \le m^2+n^2\,,$$

$$T_{0,0}f(z) = f(z)\,.$$

This is a hierarchy of integral operators with the Pompeiu operators as basic elements, namely

$$T_{0,1} = T\,, \quad T_{1,0} = \overline{T}\,, \quad T_{-1,1} = \Pi\,, \quad T_{1,-1} = \overline{\Pi}\,.$$

$T_{m,n}$ is a weakly singular integral operator for $0 < m+n$ but strongly singular of Calderon–Zygmund type to be understood as a Cauchy principal value integral operator if $m+n=0$ but $0<m^2+n^2$. Moreover,

$$|T_{m,n}f(z)| \le M\ \|f\|_p\,, \quad 1<p\,,\ \ 1\le m+n\,,\ \ |z|\le R\,,$$

$$|T_{m,n}f(z_1)-T_{m,n}f(z_2)| \le M\ |z_1-z_2|^\alpha\,,\ \ |z_1|,|z_2|\le R\,,$$

$$\alpha = \begin{cases} (p-2)/p\ , m+n=1\,,\ \ 2<p\,, \\ \quad 1 \qquad , m+n=2\,,\ \ 2<p\,;\ \ 3\le m+n\,,\ \ 1\le p\,, \end{cases}$$

$$\|T_{m,-m}f\|_{p,\mathbb{C}} \le M(p)\ \|f\|_{p,D}\,, \quad m\neq 0\,,$$

$$\|T_{m,-m}f\|_{2,\mathbb{C}} \le \|f\|_{2,\mathbb{C}}\,.$$

For details see [8].

There is a higher order Cauchy–Pompeiu formula related to the differential operator $\partial_z^m \partial_{\bar z}^n$. For simplicity only a particular case is presented here.

A higher–order Cauchy–Pompeiu formula *Let $D\subset\mathbb{C}$ be a regular domain and $w\in C^m(D;\mathbb{C})\cap C^{m-1}(\overline{D};\mathbb{C})$, $1\le m$. Then*

$$w(z) = \sum_{\mu=0}^{m-1} \frac{1}{2\pi i}\int_{\partial D} \frac{1}{\mu!}\ \frac{(\overline{z-\zeta})^\mu}{\zeta - z}\,\partial_{\bar\zeta}^\mu w(\zeta)\,d\zeta \\ -\frac{1}{\pi}\int_D \frac{1}{(m-1)!}\ \frac{(\overline{z-\zeta})^{m-1}}{\zeta-z}\,\partial_{\bar\zeta}^m w(\zeta)\,d\xi\,d\eta\,.$$

Proof. (1) For $m=1$ the formula coincides with one of the basic Cauchy–Pompeiu formulas.
(2) Assuming the formula holds for some m, $1\le m$, from the Gauss

theorem

$$\begin{aligned}
&\frac{1}{\pi}\int_D \frac{1}{m!}\,\frac{(\overline{z-\zeta})^m}{\zeta - z}\,\partial_{\overline{\zeta}}^{m+1} w(\zeta)\,d\xi\,d\eta \\
&\quad = \frac{1}{\pi}\int_D \Big\{\partial_{\overline{\zeta}}\Big[\frac{1}{m!}\,\frac{(\overline{z-\zeta})^m}{\zeta - z}\,\partial_{\overline{\zeta}}^{m} w(\zeta)\Big] \\
&\qquad\qquad + \frac{1}{(m-1)!}\,\frac{(\overline{z-\zeta})^{m-1}}{\zeta - z}\,\partial_{\overline{\zeta}}^{m} w(\zeta)\Big\}\,d\xi\,d\eta \\
&\quad = \frac{1}{2\pi i}\int_{\partial D} \frac{1}{m!}\,\frac{(\overline{z-\zeta})^m}{\zeta - z}\,\partial_{\overline{\zeta}}^{m} w(\zeta)\,d\zeta \\
&\qquad\qquad + \frac{1}{\pi}\int_D \frac{1}{(m-1)!}\,\frac{(\overline{z-\zeta})^{m-1}}{\zeta - z}\,\partial_{\overline{\zeta}}^{m} w(\zeta)\,d\xi\,d\eta\,.
\end{aligned}$$

This identity gives the formula for $m+1$ rather than m. □

Corollary 1.1. *Any $w \in C^m(D;\mathbb{C}) \cap C^{m-1}(\overline{D};\mathbb{C})$, $1 \le m$, with $\partial_{\overline{z}}^m w = f$ is representable as*

$$w(z) = \sum_{\mu=0}^{m-1} \varphi_\mu(z)\,\overline{z}^\mu + T_{0,m} f(z)\,. \tag{1.7}$$

Here φ_μ, $0 \le \mu \le m-1$, is analytic. $\sum_{\mu=0}^{m-1} \varphi_\mu(z)\,\overline{z}^\mu$ is as a polyanalytic function of order m, the general solution to the homogeneous equation $\partial_{\overline{z}}^m \varphi = 0$.

A complex m–th order equation of the form

$$\partial_{\overline{z}}^m w + \sum_{\substack{\rho+\sigma=m\\ \sigma\neq 0}} \mu_{\rho\sigma}\partial_{\overline{z}}^{\rho}\partial_z^{\sigma} w = F(z, \partial_{\overline{z}}^{\rho}\partial_z^{\sigma}\ w\,(0 \le \rho+\sigma < m))$$

with

$$\sum_{\substack{\rho+\sigma=m\\ \sigma\neq 0}} |\mu_{\rho\sigma}(z)| \le q_0 < 1$$

is transformed through (1.7) into a singular integral equation of the form

$$\begin{aligned}
f + \sum_{\substack{\rho+\sigma=m\\ \sigma\neq 0}} \mu_{\rho\sigma} T_{-\sigma,m-\rho} f - F(z, T_{-\sigma,m-\rho} f\ \ (0 \le \rho+\sigma < m)) \\
= G(\varphi_\mu(0 \le \mu \le m-1))\,.
\end{aligned}$$

Proper boundary conditions serve to determine the φ_μ through f and the boundary data.

1.2.4. *Orthogonal decomposition of $L_2(D;\mathbb{C})$*

The inner product for the Hilbert space of square integrable functions in D is defined as

$$(f,g) = \int_D \overline{f(z)}\, g(z)\, dx\, dy\,.$$

As before $D \subset \mathbb{C}$ is a regular domain.

