

## EDITORIAL PREFACE

The *Second Perspectives on Mathematical Practices Conference* (PMP2007) was held at the Free University of Brussels (VUB), Belgium, from 26 to 28 March, 2007. This volume contains texts issuing from talks delivered at that occasion which particularly focused on the historical dimension of mathematical practice, the core subject of the conference. All papers gathered here address aspects of the question how the philosophy of mathematics relates to the history of mathematics. Nature and goals of this type of inquiry have been most clearly stated by José Ferreirós and Jeremy Gray in the introduction to their seminal reader *The Architecture of Modern Mathematics* (Oxford University Press, 2006), opposing with great sensitivity the ahistorical received view in the philosophy of mathematics to a recently emerging trend of studies in contextualized mathematical practices. We subscribe to the programme set out by them, and hope to provide here with a modest contribution to it. Incidentally, both editors have participated to PMP2007, and one may find Gray's paper included here, while that by Ferreirós is part of a companion volume consisting of rather philosophically laden texts, to appear with *College Publications*.

Let me give, at the outset of this collection, an overview of the articles included, which have been loosely organized in chronological order of the period covered by them. Algebra is the topic of the first triple of papers, opening with that by *Albrecht Heeffer* (Ghent), in which it is contended that the traditional three-stage division of the development of algebra, viz. into rhetorical, syncopated and symbolic phases, is not adequate. As is argued, it would better be replaced by an alternative account, in which non-symbolic, proto-symbolic and symbolic phases succeed one another. The first covers the algorithmic type of algebra dealing with numerical values or a non-symbolic model, e.g. Greek geometrical algebra. The second is home to algebras employing words or abbreviations for the unknown but not therefore being symbolic in character, such as Diophantian and early Abbacus algebra. The third and last phase, that of (truly) symbolic algebra, viz. allowing for manipulations on the symbolic level of symbols only, starts around 1560.

*Ad Meskens* (Antwerp) in his contribution explains that Diophantos had complete mastery over the methods of solution for solving linear equations, for indeterminate quadratic equations and for systems of equations of first and second degree, while for higher degree equations he only sometimes had a solution method. When in 1971 an Arabic version of Diophantos was found, it came as a shock that in this book some previously unknown material was found. In it, Diophantos apparently used the methods for solving higher-degree problems described in other books to their limits. The structure of a Diophantine problem follows a general rule set out by Proklos. Problems are put a general way, using indeterminate numbers. But solutions are provided for specific numbers given at the outset. We can see a partial analogy with geometrical construction: applied to a specific figure, though posed generally. Even if the specific example allows it, Diophantos never gives general methods. During the elaboration of the example he sometimes adds a restriction. In some other cases, where there is a need for restrictions, he does not impose them. It is unclear whether this is due to ignorance about the need of such a restriction or to the impossibility to correctly formulate it.

According to *Jens Høygrup* (Roskilde), Italian fourteenth- and fifteenth-century abacus algebra presents us with a number of deviations from what we would consider normal mathematical practice and proper mathematical behaviour: the invention of completely false algebraic rules for the solution of cubic and quartic equations, and of rules that pretend to be generally valid but in fact only hold in very special cases; and (in modern terms) an attempt to expand the multiplicative semi-group of non-negative algebraic powers into a complete group by identifying roots with negative powers. In both false-rule cases, the authors of the fallacies must have known they were cheating. Certain abacus writers seem to have discovered, however, that something was wrong, and devised alternative approaches to the cubics and quartics; they also developed safeguards against the misconceived extension. In his paper, Høygrup analyses both phenomena, and correlates them with the general practice and norm system of abacus mathematics as this can be extracted from the more elementary level of the abacus treatises.

*Matthew Parker's* (London) contribution considers Cantor's extension of the concept of number to the transfinite, and the resolution this supplies for what has been called "Galileo's Paradox", namely that the square numbers seem to be at once fewer than and equal to the positive integers. Galileo's Paradox is held to have been resolved by the articulation of numerosity into distinct concepts, including those of proper inclusion,

*Anzahl*, and power. Power has become the basis of an elegant and useful theory and has proven especially useful in addressing the motivations common to Galileo, Bolzano and Cantor, namely, to grasp the relations between numerosity and geometric magnitude, to defend the analysis of the continuum into points, and to explain physical phenomena. As Parker explains, it is in virtue of its success in serving such motivations that Cantor's theory of transfinite numbers constitutes a solution to some of the deeper philosophical problems posed by Galileo's Paradox. But there are alternatives. For example, *Anzahl* too can be considered as a notion of numerosity. In order to analyze this matter, Parker proposes a Method of Conceptual Articulation.

