

Chapter 1

Introduction

1.1 Critical exponents of equilibrium (thermal) systems

In this section I briefly summarize the definition of well known critical exponents of homogeneous equilibrium systems and show some scaling relations [Fisher (1967); Kadanoff *et al.* (1967); Stanley (1971); Ma (1976); Amit (1984)]. The basic thermal exponents (denoted by subscript ‘ H ’ to avoid confusion with the some nonequilibrium ones defined later) are defined via the scaling laws:

$$c_H \propto \alpha_H^{-1} \left((|T - T_c|/T_c)^{-\alpha_H} - 1 \right) , \quad (1.1)$$

$$m \propto (T_c - T)^\beta , \quad (1.2)$$

$$\chi \propto |T - T_c|^{-\gamma} , \quad (1.3)$$

$$m \propto H^{1/\delta_H} , \quad (1.4)$$

$$G_c^{(2)}(r) \propto r^{2-d-\eta_\perp} , \quad (1.5)$$

$$\xi \propto |T - T_c|^{-\nu_\perp} . \quad (1.6)$$

Here c_H denotes the specific heat, m the order parameter, χ the susceptibility and ξ the correlation length. Note that the anomalous dimension exponent in the spatial two-point correlation scaling-law is denoted by η_\perp in nonequilibrium systems — where \perp corresponds to perpendicular to the time direction — therefore in the time being I shall use this notation. The presence of another degree of freedom besides the temperature T , like a (small) external field (labeled by H), leads to other interesting power laws when $H \rightarrow 0$. The d present in the expression of two-point correlation function $G_c^{(2)}(r)$ is the space dimension of the system.

Some laws are valid both to the right and to the left of the critical point; the values of the relative proportionality constants, or *amplitudes*,

are in general different for the two branches of the functions, whereas the exponent is the same. However there are universal amplitude relations among them. We can see that there are altogether six basic exponents. Nevertheless they are not independent of each other, but related by some simple scaling relations:

$$\begin{aligned} \alpha_H + 2\beta + \gamma = 2, \quad \alpha_H + \beta(\delta_H + 1) = 2, \\ (2 - \eta_\perp)\nu_\perp = \gamma_H, \quad \nu_\perp d = 2 - \alpha_H. \end{aligned} \quad (1.7)$$

The last relation is a so-called **hyper-scaling law**, which depends on the spatial dimension d and is **not valid above the upper critical dimension** d_c , for example by the Gaussian theory. Therefore below d_c there are only two independent exponents in equilibrium.

Below the **lower critical dimension** (d_c^-) the fluctuations are so strong that they completely destroy the ordered state, hence no order-disorder phase transition can exist. In equilibrium models finite-range interactions cannot maintain long-range order below $d_c^- = 2$ [Landau and Lifshitz (1981)].

One of the most interesting aspects of second order phase transitions is the so-called **universality**, i.e., the fact that systems which can be very different from each other share the same set of critical indices (exponents and some amplitude ratios). One can therefore hope to assign all systems to **classes** each of them being identified by a set of critical indices.

1.2 Static percolation cluster exponents

Universal behavior may occur at percolation [Stauffer and Aharony (1994); Grimmett (1999)], which can be considered a **purely geometrical phenomenon** describing the occurrence of infinitely large connected clusters on lattices. On the other hand such clusters emerge at critical phase transition of lattice models indeed. The definition of connected clusters is not unambiguous. It may mean the set of sites or bonds with variables in the same state, or connected by bonds with probability $b = 1 - \exp(-2J/kT)$. The differences in correlated cluster behavior of critical domains will be discussed in Chapter 2. following the definition of basic models.

By changing the system control parameters ($p \rightarrow p_p$) (that usually is the temperature in equilibrium systems) the coherence length between sites may diverge as

$$\xi(p) \propto |p - p_p|^{-\nu_\perp}, \quad (1.8)$$

hence percolation at p_p like standard critical phenomena exhibits renormalizability and universality of critical exponents. At p_p the cluster size (s) distribution (the number of s -clusters divided by the total number of lattice sites) follows the scaling law:

$$n_s \propto s^{-\tau} f(|p - p_p|s^\sigma) . \quad (1.9)$$

Formally we may define a free energy F as a generating function of a ghost field (in analogy with equilibrium statistical physics)

$$F(h) = \sum_s n_s e^{-hs} , \quad (1.10)$$

such that its h derivatives, the moments of 1.9 exhibit the leading singular behavior, characterized by the exponents:

$$\sum_s n_s(p) \propto |p - p_p|^{2-\alpha_p} , \quad (1.11)$$

$$\sum_s s n_s(p) \propto |p - p_p|^\beta , \quad (1.12)$$

$$\sum_s s^2 n_s(p) \propto |p - p_p|^{-\gamma} , \quad (1.13)$$

$$\sum_s s n_s(p) e^{-hs} \propto h^{1/\delta_p} . \quad (1.14)$$

These critical exponents are formally related by the same scaling laws as the thermal ones (1.7). The cluster exponents τ , σ are connected to them via

$$\tau = 2 + 1/\delta_p, \quad 1/\sigma = \beta\delta_p , \quad (1.15)$$

hence only two of them are independent. Note that for exponents α and δ I introduced the subscript ‘ p ’ here, referring to percolation (contrary to the subscript ‘ H ’ of analogous exponents characterizing thermal properties in the previous section) and there are also plain α and δ (for historical reasons) to be introduced in Sect. 1.37, which describe a different scaling law.