Definition 1.3. The subset of polyholomorphic functions of order $k \geq 1$ in $L_2(D;\mathbb{C})$ is

$$\mathcal{O}_{k,2}(D;\mathbb{C}) = \{f : f \in L_2(D;\mathbb{C}),\ \partial_{\bar z}^k f = 0 \ \text{ in } \ D\}\,.$$

Its orthogonal complement is denoted by

$$\mathcal{O}_{k,2}^{\perp}(D;\mathbb{C}) = \{g : g \in L_2(D;\mathbb{C}),\ (g,f) = 0 \ \text{ for all } \ f \in \mathcal{O}_{k,2}(D;\mathbb{C})\}\,.$$

As usual

$$\overset{\circ}{W}{}_2^k(D;\mathbb{C}) = \{f : f \in W_2^k(D;\mathbb{C})\,,\ \ \partial_z^\nu f = 0 \ \text{ on } \ \partial D,\ 0 \leq \nu \leq k-1\}$$

denotes the subspace of functions with vanishing boundary data of the Sobolev space $W_2^k(D;\mathbb{C})$. To the latter belong all functions with weak derivatives up to the k–th order in $L_2(D;\mathbb{C})$.

Lemma 1.1. *For $q \in \mathcal{O}_{k,2}^{\perp}(D;\mathbb{C})$ the problem*

$$\partial_z^k r = q \ \ in \ \ D\,, \quad \partial_z^\nu r = 0 \ \ on \ \ \partial D \ \ for \ \ 0 \leq \nu \leq k-1$$

is equivalent to the problem

$$\Delta^k r = 4^k \partial_{\bar z}^k q \ \ in \ \ D,\ \partial_z^\nu r = 0 \ \ on\ \partial D \ \ for \ \ 0 \leq \nu \leq k-1\,.$$

Proof. (1) Applying $4^k\partial_{\bar z}^k$ to $\partial_z^k r = q$ shows $\Delta^k r = 4^k \partial_{\bar z}^k q$.
(2) A solution r to the second problem satisfies

$$\partial_{\bar z}^k(\partial_z^k r - q) = 0\,.$$

Hence, $\partial_z^k r - q \in \mathcal{O}_{k,2}(D;\mathbb{C})$.

Let now $r \in \overset{\circ}{W}{}^k_2(D;\mathbb{C})$ and $\varphi \in \mathcal{O}_{k,2}(D;\mathbb{C})$ then

$$
\begin{aligned}
(\partial_z^k r, \varphi) &= \int_D \partial_{\overline{z}}^k \overline{r(z)}\, \varphi(z)\, dx\, dy \\
&= \int_D \left\{ \partial_{\overline{z}} [\partial_{\overline{z}}^{k-1} \overline{r(z)}\, \varphi(z)] - \partial_{\overline{z}}^{k-1} \overline{r(z)}\, \partial_{\overline{z}} \varphi(z) \right\} dx\, dy \\
&= \frac{1}{2i} \int_{\partial D} \overline{\partial_z^{k-1} r(z)}\ \varphi(z)\, dz - \int_D \partial_{\overline{z}}^{k-1} \overline{r(z)}\, \partial_{\overline{z}} \varphi(z)\, dx\, dy = \cdots \\
&= (-1)^k \int_D \overline{r(z)}\, \partial_{\overline{z}}^k \varphi(z)\, dx\, dy = 0\,.
\end{aligned}
$$

Thus, $\partial_z^k r \in \mathcal{O}_{k,2}^{\perp}(D;\mathbb{C})$. As also $q \in \mathcal{O}_{k,2}^{\perp}(D;\mathbb{C})$ then $\partial_z^k r - q \in \mathcal{O}_{k,2}^{\perp}(D;\mathbb{C})$. Therefore $\partial_z^k r - q = 0$. □

Remark. While the problem

$$\partial_z^k r = q \ \text{ in } \ D\,, \quad \partial_z^\nu r = 0 \ \text{ on } \ \partial D \ \text{ for } 0 \le \nu \le k-1$$

is overdetermined, the problem

$$\Delta^k r = \widetilde{q} \ \text{ in } \ D\,, \quad \partial_z^\nu r = 0 \ \text{ on } \ \partial D \ \text{ for } 0 \le \nu \le k-1$$

is well–posed.

Theorem 1.2. *For regular domains $D \subset \mathbb{C}$*

$$\mathcal{O}_{k,2}^{\perp}(D;\mathbb{C}) = \partial_z^k \overset{\circ}{W}{}^k_2(D;\mathbb{C})\,.$$

Proof. (1) Consider for $q \in \mathcal{O}_{k,2}^{\perp}(D;\mathbb{C})$ the problem

$$\partial_z^k r = q \ \text{ in } \ D\,, \quad \partial_z^\nu r = 0 \ \text{ on } \ \partial D \ \text{ for } \ 0 \le \nu \le k-1\,.$$

This problem is solvable according to the preceding lemma. The solution is representable via the Cauchy–Pompeiu formula

$$
\begin{aligned}
r(z) = -\sum_{\mu=0}^{k-1} \frac{1}{2\pi i} \int_{\partial D} \frac{1}{\mu!}\, \frac{(z-\zeta)^\mu}{\overline{\zeta - z}}\ \partial_\zeta^\mu r(\zeta)\, d\overline{\zeta} \\
- \frac{1}{\pi} \int_D \frac{1}{(k-1)!}\, \frac{(z-\zeta)^{k-1}}{\overline{\zeta - z}}\ \partial_\zeta^k r(\zeta)\, d\xi\, d\eta
\end{aligned}
$$

so that

$$r(z) = -\frac{1}{\pi}\int_D \frac{1}{(k-1)!}\ \frac{(z-\zeta)^{k-1}}{\overline{\zeta - z}}\ q(\zeta)\,d\xi\,d\eta\,.$$