During the first part of the nineteenth century, the mathematical disciplines of analysis and algebra developed tremendously, exploring new techniques and questions to arrive at far-reaching and sometimes surprising new results. According to *Henrik Kragh Sørensen* (Aarhus), a central part of this development involved changes in the role and use of representations. Mathematicians often work with representations in order to access and manipulate mathematical objects such as functions. These uses can satisfy a variety of demands. For instance, representations of implicitly defined functions as infinite series can add to the familiarity of these new functions by anchoring them within existing ways of accessing and manipulating functions. In different contexts, the question of whether a given function can or cannot be represented in a specific form opens the door for new results such as impossibility proofs. In his paper, Kragh Sørensen analyses such multiple roles of representations of functions from a Wittgenstein-inspired perspective as "aspects" of functions. By comparing important results from algebra (algebraic unsolvability of the general quintic equation) and analysis (representations of elliptic functions) in the context of Abel's mathematics, representations are highlighted both as means and ends in themselves.

Point of departure of the next article, by *Jeremy J. Gray* (Milton Keynes), is the observation that, on the one hand, historians and philosophers of mathematics share an interest in the nature of mathematics (what it is, what features affect its growth, how it informs other disciplines), but that, on the other hand, much of the work done in history and philosophy of mathematics shows that the two groups largely work in isolation. A reconsideration of the history of mathematical analysis in the nineteenth century, according to Gray, suggests that history and philosophy of mathematics can be done together to the advantage of both, and also how legitimately different enquiries need not drive them apart.

The last few decades have witnessed a broadening of the philosophy of mathematics, beyond narrowly foundational and metaphysical issues, and towards the inclusion of more general questions concerning methodology and practice. Part of this broadening, although a part that remains relatively close to foundational and metaphysical issues, is the turn towards a “new epistemology” for mathematics, including the study of topics such as the role of visualization in mathematics, the use of computers in proving mathematical theorems, and the notion of explanation as applied to mathematics. *Erich H. Reck*’s (Riverside, CA) paper is a contribution to such a new epistemology. More particularly, it is an attempt to bring into sharper focus, and to argue for the relevance of, two related themes: “structural reasoning” and “mathematical understanding”. As the notion of understanding is vague and slippery in general, as well as very loaded in philosophical discussions of the sciences, the label is handled with care. Similarly, while talking about “structural” reasoning in mathematics may be suggestive, that term too requires further elaboration. Reck’s clarifications and elaborations are tied to a specific historical figure and period, Richard Dedekind, and his contributions to algebraic number theory in the nineteenth century, which proves to be all but an incidental choice.

*Eduard Glas* (Delft) undertakes a comparison between mathematician Felix Klein and philosopher Imre Lakatos. Klein, Glas argues, is perhaps the most outstanding example of an eminently fruitful mathematician opposing the one-sided obsession of most mathematicians of his generation with purity and rigor, an obsession through which the discipline increasingly tended to fall apart into disparate, self-contained specialties. In contrast to the adepts of rigor and purity within the leading schools, who eschewed reliance on intuitive or quasi-empirical insights, Klein’s methodology was based on the use of geometric and even physical models and thought experiments, a methodology which certainly qualifies as ‘quasi-empirical’. Klein’s successes depended in large measure on his exceptional versatility in the mental visualisation even of the most abstract mathematical objects and relations. Throughout his career, Klein kept insisting that intuition, especially spatial intuition, is indispensable in all mathematical endeavours, which also makes for their rootedness in concrete experience. According to Glas, Klein was as much a maverick in the eyes of ‘pure’ mathematicians as Imre Lakatos would become in the eyes of mainstream philosophers of mathematics. Like Lakatos, Klein insisted that progress in mathematics relies on methods that are very much akin to those of natural science, especially as concerns the use of models and (thought) experiments. He in

fact practised a model-based, quasi-empirical method of investigation that indeed tallies nicely with Lakatos' quasi-empiricist methodology.