In case of completely random placement of (sites, bonds, etc) variables (with probability p) on lattices we find **random isotropic (ordinary) percolation** (see Sect. 4.4.1). Percolating clusters may arise at critical, thermal transitions or by nonequilibrium processes. If the critical point (p_c) of the order parameter does not coincide with p_p than at the percolation transition the order parameter coherence length is finite and does not influence the percolation properties. We observe random percolation

in that case. In contrast if $p_p = p_c$ percolation is influenced by the order parameter behavior and we find different, **correlated percolation** universality [Fortuin and Kasteleyn (1972); Coniglio and Klein (1980); Stauffer and Aharony (1994)] whose exponents may coincide with those of the order parameter.

According to the Fortuin-Kasteleyn construction of clusters [Fortuin and Kasteleyn (1972)] two nearest-neighbor spins of the same state belong to the same cluster with probability $b = 1 - \exp(-2J/kT)$. It was shown that using this prescription for Z_n and $O(n)$ and symmetric models [Coniglio and Klein (1980); Bialas *et al.* (2000); Fortunato and Satz (2001); Blanchard *et al.* (2000)] the **thermal phase transition** point coincides with the percolation limits of such clusters. On the other hand in case of “pure-site clusters” ($b = 1$) different, universal cluster exponents are reported in two dimensional models [Fortunato and Satz (2001); Fortunato (2002)] (see Sects. 2.3.1, 2.4.1, 2.6.1).

1.3 Dynamical critical exponents

Nonequilibrium systems were first introduced to study relaxation to equilibrium states [Hohenberg and Halperin (1977)] and phase ordering kinetics [Binder and Stauffer (1974); Marro *et al.* (1979)] (see Fig. 1.1). Since at the critical point (T_c) and at the ordered $T = 0$ situation the the spatial correlation length and time are divergent (or they are much larger than the microscopic length and time scales) universal dynamical scaling is natural. Power-law time dependences were investigated away from the critical point as well, example by the domain growth in a quench to low temperature $0 < T < T_c$. Later the combination of different heat-baths, different dynamics, external currents became popular investigation tools of fully nonequilibrium models.

To describe the dynamical behavior of a critical system scaling hypothesis was introduced [Ferrell *et al.* (1967); Halperin and Hohenberg (1967)] and additional exponents were introduced. For example the relation of the divergences of the relaxation time t_R and correlation length ξ is described by the **dynamical exponent** Z

$$t_R \propto \xi^Z . \quad (1.16)$$

Systems out of equilibrium may show anisotropic scaling of two (and n) point functions

$$G(b\mathbf{r}, b^\zeta t) = b^{-2x} G(\mathbf{r}, t) \quad (1.17)$$

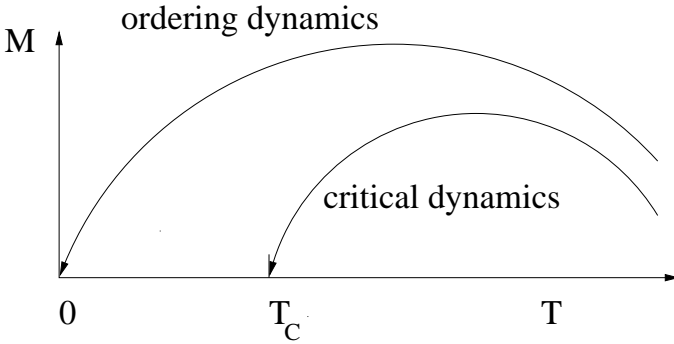


Fig. 1.1 Different dynamics considered by a quench from high temperature to $T = 0$ (ordering) or to T_c (critical).

where \mathbf{r} and t denote spatial and temporal coordinates, x is the scaling dimension and ζ is the anisotropy exponent. As a consequence the temporal (ν_{\parallel}) and spatial (ν_{\perp}) correlation length exponents may be different, described by $\zeta = Z$.