The Green function

$$G_k(z,\zeta) = \frac{1}{(k-1)!^2}|\zeta - z|^{2(k-1)}[\log|\zeta - z|^2 - 2\sum_{\rho=1}^{k-1}\frac{1}{\rho}] + h_k(z,\zeta)$$

of the differential operator Δ^k where $h_k \in C^{2k}(D\times D;\mathbb{C})\cap C^{2k-1}(\overline{D}\times\overline{D};\mathbb{C})$ with $\Delta_z^k h_k(z,\zeta) = 0$ in $D\times D$, $\partial_z^\rho G_k(z,\zeta) = 0$, $\partial_{\overline{z}}^\rho G_k(z,\zeta) = 0$ on $\partial D\times D$ for $0\le\rho\le k-1$ satisfies

$$\partial_{\overline{\zeta}}^k G_k(z,\zeta) = \frac{(\zeta - z)^{k-1}}{(k-1)!}\ \frac{1}{\overline{\overline{\zeta - z}}} + \partial_{\overline{\zeta}}^k h_k(z,\zeta)\,.$$

From the Gauss theorem

$$\begin{aligned}
&\int_D \partial_{\overline{\zeta}}^k h_k(z,\zeta)\,\partial_\zeta^k r(\zeta)\,d\xi\,d\eta \\
&= \int_D \{\partial_\zeta[\partial_{\overline{\zeta}}^k h_k(z,\zeta)\,\partial_\zeta^{k-1} r(\zeta)] - \partial_\zeta\partial_{\overline{\zeta}}^k h_k(z,\zeta)\,\partial_\zeta^{k-1} r(\zeta)\}\ d\xi\,d\eta \\
&= -\frac{1}{2i}\int_{\partial D} \partial_{\overline{\zeta}}^k h_k(z,\zeta)\,\partial_\zeta^{k-1} r(\zeta)\,d\overline{\zeta} - \int_D \partial_\zeta\partial_{\overline{\zeta}}^k h_k(z,\zeta)\,\partial_\zeta^{k-1} r(\zeta)\,d\xi\,d\eta \\
&= \cdots = (-1)^k\int_D \partial_\zeta^k\partial_{\overline{\zeta}}^k h_k(z,\zeta)\,r(\zeta)\,d\xi\,d\eta = 0
\end{aligned}$$

follows. Adding this to the representation of r gives

$$\begin{aligned}
r(z) &= \frac{(-1)^k}{\pi}\int_D \partial_{\overline{\zeta}}^k G_k(z,\zeta)\,q(\zeta)\,d\xi\,d\eta \\
&= -\frac{1}{\pi}\int_D \frac{1}{(k-1)!}\ \frac{(z-\zeta)^{k-1}}{\overline{\zeta - z}}\ q(\zeta)\,d\xi\,d\eta\,.
\end{aligned}$$

By differentiation

$$\begin{aligned}
\partial_z^{k-\nu} r(z) &= \frac{(-1)^k}{\pi}\int_D \partial_{\overline{\zeta}}^k\partial_z^{k-\nu} G_k(z,\zeta)\,q(\zeta)\,d\xi\,d\eta \\
&= -\frac{1}{\pi}\int_D \frac{1}{(\nu-1)!}\ \frac{(z-\zeta)^{\nu-1}}{\overline{\zeta - z}}\ q(\zeta)\,d\xi\,d\eta
\end{aligned}$$

follows for $1\le\nu\le k$. From here

$$\|r\|_{W_2^k} \le M\ \|q\|_{L_2}$$

can be shown. As $r \in \overset{\circ}{W}{}_2^k(D;\mathbb{C})$ satisfies $\partial_z^k r = q$ one has $q \in \partial_z^k \overset{\circ}{W}{}_2^k(D;\mathbb{C})$.

(2) Let $q \in \partial_z^k \overset{\circ}{W}{}_2^k(D;\mathbb{C})$. Then there exists an $r \in \overset{\circ}{W}{}_2^k(D;\mathbb{C})$ such that $\partial_z^k r = q$ in D, $\partial_z^\nu r = 0$ on ∂D for $0 \le \nu \le k-1$. Let now $\varphi \in \mathcal{O}_{k,2}(D;\mathbb{C})$. Then as before

$$(q,\varphi) = \int_D \partial_{\overline{z}}^k \overline{r(z)}\, \varphi(z)\, dx\, dy = (-1)^k \int_D \overline{r(z)}\, \partial_{\overline{z}}^k \varphi(z)\, dx\, dy = 0\,.$$

Thus $q \in \mathcal{O}_{k,2}^\perp(D;\mathbb{C})$. □

Corollary 1.2. *For regular domains $D \subset \mathbb{C}$*

$$L_2(D;\mathbb{C}) = \mathcal{O}_{k,2}(D;\mathbb{C}) \oplus \partial_z^k \overset{\circ}{W}{}_2^k(D;\mathbb{C})\,.$$

By interchanging the roles of z and $\overline{z}$ a dual result is available. In the same way $L_2(D;\mathbb{C})$ can be orthogonally decomposed with respect to polyharmonic functions.

Definition 1.4. The subset of polyharmonic functions of order $k \ge 1$ in $L_2(D;\mathbb{C})$ is

$$\mathbb{H}_{k,2}(D;\mathbb{C}) = \{f : f \in L_2(D;\mathbb{C}), \partial_z^k \partial_{\overline{z}}^k f = 0 \ \text{ in } \ D\}\,.$$

Its orthogonal complement is denoted by

$$\mathbb{H}_{k,2}^\perp(D;\mathbb{C}) = \{g : g \in L_2(D;\mathbb{C}), (g,f) = 0 \ \text{ for all } \ f \in \mathbb{H}_{k,2}(D;\mathbb{C})\}\,.$$

Moreover,

$$\overset{\circ}{W}{}_{\Delta^k,2}^{2k}(D;\mathbb{C}) = \{f : f \in W_2^{2k}(D;\mathbb{C}), \partial_z^\nu \partial_{\overline{z}}^\nu f = 0,\ \partial_z^{\nu+1} \partial_{\overline{z}}^\nu f = 0\,,$$
$$\partial_z^\nu \partial_{\overline{z}}^{\nu+1} f = 0 \ \text{ on } \ \partial G \ \text{ for } \ 0 \le \nu \le k-1\}\,.$$

Theorem 1.3. *For regular domains $D \subset \mathbb{C}$*

$$\mathbb{H}_{k,2}^\perp(D;\mathbb{C}) = \partial_z^k \partial_{\overline{z}}^k \overset{\circ}{W}{}_{\Delta^k,2}^{2k}(D;\mathbb{C})$$

and

$$L_2(D;\mathbb{C}) = \mathbb{H}_{k,2}(D;\mathbb{C}) \oplus \partial_z^k \partial_{\overline{z}}^k \overset{\circ}{W}{}_{\Delta^k,2}^{2k}(D;\mathbb{C})\,.$$

For a proof see [7].

