

## Chapter 1

# Complex Numbers

Let  $\mathbb{R}$  be the set of all real numbers. Then a complex number is of the form  $a + ib$ , where  $a$  and  $b$  are in  $\mathbb{R}$  and  $i^2 = -1$ . We denote the set of all complex numbers by  $\mathbb{C}$ .

**Definition 1.1.** Let  $a + ib$  and  $c + id$  be complex numbers. Then we say that  $a + ib = c + id$  if and only if  $a = c$  and  $b = d$ .

**Remark 1.2.** Using the formula  $i^2 = -1$ , complex numbers are added, subtracted, multiplied and divided like real numbers.

**Example 1.3.** Compute  $(2 + i3)(3 - i4)$ .

**Solution**  $(2 + i3)(3 - i4) = 6 - i8 + i9 - 12i^2 = 6 + i + 12 = 18 + i$ .

**Example 1.4.** Compute  $\frac{2+i3}{1+i2}$ .

**Solution** We rationalize the denominator and we get

$$\frac{2 + i3}{1 + i2} = \frac{(2 + i3)(1 - i2)}{(1 + i2)(1 - i2)} = \frac{2 - i4 + i3 - 6i^2}{1 - 4i^2} = \frac{8 - i}{5} = \frac{8}{5} - i\frac{1}{5}.$$

Let  $z = a + ib$  be a complex number. Then we call  $a$  the real part of  $z$  and  $b$  the imaginary part of  $z$ . We sometimes write  $a = \operatorname{Re} z$  and  $b = \operatorname{Im} z$ . The complex conjugate  $\bar{z}$  of  $z$  is defined by

$$\bar{z} = a - ib.$$

The absolute value  $|z|$  of  $z$  is defined by

$$|z| = \sqrt{a^2 + b^2}.$$

We can now give a geometric interpretation of complex numbers. We identify  $\mathbb{C}$  as the  $xy$ -plane  $\mathbb{R}^2$  and we identify the complex number  $z = a + ib$

as the point  $(a, b)$  in  $\mathbb{R}^2$ . Then  $\bar{z}$  is the point  $(a, -b)$ , which is the mirror image of  $z$  with respect to the  $x$ -axis.  $|z|$  is then simply the distance between  $z$  and the origin.

**Theorem 1.5.** *Let  $z$  be a complex number. Then*

- (1)  $|z|^2 = z\bar{z}$ ,
- (2)  $\operatorname{Re} z = \frac{z+\bar{z}}{2}$ ,
- (3)  $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$ ,
- (4)  $\overline{\bar{z}} = z$ ,
- (5)  $|\bar{z}| = |z|$ .

**Proof** Let  $z = a + ib$ . Then

$$z\bar{z} = (a + ib)(a - ib) = a^2 - b^2i^2 = a^2 + b^2 = |z|^2.$$

This is Part (1). For Parts (2) and (3), we note that

$$\frac{z + \bar{z}}{2} = \frac{a + ib + a - ib}{2} = a = \operatorname{Re} z$$

and

$$\frac{z - \bar{z}}{2i} = \frac{a + ib - a + ib}{2i} = b = \operatorname{Im} z.$$

The last two parts can be seen most easily from the geometric interpretation of complex numbers.  $\square$

**Theorem 1.6.** *Let  $z_1$  and  $z_2$  be complex numbers. Then*

- (1)  $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$ ,
- (2)  $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ ,
- (3)  $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ .

The proof of Theorem 1.6 is easy and hence omitted.

### Exercises

- (1) Prove that for all complex numbers  $z$ ,

$$\operatorname{Re}(iz) = -\operatorname{Im} z.$$

- (2) Prove that if  $z$  is a complex number such that  $\operatorname{Im} z > 0$ , then

$$\operatorname{Im}\left(\frac{1}{z}\right) < 0.$$

- (3) Prove that if  $z$  and  $w$  are complex numbers such that  $z + w$  is a real number and  $zw$  is a negative real number, then  $z$  and  $w$  are both real numbers.
- (4) Prove that if  $z$  is a complex number such that  $|z| = \operatorname{Re} z$ , then  $z$  is a nonnegative real number.
- (5) Draw the set of all the points  $z$  in  $\mathbb{C}$  satisfying each of the following relations.
- (a)  $|2z - i| = 4$
  - (b)  $|z| = \operatorname{Re} z + 2$
  - (c)  $|3z + i| < 2$
  - (d)  $|z - 1| + |z + 1| = 7$ .