$$Z = \zeta = \frac{\nu_{\parallel}}{\nu_{\perp}} . \quad (1.18)$$

For some years it was believed that dynamical critical phenomena are characterized by a set of three critical exponents, comprising two independent static exponents (other static exponents being related to these by scaling laws) and the dynamical exponent Z . However, the large correlation time induces a memory effect. In a magnetic system exhibiting a small initial magnetization m_0 a quench to T_c (Fig. 1.1) a short time scaling emerges (see Fig. 1.2). First the magnetization increases as $\propto t^{\eta}$ until the ordered domains begin to feel each other ($t < \sim m_0^{Z/\nu_{\parallel}}$), then this crosses over to the long-time power-law decay ($\propto t^{-\beta/\nu_{\parallel}}$), finally exponential cutoff may occur due to finite size effects ($t > \sim L^Z$). Note that in many papers η is denoted by θ . This is very similar to the magnetization profile near a surface with different bulk and surface magnetizations (caused by an external magnetic field), because a d dimensional dynamical system is like a $d + 1$ dimensional static one. In this sense the initial state corresponds to the surface state of a $d + 1$ static model. This kind of so-called ‘initial slip’ behavior occurs at spreading from an initial active site in reaction-diffusion models as well (see. [Grassberger and de la Torre (1979)], Sect. 1.5).

A universal dynamic scaling form, which sets in right after a time scale t_{mic} which is enough large in *microscopic* sense but still very small in *macro-*

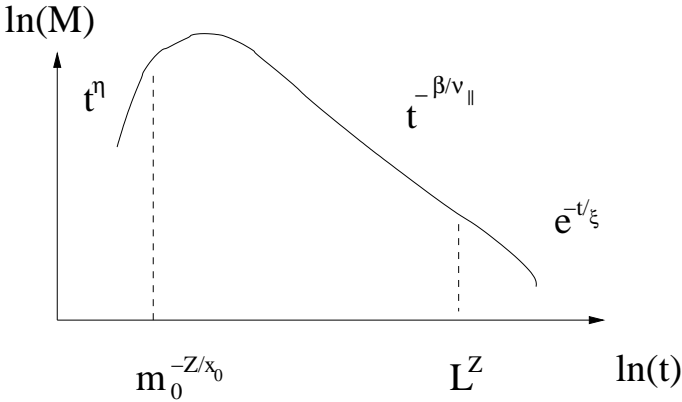


Fig. 1.2 Sketch of the evolution of magnetization in a quench to T_c from a high temperature state with initial magnetization m_0 .

scopic sense, has been derived with an ϵ -expansion [Janssen *et al.* (1989)]. Important is that extra critical exponents should be introduced to describe the dependence of the scaling behavior on the initial conditions. For example, for the k -th moment of the magnetization, the finite size scaling form is written as [Janssen *et al.* (1989)]

$$M^{(k)}(t, \tau, L, m_0) = b^{-k\beta/\nu_{\perp}} M^{(k)}(b^{-Z}t, b^{1/\nu_{\perp}}\tau, b^{-1}L, b^{x_0}m_0) \quad (1.19)$$

here the **new independent** exponent x_0 is the scaling dimension of the initial magnetization m_0 . For $\tau = 0$ in a large lattice ($L \rightarrow \infty$) with small t and m_0 we can derive

$$M(t) \sim m_0 t^{\eta}, \quad \eta = (x_0 - \beta/\nu_{\perp})/Z. \quad (1.20)$$

For almost all statistical systems studied up to now, the exponent η is *positive*, i.e. the magnetization undergoes surprisingly a **critical initial increase**. The time scale of this increase is $t_0 \sim m_0^{-Z/x_0}$. As we shall see later similar initial slip may occur in genuinely nonequilibrium systems as well. The slip exponents may be related to others, depending on the symmetries of the system.

By studying phase ordering kinetics of magnets it was discovered that in case of a quench to critical or zero temperature magnets exhibit slow, non-exponential relaxation, dynamical scaling, and time dependent observable depend on the **age** of the system (for a review see [Bray (1994)]). **Physical ageing** was first identified in glassy systems [Struik (1978)] (for a recent

review of ageing see [Cugliandolo (2003)]). For characterizing ageing the introduction of two-point functions are need, where in general **translational invariance in time is not satisfied**.

First it was discovered that there is another dynamical exponent, the ‘nonequilibrium’ or **short-time exponent** λ_C , needed to describe **two-time correlations** in a spin system ($\{s_i\}$) of size L relaxing to the critical state from a random initial condition ($m_0 = 0$) [Janssen *et al.* (1989); Huse (1989)].

$$A(t, 0) = \frac{1}{L^d} \langle \sum_i s_i(0) s_i(t) \rangle - \langle s_i(0) \rangle \langle s_i(t) \rangle \propto t^{-\lambda_C/Z}. \quad (1.21)$$

Generally the λ_C autocorrelation exponent may or may not be independent from other dynamical exponents (see for example Sects. 2.3.2, 2.6, 4.2.1). But even for the same model it depends on the initial conditions. For example in case of $O(N)$ models with $m_0 \neq 0$ magnetized initial state it is not independent [Calabrese and Gambassi (2007)].