1.3. Several complex variables

Quite a natural extension of the representation formulas for one complex variable to higher dimensions is available for polydomains. In principle the unit ball can be treated too, see [4, 5], but because of the complicated structure of the Pompeiu operator, see [12], iterations cannot yet be given explictly. In order to present the concept of the procedure for polydomains just bidomains are studied in $\mathbb{C}^2$. The extension to higher dimensions is then obvious, see [6].

Theorem 1.4. *Let D_1 and D_2 be regular domains in $\mathbb{C}$ and $D = D_1 \times D_2 \subset \mathbb{C}^2$ be the bidomain composed by D_1 and D_2 and $\partial_0 D = \partial D_1 \times \partial D_2$ the distinguished boundary of D. Any $w \in C^1(D;\mathbb{C}) \cap C(D \cup \partial_0 D;\mathbb{C})$ can be represented as*

$$\begin{aligned} w(z_1,z_2) \;=\; & \frac{1}{(2\pi i)^2}\int_{\partial D_1}\int_{\partial D_2} w(\zeta_1,\zeta_2)\,\frac{d\zeta_1}{\zeta_1 - z_1}\,\frac{d\zeta_2}{\zeta_2 - z_2} \\ & -\frac{1}{\pi}\int_{D_1} w_{\overline{\zeta_1}}(\zeta_1,z_2)\,\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1} - \frac{1}{\pi}\int_{D_2} w_{\overline{\zeta_2}}(z_1,\zeta_2)\,\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2} \\ & -\frac{1}{\pi^2}\int_{D_1}\int_{D_2} w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\,\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}\,\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}\,. \end{aligned} \tag{1.8}$$

Proof. Applying the Cauchy–Pompeiu formula for D_1 and then for D_2 gives

$$\begin{aligned} w(z_1,z_2) &= \frac{1}{2\pi i}\int_{\partial D_1} w(\zeta_1,z_2)\,\frac{d\zeta_1}{\zeta_1 - z_1} - \frac{1}{\pi}\int_{D_1} w_{\overline{\zeta_1}}(\zeta_1,z_2)\,\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1} \\ &= \frac{1}{(2\pi i)^2}\int_{\partial D_1}\int_{\partial D_2} w(\zeta_1,\zeta_2)\,\frac{d\zeta_1}{\zeta_1 - z_1}\,\frac{d\zeta_2}{\zeta_2 - z_2} \\ &\quad -\frac{1}{2\pi^2 i}\int_{\partial D_1}\int_{D_2} w_{\overline{\zeta_2}}(\zeta_1,\zeta_2)\,\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}\,\frac{d\zeta_1}{\zeta_1 - z_1} \\ &\quad -\frac{1}{2\pi^2 i}\int_{D_1}\int_{\partial D_2} w_{\overline{\zeta_1}}(\zeta_1,\zeta_2)\,\frac{d\zeta_2}{\zeta_2 - z_2}\,\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1} \\ &\quad +\frac{1}{\pi^2}\int_{D_1}\int_{D_2} w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\,\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}\,\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}\,. \end{aligned}$$

With

$$\frac{1}{\pi}\int_{D_1} w_{\overline{\zeta_1}}(\zeta_1, z_2)\,\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1} = \frac{1}{2\pi^2 i}\int_{D_1}\int_{\partial D_2} w_{\overline{\zeta_1}}(\zeta_1,\zeta_2)\,\frac{d\zeta_2}{\zeta_2 - z_2}\,\frac{d\xi_1 d\eta_1}{\zeta_1 - z_1}$$
$$-\frac{1}{\pi^2}\int_{D_1}\int_{D_2} w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\,\frac{d\xi_1 d\eta_1}{\zeta_1 - z_2}\,\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2},$$

$$\frac{1}{\pi}\int_{D_2} w_{\overline{\zeta_2}}(z_1, \zeta_2)\,\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2} = \frac{1}{2\pi^2 i}\int_{\partial D_1}\int_{D_2} w_{\overline{\zeta_2}}(\zeta_1,\zeta_2)\,\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}\,\frac{d\zeta_1}{\zeta_1 - z_1}$$
$$-\frac{1}{\pi^2}\int_{D_1}\int_{D_2} w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\,\frac{d\xi_1 d\eta_1}{\zeta_1 - z_2}\,\frac{d\xi_2 d\eta_2}{\zeta_2 - z_2}$$

the above formula follows. □

Remarks. For polydomains $C^k(D;\mathbb{C})$ denotes the set of complex functions in D having continuous derivatives with respect to any single variable up to order k. E.g. for a bidomain D and $k = 1$ the functions w have continuous derivatives w_{z_1}, $w_{\overline{z_1}}$, w_{z_2}, $w_{\overline{z_2}}$, $w_{z_1 z_2}$, $w_{z_1\overline{z_2}}$, $w_{\overline{z_1}z_2}$, $w_{\overline{z_1}\,\overline{z_2}}$. Other representation formulas are available by replacing ζ_k by $\overline{\zeta_k}$ for one index or both indices $k = 1, 2$. The respective representation in the case of $\mathbb{C}^n$ is

$$w(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0 D} w(\zeta)\prod_{\nu=1}^{n}\frac{d\zeta_\nu}{\zeta_\nu - z_\nu}$$
$$-\sum_{\nu=1}^{n}\sum_{1\le\rho_1<\cdots<\rho_\nu\le n}\frac{1}{\pi^\nu}\int_{D_{\rho_1}}\cdots\int_{D_{\rho_\nu}} w_{\overline{\zeta_{\rho_1}}\cdots\overline{\zeta_{\rho_\nu}}}(\zeta)\prod_{\mu=1}^{\nu}\frac{d\xi_{\rho_\mu} d\eta_{\rho_\mu}}{\zeta_{\rho_\mu} - z_{\rho_\mu}}\,.$$

In this formula components of ζ not integrated upon have to be interpreted as z–components.