The idea that formal geometry derives from intuitive notions of space has appeared in many guises, most notably in Kant's argument from geometry. Kant claimed that a priori knowledge of spatial relationships both allows and constrains formal geometry: it serves as the actual source of our cognition of the principles of geometry and as a basis for its further cultural development. The development of non-Euclidean geometries, however, undermined the idea that there is some privileged relationship between our spatial intuitions and mathematical theory. The aim of the paper by *Helen De Cruz* (Leuven) is to look at this longstanding philosophical issue through the lens of cognitive science. Drawing on recent evidence from cognitive ethology, developmental psychology, neuroscience and anthropology, she argues for an enhanced, more informed version of the argument from geometry: humans share with other species evolved, innate intuitions of space which serve as a vital precondition for geometry as a formal science.

In line with the general spirit of the underlying conference, *Ronny Desmet* (Brussels) observes that it is part of the growing awareness that historical, social and psychophysical processes precede the cut and dried results of mathematics, even those which have been presented as the obvious starting points of all pure mathematics. And as with all fashionable currents, he continues, the shift from foundations to practices in the philosophy of mathematics has its heroes. Indeed, Lakatos and Wittgenstein immediately come to mind in this respect. If an author were to say that, given their mutual influence, Wittgenstein's view on mathematics can be identified with Russell's, dissent would follow. But if he were to say that, given their intense collaboration Whitehead's view on mathematics can be identified with Russell's, this claim would normally pass without much protest. Desmet, however, could not disagree more. In his paper, he argues that Whitehead, like Wittgenstein, should be differentiated from Russell, and given his own niche in philosophy of mathematics, and that, furthermore, Whitehead's writings, which were based on his own mathematical experience, offer a perspective on mathematical practices equalling, or even surpassing, that provided by Wittgenstein.

In discussions of mathematical practice, *Dirk Schlimm* (Montréal) points out, the role axiomatics has often been confined to providing the starting points for formal proofs, with little or no effect on the discovery or creation of new mathematics. Nevertheless, it is undeniable that axiomatic systems have played an essential role in a number of mathematical innova-

tions. Moreover, it was not only through the investigation and modification of given systems of axioms that new mathematical notions were introduced, but also by using axiomatic characterizations to express analogies and to discover new ones. In his contribution, which closes this volume, Schlimm however draws our attention to a different use of axiomatics in mathematical practice, namely that of being a vehicle for bridging theories belonging to previously unrelated areas. How axioms have been instrumental in linking mathematical theories is illustrated by the investigations of Boole, Stone, and Tarski, all of which revolve around the notion of Boolean algebra.

As already mentioned, a companion volume of (more philosophically oriented) PMP2007 proceedings papers is to be published by College Publications, London. At the outset of the present one, allow me to express my gratitude to an array of people. For naturally, this proceedings volume did not come about because of the efforts of the editor alone. To begin with, I am extremely grateful for all institutional and personal contributions to what in my eyes was a very successful PMP2007 conference. This includes the generous sponsors of the event: Research Foundation — Flanders, Brussels Capital-Region, National Centre for Research in Logic — Belgium, as well as its (co-)organizers Belgian Society for Logic and Philosophy of Science, Wissenschaftliches Netzwerk PhiMSAMP, and Centre for Logic and Philosophy of Science at Free University of Brussels. Further, I was very fortunate to leave local organization largely in the safe hands of my precious colleagues Patrick Allo, Ronny Desmet, and Karen François. And as there is simply no event whatsoever without an interested and interesting audience, let me hereby also thank all participants to PMP2007 (including the authors of this book). I sincerely hope that this conference series, started in 2002 with the initial PMP, may live on, possibly at other locations. On a more personal level, I also feel indebted to the bodies that have funded my academic work during the past years: Research Foundation — Flanders, Free University of Brussels (through BAP and GOA49), and Alexander von Humboldt-Foundation — Germany (with special thanks to my dear colleague and friend Thomas Müller). Finally, I would love to dedicate this volume to Jean Paul Van Bendegem, who has been my faithful and trusting mentor for a decade now, and to whom, philosophically but also beyond, I owe so much.

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