If the initial time of the auto-correlator $A(t, s)$ is not zero ($s > 0$) in the asymptotic (ageing) limit: $t, s \rightarrow \infty$ and $y = t/s > 1$ one expects the scaling form

$$A(t, s) = s^{-b} f_C(t/s), \quad (1.22)$$

where b is the ‘‘ageing’’ exponent and f_C is the corresponding scaling function such that

$$f_C(y) \sim y^{-\lambda_C/Z} \quad (1.23)$$

for $y \gg 1$.

Another two-point function characterizing ageing is the linear **autore-sponse function**, which describes the system response for an applied conjugated magnetic field at time s and location τ . In the limits $t, s \rightarrow \infty$ and $y = t/s > 1$ (ageing regime) it exhibits the scaling

$$R(t, s) = s^{-1-a} f_R(t/s) \quad (1.24)$$

where a is the autore-sponse ageing exponent and f_R is a scaling function such that

$$f_R(y) \sim y^{-\lambda_R/Z} \quad (1.25)$$

for $y \gg 1$. For **non-disordered magnets**, with short-ranged initial correlations

$$\lambda_C = \lambda_R \quad (1.26)$$

usually (for more discussion see Sect. 2.1).

For **quenches to** $T = T_c$ equilibrium state scaling arguments [Godrèche and Luck (2000a)] and renormalization approach [Janssen *et al.* (1989); Janssen (1992)] predict that the relevant length-scale is set by the time dependent correlation length

$$L(t) \sim \xi(t) \sim t^{1/Z}, \quad (1.27)$$

hence the ageing exponents are related to the equilibrium ones via the spatial two-point correlator $G(\mathbf{r}, t)$

$$a = b = (d - 2 - \eta_{\perp})/Z, \quad (1.28)$$

and the autoresponse function follows the form **for every t and s**

$$R(t, s) = A_R s^{-1-a} (t/s)^{\eta} (t/s - 1)^{-1-a} f_R(t/s), \quad (1.29)$$

where A_R is a non-universal constant, and $f_R(t/s)$ is a universal scaling function such that $f_R(t/s \gg 1) = 1$.

On the other hand for **quenches to** $T < T_c$ one usually observes simple scaling of $A(t, s)$ with $b = 0$. The value of the response function ageing exponent a depends on whether the equilibrium correlator is short (S) or long (L) (see Table 1.1) and expression (1.29) **holds in the ageing regime only**.

Table 1.1 Nonequilibrium exponents for non-conserved ferromagnets with $T_c > 0$ and $m_0 \neq 0$

	Z	a	b	Eq. corr. range
$T = T_c$	dynamics dep.	$(d - 2 + \eta_{\perp})/Z$	$(d - 2 + \eta_{\perp})/Z$	L
$T < T_c$	2	$(d - 2 + \eta_{\perp})/Z$	0	L
$T < T_c$	2	$1/Z$	0	S

More recently the **persistence exponents** θ_l and θ_g were introduced by [Derrida *et al.* (1994); Majumdar *et al.* (1996)]. These are associated with the probability $p(t)$, that the local or global order parameter has not changed sign in time t following a quench to the critical point. In many systems of physical interest these exponents decay algebraically as

$$p(t) \propto t^{-\theta} \quad (1.30)$$

(see however example Sect. 6.1). It turned out that in systems where the scaling relation

$$\theta_g Z = \lambda_c - d + 1 - \eta_{\perp}/2 \quad (1.31)$$

is satisfied the dynamics of the global order parameter is a Markov process. In contrast in systems with non-Markovian global order parameter θ_g is in general a new, non-trivial critical exponent [Majumdar *et al.* (1996)], meaning that in this case **the persistence probability cannot be expressed by finite n -point correlators**. For example it was shown that while in the $d = 1$ Glauber Ising model the magnetization is Markovian and the scaling relation (1.31) is fulfilled, at the critical point of the $d = 1$ NEKIM Ising model this is not satisfied and the persistence behavior there is characterized by a different, non-trivial θ_g exponent [Menyhárd and Ódor (1997)] (see discussion in Sect. 4.6.2). As we can see the *universality classes of static models are split by the dynamical exponents*.

1.4 Crossover between classes

A model with some additional condition like an external field, surface ... may exhibit different universal scaling behavior by varying the parameters (in the field theoretical language (see Sect. 1.6) it may possess several competing fixed points in the phase space).¹ By tuning the control parameters close to these points we may see corrections to scaling. In terms of scaling forms, crossover phenomena are described by additional relevant scaling fields, characterized by a so-called crossover exponent [Riedel and Wegner (1969)].

For example let us consider a RD model exhibiting a phase transition at the control parameter value $p = p_0$, and add an additional reaction with a small probability w . If we define the relative distances in the phase diagram $\Delta = (p_0 - p)/p_0$ and $\Delta_c(w) = (p_0 - p_c(w))/p_0$ the phase boundary is well fitted by

$$\Delta_c \sim w^{1/\phi} \quad (1.32)$$

which means that the crossover exponent describes the shape of the critical line as it approaches the crossover point in the phase diagram. In this case a crossover scaling is described by the scaling function (see for example [Lawrie and Sarbach (1984)])

$$\rho(\Delta, w; t) = t^{-\alpha} F(\Delta^{\nu_{\parallel}} t, w^{\mu_{\parallel}} t), \quad (1.33)$$

where $\mu_{\parallel} = \nu_{\parallel}/\phi$ with ϕ estimated in the above. Further scaling forms and relations to other exponents are discussed later.