Iterating (1.8) leads to a second order representation.

Theorem 1.5. *Any* $w \in C^2(D;\mathbb{C}) \cap C^1(D \cup \partial_0 D;\mathbb{C})$ *can be represented as well as*

$$
\begin{aligned}
w(z_1,z_2) = \frac{1}{(2\pi i)^2} \int_{\partial D_1} \int_{\partial D_2} \bigg\{ & \frac{w(\zeta_1,\zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} - \frac{w_{\overline{\zeta_1}}(\zeta_1,\zeta_2)}{\zeta_2 - z_2} \frac{\overline{\zeta_1 - z_1}}{\zeta_1 - z_1} \\
& - \frac{w_{\overline{\zeta_2}}(\zeta_1,\zeta_2)}{\zeta_1 - z_1} \frac{\overline{\zeta_2 - z_2}}{\zeta_2 - z_2} + w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2) \frac{\overline{\zeta_1 - z_1}}{\zeta_1 - z_1} \frac{\overline{\zeta_2 - z_2}}{\zeta_2 - z_2} \bigg\} d\zeta_1 d\zeta_2 \\
& + \frac{1}{\pi} \int_{D_1} w_{\overline{\zeta_1}\,\overline{\zeta_1}}(\zeta_1,z_2) \frac{\overline{\zeta_1 - z_1}}{\zeta_1 - z_1} d\xi_1 d\eta_1 \qquad (1.9) \\
& + \frac{1}{\pi} \int_{D_2} w_{\overline{\zeta_2}\,\overline{\zeta_2}}(z_1,\zeta_2) \frac{\overline{\zeta_2 - z_2}}{\zeta_2 - z_2} d\xi_2 d\eta_2 \\
& - \frac{1}{\pi^2} \int_{D_1} \int_{D_2} w_{\overline{\zeta_1}\,\overline{\zeta_1}\,\overline{\zeta_2}\,\overline{\zeta_2}}(\zeta_1,\zeta_2) \frac{\overline{\zeta_1 - z_1}}{\zeta_1 - z_1} \frac{\overline{\zeta_2 - z_2}}{\zeta_2 - z_2} d\xi_1 d\eta_1 d\xi_2 d\eta_2
\end{aligned}
$$

as via

$$
\begin{aligned}
w(z_1,z_2) = \frac{1}{(2\pi i)^2} \int_{\partial D_1} \int_{\partial D_2} \bigg\{ & \frac{w(\zeta_1,\zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} - \frac{w_{\overline{\zeta_1}}(\zeta_1,\zeta_2)}{\zeta_2 - z_2} \frac{\overline{\zeta_1 - z_1}}{\zeta_1 - z_1} \\
& - \frac{w_{\overline{\zeta_2}}(\zeta_1,\zeta_2)}{\zeta_1 - z_1} \frac{\overline{\zeta_2 - z_2}}{\zeta_2 - z_2} \bigg\} d\zeta_1 d\zeta_2 \qquad (1.10) \\
& + \frac{1}{\pi} \int_{D_1} w_{\overline{\zeta_1}\,\overline{\zeta_1}}(\zeta_1,z_2) \frac{\overline{\zeta_1 - z_1}}{\zeta_1 - z_1} d\xi_1 d\eta_1 \\
& + \frac{1}{\pi} \int_{D_2} w_{\overline{\zeta_2}\,\overline{\zeta_2}}(z_1,\zeta_2) \frac{\overline{\zeta_2 - z_2}}{\zeta_2 - z_2} d\xi_2 d\eta_2 \\
& + \frac{1}{\pi^2} \int_{D_1} \int_{D_2} \bigg\{ \frac{w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} + \frac{w_{\overline{\zeta_1}\,\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)}{\zeta_2 - z_2} \frac{\overline{\zeta_1 - z_1}}{\zeta_1 - z_1} \\
& + \frac{w_{\overline{\zeta_1}\,\overline{\zeta_2}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)}{\zeta_1 - z_1} \frac{\overline{\zeta_1 - z_2}}{\zeta_2 - z_2} \bigg\} d\xi_1 d\eta_1 d\xi_2 d\eta_2 \,.
\end{aligned}
$$

Proof. Iterating instead of (1.8) the respective second order representations for one complex variable applied to z_1 and z_2 leads to (1.10). The equivalent formula (1.10) then follows as by the Gauss theorem

$$\frac{1}{(2\pi i)^2}\int_{\partial D_1}\int_{\partial D_2} w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\,\frac{\overline{\zeta_1-z_1}}{\zeta_1-z_1}\,\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}\,d\zeta_1 d\zeta_2$$

$$=\frac{1}{\pi^2}\int_{D_1}\int_{D_2}\partial_{\overline{\zeta_1}}\partial_{\overline{\zeta_2}}\Big[w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\,\frac{\overline{\zeta_1-z_1}}{\zeta_1-z_1}\,\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}\Big]d\xi_1 d\eta_1 d\xi_2 d\eta_2\,.$$

□

Remarks There are other second order representations by interchanging $\overline{\zeta}_k$–derivatives with ζ_k–derivatives for $k=1$ and/or $k=2$ and at the same time the $(\overline{\zeta_k - z_k})/(\zeta_k - z_k)$–kernels with the $\log|\zeta_k - z_k|^2$–kernels.

Although these representation formulas express the function through boundary values of proper lower–order derivatives this does not imply that the related boundary value problem is solvable and the solution is given by this formula. Only if the solvability is guaranteed then this representation formula may be used for representing the solution.

Of course, there are also mixed order representations. Again only one example is formulated.