¹It is matter of taste to call a model modified in such a way a different one.

Although crossover is well understood by field theory, several aspects are still open and are discussed in the literature. For example, the question whether the so-called effective exponents fulfill scaling laws over the entire crossover region has been revisited several times.

1.5 Critical exponents and relations of spreading processes

In the previous section I defined quantities describing dynamical properties of the bulk of a material. In a dual way to this one may also consider cluster properties arising by initiating a process from an ordered (correlated) state with a small cluster of activity. Here I define a basic set of critical exponents that occur in spreading processes and show the scaling relations among them. In such processes phase transition may exist to absorbing state(s) where the density of spreading entity (particle, agent, epidemic etc.) disappears. The order parameter is usually the density of active sites $\{s_i\}$ (particles, kinks, spins ... etc.) of a d -dimensional lattice of size L

$$\rho(t) = \frac{1}{L^d} \left\langle \sum_i s_i(t) \right\rangle, \quad (1.34)$$

which in the supercritical phase vanishes with the leading power behavior

$$\rho^\infty \propto |p - p_c|^\beta, \quad (1.35)$$

as the control parameter p is varied. The “dual” quantity is the ultimate survival probability P_∞ of an infinite cluster of active sites that scales in the active phase as

$$P_\infty \propto |p - p_c|^{\beta'} \quad (1.36)$$

with some critical exponent β' [Grassberger and de la Torre (1979)]. In field theoretical description of reacting particles (see Sect. 1.6) β is associated with the annihilation, β' with the particle creation operator and in case of rapidity reversal symmetry (see Eq. (4.41)) they are equal.

The critical long-time behavior of these quantities are described by

$$\rho(t) \propto t^{-\alpha} f(\Delta t^{1/\nu_{||}}), \quad P(t) \propto t^{-\delta} g(\Delta t^{1/\nu_{||}}), \quad (1.37)$$

where α and δ are the critical exponents for decay and survival, $\Delta = |p - p_c|$, f and g are *universal scaling functions* [Grassberger and de la Torre (1979); Muñoz *et al.* (1997); Janssen (2003)]. The obvious scaling relations among them are

$$\alpha = \beta/\nu_{||}, \quad \delta = \beta'/\nu_{||}. \quad (1.38)$$

For *finite systems* (of size $N = L^d$) these quantities scale as

$$\rho(t) \propto t^{-\beta/\nu_{\parallel}} f'(\Delta t^{1/\nu_{\parallel}}, t^{d/Z}/N), \quad (1.39)$$

$$P(t) \propto t^{-\beta'/\nu_{\parallel}} g'(\Delta t^{1/\nu_{\parallel}}, t^{d/Z}/N), \quad (1.40)$$

leading to finite size scaling laws

$$t_R \propto L^Z, \quad \rho_s \propto L^{-\beta/\nu_{\perp}}, \quad \chi_s \propto L^{-\gamma/\nu_{\perp}}, \quad (1.41)$$

where ρ_s is the steady state density and χ_s is its fluctuation averaged over the surviving samples.

For “*relatively short times*” or for initial conditions with a single active seed the the number of active sites $N(t)$ and its mean square of spreading distance (x_i) from the origin

$$R^2(t) = \frac{1}{N(t)} \langle \sum_i x_i^2(t) \rangle \quad (1.42)$$

follow the “initial slip” scaling laws [Grassberger and de la Torre (1979)]

$$N(t) \propto t^{\eta}, \quad (1.43)$$

$$R^2(t) \propto t^z. \quad (1.44)$$

The spreading exponents in general are related by the **hyperscaling relation** derived in [Mendes *et al.* (1994)]

$$2 \left(1 + \frac{\beta}{\beta'} \right) \delta + 2\eta = dz \quad (1.45)$$

but in case of symmetries special versions of it hold (see example (4.43)).

The derivation of (1.45) goes along the following lines. It starts with the general expression for the density of particles at space-point r in the absorbing phase ($\Delta < 0$) and at large fixed value of t (for $d = 1$):

$$\rho(r, t) = t^{\eta-z/2} F(r^2/t^z, \Delta t^{1/\nu_{\parallel}}) \quad (1.46)$$

and with the relations (1.36),(1.37),(1.38) for the survival probability. Since the stationary distribution is unique

$$\rho(x, t) \rightarrow P_{\infty} \Delta^{\beta} \sim \Delta^{\beta+\beta'} \quad (1.47)$$

as $t \rightarrow \infty$. Hence $F(0, y) \sim y^{\beta+\beta'}$ which entails Eq. (1.45). If for example $\beta' = \beta$ one gets the special form of Eq.(1.45)

$$2\delta + \eta = z/2. \quad (1.48)$$

In the absorbing phase $\rho(r, t)$ is expected to decrease exponentially as $\rho(r, t) \propto \exp(-r/\xi)$ usually. This implies for $F(u, v)$ (with $v < 0$) the form

$$F(u, v) \rightarrow \exp(-C\sqrt{|u|}|v|^{\nu_{\perp}}) \quad (1.49)$$

where $C > 0$ is constant. For ξ to be time-independent the scaling law is required:

$$z = \frac{2\nu_{\perp}}{\nu_{\parallel}} = \frac{2}{Z}, \quad (1.50)$$

i.e. the dynamical exponent of the model is related to the cluster spreading exponents of its components. In some anisotropic system (see example Sect. 4.6.9) this relation was found to be broken [Menyhárd and Ódor (2002)].

Another interesting quantity, which can be observed in nature relatively easily is the **fractal dimension** of spreading clusters. The average number of active sites per surviving clusters scales as $N_{sur}(t) = N(t)/P(t) \propto t^{\eta+\delta}$ and the fractal dimension d_f is defined via

$$N_{sur}(t) \propto R(t)^{d_f} \quad (1.51)$$

resulting in the scaling law

$$d_f = Z(\eta + \delta) = d - \beta/\nu_{\perp}. \quad (1.52)$$

1.5.1 Damage spreading exponents

Phase transitions between chaotic and non-chaotic states may be described by damage spreading (DS). While DS was first introduced in biology [Kauffman (1969)] it has become an interesting topic in physics as well [Creutz (1986); Stanley *et al.* (1986); Derrida and Weisbuch (1987)]. The main question is if a damage introduced in a dynamical system survives or disappears. To investigate this the usual technique is to make replica(s) of the original system and let them evolve with the same dynamics and external noise. This method has been found to be very useful to measure accurately dynamical exponents of equilibrium systems [Grassberger (1995a)]. It has turned out however, that DS properties do depend on the applied dynamics. An example is the case of the two-dimensional Ising model with heat-bath algorithm versus Glauber dynamics [Mariz *et al.* (1990); Jan and de Arcangelis (1994); Grassberger (1995b)].

To avoid dependency on the dynamics a definition of “physical” family of DS dynamics was suggested by [Hinrichsen *et al.* (1997b)] according to

the active phase may be divided to a sub-phase in which DS occurs for every member of the family, another sub-phase where the damage heals for every member of the family and a third possible sub-phase, where DS is possible for some members and the damage disappears for other members. The family of possible DS dynamics is defined to be consistent with the physics of the single replicas (symmetries, interaction ranges etc.).

Usually the order parameter of the damage is the Hamming distance between replicas

$$D(t) = \left\langle \frac{1}{L} \sum_{i=1}^L |s(i) - s'(i)| \right\rangle \quad (1.53)$$

where $s(i)$ and $s'(i)$ denote variables of the replicas. At continuous DS transitions D exhibits power-law singularities as physical quantities at the critical point. For example one can follow the fate of a single difference between two (or more) replicas and measure the spreading exponents:

$$D(t) \propto t^{\eta_d} \quad (1.54)$$

Similarly the survival probability of damage variables behaves as:

$$P_D(t) \propto t^{-\delta_d} \quad (1.55)$$

and similarly to Eq. (1.42) the average mean square spreading distance of damage variables from the center scales as:

$$R_D^2(t) \propto t^{z_d} . \quad (1.56)$$

Grassberger conjectured, that DS transitions should belong to DP class (see Sect. 4.2) unless they coincide with other transition points and provided the probability for a locally damaged state to become healed is not zero [Grassberger (1995c)]. This hypothesis has been confirmed by simulations of many different systems.

1.6 Field theoretical approach to reaction-diffusion systems

In this review I define nonequilibrium systems formally by their (bosonic) field theoretical action where it is possible. Therefore in this subsection I give a brief introduction to the field theoretical formalism. This will be through the simplest example of reaction-diffusion systems, via the: $A + A \rightarrow \emptyset$ annihilating random walk (ARW) (see Sect. 4.5.1). Similar stochastic differential equation formalism can also be set up for growth

processes in most cases. For a more complete introduction see [Cardy (1996, 1997); Täuber *et al.* (2005)].