Theorem 1.6. *Let w be defined and complex–valued in the regular bidomain D such that $w_{\overline{z_1}}$ and $w_{\overline{z_1}\,\overline{z_2}}$ are continuous and $w_{\overline{z_1}\,\overline{z_2}\,\overline{z_2}} = w_{\overline{z_2}\,\overline{z_2}\,\overline{z_1}}$. Then*

$$w(z_1,z_2)$$

$$=\frac{1}{(2\pi i)^2}\int_{\partial D_1}\int_{\partial D_2}\left\{\frac{w(\zeta_1,\zeta_2)}{(\zeta_1-z_1)(\zeta_2-z_2)}-\frac{w_{\overline{\zeta_2}}(\zeta_1,\zeta_2)}{\zeta_1-z_1}\,\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}\right\}d\zeta_1 d\zeta_2$$

$$-\frac{1}{\pi}\int_{D_1} w_{\overline{\zeta_1}}(\zeta_1,z_2)\,\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}+\frac{1}{\pi}\int_{D_2} w_{\overline{\zeta_2}\,\overline{\zeta_2}}(z_1,\zeta_2)\,\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}\,d\xi_2 d\eta_2$$

$$+\frac{1}{\pi^2}\int_{D_1}\int_{D_2}\frac{w_{\overline{\zeta_1}\,\overline{\zeta_2}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)}{\zeta_1-z_1}\,\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}\,d\xi_1 d\eta_1 d\xi_2 d\eta_2\,. \qquad (1.11)$$

The proof again follows by proper iteration of the Cauchy–Pompeiu formulas. Other formulas of this kind with ∂_ζ–operators instead of $\partial_{\overline{\zeta}}$–operators are available. Generalization to more than two variables are obvious, see [4, 6].

1.4. Clifford analysis

Let $\{e_k : 1 \le k \le m\}$ be an orthonormal basis of $\mathbb{R}^m$ with $2 \le m$ such that $x \in \mathbb{R}^m$ is represented as $x = \sum_{\mu=1}^m x_k e_k$. Introducing a multiplication via

$$e_1 = 1\,, \quad e_j e_k + e_k e_j = -2\delta_{jk}\,, \quad 2 \le j,k \le m\,,$$

a Clifford algebra $\mathbb{C}_m$ is introduced as the set of elements $a = \sum_A a_A e_A$ where $a_A \in \mathbb{C}$

$$e_A = 1 \;\text{ if }\; A = \emptyset\,, \quad e_A = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k} \;\text{ if }\; A = \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$$

with $2 \le \alpha_1 < \alpha_2 < \cdots < \alpha_k \le m$ and the sum is taken over all subsets A of $\{2, 3, \ldots, m\}$.

If $m = 2$ the multiplication is commutative and $\mathbb{C}_2$ with $e_2 = i$ is just the field of complex numbers $\mathbb{C}$. Otherwise $\mathbb{C}_m$ is a noncommutative algebra over $\mathbb{C}$ of dimension 2^{m-1}. By

$$\overline{a} = \sum_A \overline{a_A}\,\overline{e_A} \;\text{ for }\; a = \sum_A a_A e_A$$

with

$$\overline{e_1} = e_1\,, \quad \overline{e_k} = -e_k\,, \quad 2 \le k \le m\,, \quad \overline{e_A e_B} = \overline{e_B}\,\overline{e_A}$$

a complex conjugation is introduced. Denoting

$$|a| = \Big(\sum_A |a_A|^2\Big)^{1/2} \;\text{ for }\; a = \sum_A a_A e_A$$

via $|a|_0 = 2^{m/2}|a|$ an algebra norm is defined. Identifying $x = \sum_{\mu=1}^m x_k e_k \in \mathbb{R}^m$ with $z = \sum_{\mu=1}^m x_k e_k \in \mathbb{C}_m$ the space $\mathbb{R}^m$ is embedded into $\mathbb{C}_m$. These elements satisfy $z\overline{z} = \overline{z}z = |z|^2$.

A natural basic first order differential operator for functions defined on subsets of $\mathbb{R}^m$ with values in $\mathbb{C}_m$ is the so–called Dirac operator $\partial = \sum_{\mu=1}^m e_k \partial_{x_k}$ and its complex conjugate $\overline{\partial} = \partial_{x_1} - \sum_{\mu=2}^m e_k \partial_{x_k}$. For $m = 2$ they essentially coincide with the Cauchy–Riemann operator and its complex conjugate $2\partial_{\overline{z}} = \partial_x + i\partial_y$, $2\partial_z = \partial_x - i\partial_y$. The importance of these operators is the connection with the Laplace operator $\partial\overline{\partial} = \overline{\partial}\partial = \Delta$. This factorization makes Clifford analysis important for mathematical physics.

Some basic differentiation rules are

$$\partial z = z\partial = 2 - m\,, \quad \partial\overline{z} = \overline{z}\partial = m\,, \quad \partial|z|^2 = |z|^2\partial = 2z\,,$$

$$\partial|z|^\alpha = |z|^\alpha\partial = \alpha\,|z|^{\alpha-2}z\,, \quad \partial(\overline{z}/|z|^m) = (\overline{z}/|z|^m)\,\partial = 0\,.$$

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The last rule identifies $\overline{z}/|z|^m$ as the fundamental solution to the Dirac equation. For more detailed information see [10, 11].

Basic for representation formulas for functions in Clifford algebra is a version of the Gauss theorem. Because of the anticommutativity of multiplication two functions are involved here.

Gauss Theorem. *Let $D \subset \mathbb{R}^m$ be a regular domain and $f, g \in C^1(D; \mathbb{C}_m) \cap C(\overline{D}; \mathbb{C}_m)$. Then*

$$\int_D [(f\partial)\, g + f(\partial g)]\, dv = \int_{\partial G} f\, d\vec{\sigma}\, g\,,$$

$$\int_D [(f\overline{\partial})\, g + f(\overline{\partial} g)]\, dv = \int_{\partial G} f\, d\overline{\vec{\sigma}}\, g\,.$$

Here dv denotes the volume element of D, $d\sigma$ the area element of ∂D, $n = (n_1, \cdots, n_m)$ the outward normal vector on ∂D, $\vec{n} = \sum_{\mu=1}^m n_\mu e_\mu$ the corresponding element in $\mathbb{C}_m$, and $d\vec{\sigma} = d\sigma\, \vec{n}$ the directed area element on ∂D, $d\overline{\vec{\sigma}} = d\sigma\, \overline{\vec{n}}$ its complex conjugate.