A proper field theoretical treatment should start from the **Master equation** for the microscopic time evolution of probabilities $p(\alpha; t)$ of states α

$$\frac{dp(\alpha; t)}{dt} = \sum_{\beta} R_{\beta \rightarrow \alpha} p(\beta; t) - \sum_{\beta} R_{\alpha \rightarrow \beta} p(\alpha; t) , \quad (1.57)$$

where $R_{\alpha \rightarrow \beta}$ denotes the transition matrix from state α to β . In field theory this can be expressed in Fock space formalism with annihilation (a_i) and creation (c_i) operators satisfying the commutation relation [Doi (1976)]

$$[a_i, c_j] = \delta_{ij} , \quad [a_i, a_j] = [c_i, c_j] = 0 . \quad (1.58)$$

The states are built up from the vacuum $|0\rangle$ as the linear superposition

$$\Psi(t) = \sum_{\alpha} p(n_1, n_2, \dots; t) c_1^{n_1} c_2^{n_2} \dots |0\rangle , \quad (1.59)$$

with occupation number coefficients $p(n_1, n_2, \dots; t)$. The evolution of states can be described by a Schrödinger-like equation

$$\frac{d\Psi(t)}{dt} = -H\Psi(t) \quad (1.60)$$

with a generally non-hermitian (quasi-) Hamiltonian, which in case of the ARW process looks like

$$H = D \sum_{ij} (c_i - c_j)(a_i - a_j) - \lambda \sum_j (a_j^2 - c_j^2 a_j^2) , \quad (1.61)$$

here D denotes the diffusion strength and λ the annihilation rate. If we wanted to treat site restricted (sometimes called fermionic) models precisely — and in low dimensions particle exclusion turns out to be relevant in many cases (see Sect. 6.13) — we should use Pauli spin matrixes instead of bosonic operators. Such models are often integrable in one dimension [Alcaraz *et al.* (1994); Henkel *et al.* (1997); Schütz (2001)]. However fermionic field theories are much more difficult to solve. They involve non-commutative, Grassmann algebra and since near a transition to the absorbing state (which is very common in nonequilibrium models) the particle density is very low, they are expected to be equivalent to the bosonic theories.

By going to the continuum limit Eq.(1.61) turns into

$$H = \int d^d x [D(\nabla\psi)(\nabla\phi) - \lambda(\phi^2 - \psi^2\phi^2)] , \quad (1.62)$$

and in the path integral formalism [Peliti (1985)] one can introduce a partition function over fields $\phi(x, t)$ and “response fields” $\psi(x, t)$ with the statistical weight $e^{-S(\phi, \psi)}$. The “response field $\psi(x, t)$ [Martin *et al.* (1973)] is defined by the two-point response function

$$R(t, s) = \left. \frac{\delta \langle \phi(t, \vec{r}) \rangle}{\delta h(s, \vec{r})} \right|_{h=0} = \langle \phi(t, \vec{r}) \psi(s, \vec{r}) \rangle \quad (1.63)$$

where h is the ‘magnetic’ field conjugate to ϕ . For example the action $S(\phi, \psi)$ [Janssen (1976); de Dominics and Peliti (1978)], in case of ARW is

$$S = \int dt d^d x [\psi \partial_t \phi + D \nabla \psi \nabla \phi - \lambda(\phi^2 - \psi^2 \phi^2)] . \quad (1.64)$$

The action is usually analyzed by renormalization group (RG) methods at criticality [Ma (1976); Amit (1984); Zinn-Justin (2002); Täuber *et al.* (2005); Täuber (to be published)]. The most common way of RG calculations is done via perturbative epsilon expansion (PRG) below the upper critical dimension $d_c = \epsilon + d$, which is the lower limit of the validity of the mean-field (MF) behavior of the system. However in nonequilibrium statistical physics no RG calculation is available at and above three loop order, preventing to use re-summation techniques and to compute accurately universal quantities in low dimensions.

In the field theoretical formalism the symmetries of a model can be expressed in terms of the $\phi(x, t)$ field and $\psi(x, t)$ response field variables and the corresponding hyper-scaling relations can be derived [Muñoz *et al.* (1997); Janssen (2003)].

Features not accessible to PRG may also play a crucial role and then the so-called non-perturbative renormalization group (NPRG) (or exact/functional renormalization) appears as a method of choice [Wegner and Houghton (1972); Bagnuls and Bervillier (2001); Berges *et al.* (2002a); Delamotte and Canet (2005); Delamotte (2007)]. Modern functional RG approaches are based on Wilson’s idea [Wilson and Kogut (1974); Wegner and Houghton (1972)] to integrate out momentum modes within a path-integral representation of the theory. Without going deeply into the details of this method the main idea is to build a one-parameter family of models, indexed by a momentum scale k , interpolating smoothly between the short-distance physics at the (microscopic) scale $k = \Lambda$, where no fluctuation has been taken into account, and the long-distance physics at scale $k = 0$, where all fluctuations have been integrated out.