Proof. (1) Let $f, g \in C^1(D; \mathbb{C}) \cap C(\overline{D}; \mathbb{C})$. Then from the classical Gauss theorem

$$\int_D [f_{x_\mu} g + f g_{x_\mu}]\, dv = \int_D \partial_{x_\mu}(fg)\, dv = \int_{\partial D} fg\, n_\mu d\sigma$$

multiplication by e_μ and adding up gives

$$\int_D [(\partial f)\, g + f\, (\partial g)]\, dv = \int_{\partial D} fg\, d\vec{\sigma}\,.$$

Rearranging this formula and replacing f by f_A and g by g_B, $A, B \subset \{2, \dots, m\}$ gives

$$\int_D [(f_A \partial)\, g_B + f_A(\partial g_B)]\, dv = \int_{\partial D} f_A d\vec{\sigma}\, g_B\,.$$

Multiplying with e_A from the left and e_B on the right and taking sums give

$$\int_D [(f\partial)\, g + f\, (\partial g)]\, dv = \int_{\partial D} f\, d\vec{\sigma}\, g\,.$$

The second formula follows analogously or by complex conjugation of the first one. □

From this Gauss theorem Cauchy–Pompeiu representation formulas follow in the same way as in the complex case.

Cauchy–Pompeiu representation *Any* $w \in C^1(D; \mathbb{C}_m) \cap C(\overline{D}; \mathbb{C}_m)$ *can be represented as*

$$w(z) = \frac{1}{\omega_m} \int_{\partial D} \frac{\overline{\zeta - z}}{|\zeta - z|^m} \, d\vec{\sigma}(\zeta) \, w(\zeta) - \frac{1}{\omega_m} \int_D \frac{\overline{\zeta - z}}{|\zeta - z|^m} \, \partial w(\zeta) \, dv(\zeta) \,,$$

$$w(z) = \frac{1}{\omega_m} \int_{\partial D} \frac{\zeta - z}{|\zeta - z|^m} \, d\overline{\vec{\sigma}(\zeta)} \, w(\zeta) - \frac{1}{\omega_m} \int_D \frac{\zeta - z}{|\zeta - z|^m} \, \overline{\partial} w(\zeta) \, dv(\zeta) \,.$$

Here ω_m denotes the area of the unit sphere in $\mathbb{R}^m$. There are dual formulas where the function and its derivative, respectively and the kernel function are changing their positions with one another.

Proof. Let $0 < \varepsilon$ be so small that $D_\varepsilon = \{\zeta : \zeta \in D, \varepsilon < |\zeta - z|\}$ is a regular domain. Then from the Gauss theorem

$$\int_{\partial D_\varepsilon} \frac{\overline{\zeta - z}}{|\zeta - z|^m} \, d\vec{\sigma}(\zeta) \, w(\zeta) = \int_{D_\varepsilon} \frac{\overline{\zeta - z}}{|\zeta - z|^m} \, \partial w(\zeta) \, dv(\zeta) \,.$$

As

$$\begin{aligned} &\frac{1}{\omega_m} \int_{|\zeta - z| = \varepsilon} \frac{\overline{\zeta - z}}{|\zeta - z|^m} \, d\vec{\sigma}(\zeta) \, w(\zeta) \\ &\qquad = \frac{1}{\omega_m} \int_{|\omega|=1} \varepsilon^{1-m} \overline{\omega} \, \varepsilon^{m-1} \omega \, d\sigma(\omega) \, w(z + \varepsilon\omega) \\ &\qquad = \frac{1}{\omega_m} \int_{|\omega|=1} w(z + \varepsilon\omega) \, d\sigma(\omega) \end{aligned}$$

tends to $w(z)$ with ε tending to 0 and

$$\begin{aligned} &\left| \int_{|\zeta - z| < \varepsilon} \frac{\overline{\zeta - z}}{|\zeta - z|^m} \, \partial w(\zeta) \, dv(\zeta) \right|_0 \\ &\qquad \leq 2^m \int_0^\varepsilon \int_{|\omega|=1} t^{1-m} |\partial w(z + t\omega)| \, t^{m-1} dt \, d\sigma(\omega) \end{aligned}$$

tends to 0 with ε, the first representation formula follows. The second can be deduced similarly. It also could be attained from the dual of the first formula where the places of the kernel and the function w and $w\partial$, respectively are interchanged with one another, by complex conjugation.

Iterating the first formula leads to higher order Cauchy–Pompeiu formulas as before. □

Theorem 1.7. *Let $w \in C^k(D;\mathbb{C}_m) \cap C^{k-1}(\overline{D},\mathbb{C}_m)$ for $1 \le k$. Then*

$$w(z) = \sum_{\mu=0}^{k-1} \frac{1}{\omega_m} \int_{\partial D} \frac{\overline{(\zeta - z)}(\overline{z-\zeta} + z - \zeta)^\mu}{2^\mu \mu! \ \ |\zeta - z|^m} \, d\vec{\sigma}(\zeta)\, \partial^\mu w(\zeta)$$

$$- \frac{1}{\omega_m} \int_D \frac{\overline{(\zeta - z)}(\overline{z-\zeta} + z - \zeta)^{k-1}}{2^{k-1}(k-1)! \ \ |\zeta - z|^m} \, \partial^k w(\zeta)\, dv(\zeta)\,.$$

Also the two Cauchy–Pompeiu formulas can be iterated with one another leading to a formula related to the Laplacian $\Delta = \partial\overline{\partial}$.

Theorem 1.8. *Let $w \in C^2(D;\mathbb{C}_m) \cap C^1(\overline{D};\mathbb{C}_m)$. Then*

$$w(z) = \frac{1}{\omega_m} \int_{\partial D} \frac{\overline{\zeta - z}}{|\zeta - z|^m} \, d\vec{\sigma}(\zeta)\, w(\zeta) - \frac{1}{\omega_m} \int_{\partial D} \frac{|\zeta - z|^{2-m}}{2-m} \, d\overline{\vec{\sigma}(\zeta)}\, \partial w(\zeta)$$

$$+ \frac{1}{\omega_m} \int_D \frac{|\zeta - z|^{2-m}}{2-m} \, \Delta w(\zeta)\, dv(\zeta)\,.$$

Proof. From the Cauchy–Pompeiu formulas

$$w(z) = \frac{1}{\omega_m} \int_{\partial D} \frac{\zeta - z}{|\zeta - z|^m} \, d\vec{\sigma}(\zeta)\, w(\zeta) - \frac{1}{\omega_m} \int_D \frac{\overline{\widetilde{\zeta} - z}}{|\widetilde{\zeta} - z|^m} \, \partial w(\widetilde{\zeta})\, dv(\widetilde{\zeta})\,,$$

$$\partial w(\widetilde{\zeta}) = \frac{1}{\omega_m} \int_{\partial D} \frac{\zeta - \widetilde{\zeta}}{|\zeta - \widetilde{\zeta}|^m} \, d\overline{\vec{\sigma}(\zeta)}\, \partial w(\zeta) - \frac{1}{\omega_m} \int_D \frac{\zeta - \widetilde{\zeta}}{|\zeta - \widetilde{\zeta}|^m} \, \Delta w(\zeta)\, dv(\zeta)$$

it follows

$$w(z) = \frac{1}{\omega_m} \int_{\partial D} \frac{\overline{\zeta - z}}{|\zeta - z|^m} \, d\vec{\sigma}(\zeta)\, w(\zeta) + \frac{1}{\omega_m} \int_{\partial D} \psi(\zeta, z)\, d\overline{\vec{\sigma}(\zeta)}\, \partial w(\zeta)$$

$$- \frac{1}{\omega_m} \int_D \psi(\zeta, z)\, \Delta w(\zeta)\, dv(\zeta)$$

where

$$\psi(\zeta, z) = \frac{1}{\omega_m} \int_D \frac{\overline{\widetilde{\zeta} - \zeta}}{|\widetilde{\zeta} - z|^m} \, \frac{\widetilde{\zeta} - \zeta}{|\widetilde{\zeta} - \zeta|^m} \, dv(\zeta)\,.$$

By an analogue argumentation as for (2)

$$\frac{|\zeta - z|^{2-m}}{2-m} = \frac{1}{\omega_m} \int_{\partial D} \frac{\overline{\widetilde{\zeta} - z}}{|\widetilde{\zeta} - z|^m} \, d\vec{\sigma}(\widetilde{\zeta}) \, \frac{|\widetilde{\zeta} - \zeta|^{2-m}}{2-m} \tag{1.12}$$

$$-\frac{1}{\omega_m} \int_D \frac{\overline{\widetilde{\zeta} - z}}{|\widetilde{\zeta} - z|^m} \; \frac{\widetilde{\zeta} - \zeta}{|\widetilde{\zeta} - \zeta|^m} \, dv(\zeta) = \widetilde{\psi}(\zeta, z) - \psi(\zeta, z) \, .$$

Applying the Gauss theorem for $\{\widetilde{\zeta} : \widetilde{\zeta} \in D, \, \varepsilon < |\widetilde{\zeta} - z|, \, \varepsilon < |\widetilde{\zeta} - \zeta|\}$ with proper positive ε it follows

$$\begin{aligned} \widetilde{\psi}(\zeta, z) \, \overline{\partial} &= \frac{1}{\omega_m} \int_{\partial D} \frac{\overline{\widetilde{\zeta} - z}}{|\widetilde{\zeta} - z|^m} \, d\vec{\sigma}(\zeta) \, \frac{\zeta - \widetilde{\zeta}}{|\zeta - \widetilde{\zeta}|^m} \\ &= \frac{\overline{\zeta - z}}{|\zeta - z|^m} - \frac{\overline{\zeta - z}}{|\zeta - z|^m} \\ &\quad + \frac{1}{\omega_m} \int_D \left\{ \left(\frac{\overline{\widetilde{\zeta} - z}}{|\widetilde{\zeta} - z|^m} \, \partial_{\widetilde{\zeta}} \right) \frac{\overline{\zeta - \widetilde{\zeta}}}{|\zeta - \widetilde{\zeta}|^m} \right. \\ &\qquad \left. - \frac{\overline{\widetilde{\zeta} - z}}{|\widetilde{\zeta} - z|^m} \left(\partial_\zeta \frac{\overline{\widetilde{\zeta} - \zeta}}{|\widetilde{\zeta} - \zeta|^m} \right) \right\} dv(\zeta) = 0 \end{aligned}$$

when ε tends to zero. Hence, again applying the Gauss formula

$$-\frac{1}{\omega_m} \int_{\partial D} \widetilde{\psi}(\zeta, z) \, d\overline{\vec{\sigma}(\zeta)} \, \partial w(\zeta) + \frac{1}{\omega_m} \int_D \widetilde{\psi}(\zeta, z) \, \Delta w(\zeta) \, dv(\zeta) = 0 \, .$$

Adding this to the representation of w and observing (1.12) proves the representation claimed. Iterating the representation in Theorem 1.8 with itself gives the next formula. □

Theorem 1.9. *Let* $w \in C^{2k}(D; \mathbb{C}_m) \cap C^{2k-1}(\overline{D}; \mathbb{C}_m)$ *for* $1 \leq k$ *if* m *is odd* *and for* $1 \le 2k < m$ *if* m *is even. Then*

$$w(z) = \sum_{\mu=1}^{k} \left\{ \frac{1}{\omega_m} \int_{\partial D} \frac{(\overline{\zeta - z})\ |\zeta - z|^{2(\mu-1)-m}}{2^{\mu-1}(\mu-1)! \prod_{\nu=1}^{\mu-1}(2\nu - m)}\ d\vec{\sigma}(\zeta)\, \Delta^{\mu-1} w(\zeta) \right.$$

$$\left. - \frac{1}{\omega_m} \int_{\partial D} \frac{|\zeta - z|^{2\mu-m}}{2^{\mu-1}(\mu-1)! \prod_{\nu=1}^{\mu}(2\nu - m)}\ d\overline{\vec{\sigma}(\zeta)}\ \partial \Delta^{\mu-1} w(\zeta) \right\}$$

$$+ \frac{1}{\omega_m} \int_{D} \frac{|\zeta - z|^{2k-m}}{2^{k-1}(k-1)! \prod_{\nu=1}^{k}(2\nu - m)}\ \Delta^k w(\zeta)\, dv(\zeta)\,.$$

For the proofs of these results see [3]. Further representation formulas for $m \le 2k$ if m is even are given in [19]. Those related to operators of the form $\partial^\ell \overline{\partial}^k$ are the subject of a forthcoming thesis of Heinz Otto.

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