In the [Berges *et al.* (2002a)] version of the NPRG not the partition function; $Z_k[J, \bar{J}]$ itself (as suggested originally by [Wegner and Houghton

(1972)]), but its Legendre transform, the state function Γ_k (analogous to the Gibbs free energy at equilibrium)

$$\Gamma_k[\phi, \psi] + \log Z_k[J, \bar{J}] = \int J\phi + \int \bar{J}\psi - \int R_k\phi\psi \quad (1.65)$$

is followed under the infinitesimal change of the scale $s = \log(k/\Lambda)$. Here J denotes the external current and R_k describes the momentum cutoff applied. The exact functional differential equation governing the RG flow of Γ

$$\partial_s \Gamma_k = \frac{1}{2} \text{Tr} \int_{q,\omega} \partial_s \hat{R}_k \left(\hat{\Gamma}_k^{(2)} + \hat{R}_k \right)^{-1}, \quad (1.66)$$

is set up by [Berges *et al.* (2002a); Canet *et al.* (2004)], where \hat{R}_k is a symmetric, off-diagonal, 2×2 matrix of element R_k and $\hat{\Gamma}_k^{(2)}[\psi, \bar{\psi}]$ the 2×2 matrix of second derivatives of Γ_k with respect to ϕ and ψ . Usually the smooth cut-off function

$$R_k(q^2) = k^2(1 - q^2/k^2)\theta(1 - q^2/k^2) \quad (1.67)$$

is used, where $\theta(x)$ is the Heavyside step function. Since Eq. (1.66) cannot be solved exactly usually some truncation is applied. A standard truncation is the derivative expansion [Berges *et al.* (2002a)], in which Γ_k is expanded as a power series of ∇ and ∂_t . The local potential approximation (LPA) — which is the simplest such truncation — keeps only a potential term in Γ_k while neglects any field renormalization:

$$\Gamma_k^{\text{LPA}} = \int_{x,t} \{U_k(\phi, \psi) + \psi(\partial_t - D\nabla^2)\phi\}. \quad (1.68)$$

The NPRG equation for the effective potential in LPA, valid for all reaction-diffusion processes involving a single species, has been established in [Canet *et al.* (2004)]. Studying a particular model amounts to solve this equation in a sub-space defined by the symmetries of the problem, starting with the corresponding microscopic action S . The flow equation for the dimensionless potential $u = k^{d+2}U_k$, expressed in terms of the dimensionless fields $\phi \rightarrow k^{-d}\phi$ and ψ looks as

$$\partial_s u = -(d+2)u + d\phi u^{(1,0)} - V_d \left[1 - \frac{u^{(2,0)}u^{(0,2)}}{(1+u^{(1,1)})^2} \right]^{-\frac{1}{2}}, \quad (1.69)$$

where $u^{(n,p)} = \frac{\partial^{n+p} u}{\partial^n \phi \partial^p \psi}$ and $V_d = \frac{2^{-d+1}\pi^{-d/2}}{d\Gamma(d/2)}$. By inserting combinations of fields invariant under the symmetries of the model one can try to follow the flows of the couplings and locate fixed points corresponding to different

universality classes. These invariants and flows are related to those of the the topological phase spaces of the RD models discussed in Sect. 1.6.1.

By a Gaussian transformation of the action (1.64), using a suitable Gaussian distribution $P(\eta)$ one can integrate out the response field

$$e^{-\lambda\psi^2\phi^2} = \int e^{-\eta\psi} P(\eta) d\eta, \quad (1.70)$$

and we arrive to an alternative formalism: the **Langevin equation**, which in case of ARW is

$$\partial_t\phi(x, t) = D\nabla^2\phi(x, t) - 2\lambda\phi^2(x, t) + \eta(x, t) \quad (1.71)$$

with a Gaussian white noise, described by the correlations: $\langle \eta(x, t) \rangle = 0$,

$$\langle \eta(x, t)\eta(x', t') \rangle = -\Gamma\phi^2\delta^d(x - x')\delta(t - t'). \quad (1.72)$$

Here δ denotes the Dirac delta functional and Γ is the noise amplitude, so the noise disappears for zero density but is imaginary causing complex solution. Equation (1.71) is just the diffusive mean-field equation supplemented with a stochastic noise. In case of general reactions — if the Langevin equation can be deduced — the noise term can be a more complex function. Such equations are usually derived from the master equation (1.57) through some kind of coarse-graining and the exact form of the noise term is often unclear, especially when the dynamics does not constrain it via the detailed balance symmetry (2.20).

1.6.1 *Classification scheme of one-component, bosonic RD models, with short ranged interactions and memory*

The classification of nonequilibrium systems according to their critical exponents is not so straightforward as in case of equilibrium models. Especially the factors affecting a given universality class turned out to be more complex. There has been several attempt to classify universality classes of nonequilibrium phase transitions in the past decades. In Sect. 4.1 I show how the mean-field classes of one-component, bosonic reaction-diffusion systems of type $mA \rightarrow (m + k)A$, $nA \rightarrow (n - l)A$ can be obtained. Complete field theoretical description below d_c is still an open question in most cases.

Very recently a new strategy has been proposed [Elgart and Kamenev (2006)] for classifying bosonic one-component RD models exhibiting critical phase transition to absorbing states based on the topology of the phase portraits of the corresponding Hamiltonians.² These “Hamiltonians” are

²This treatment is restricted to homogeneous, isotropic, models with short ranged interactions and memory. Furthermore in low dimensional fermionic systems topological constraints can also be relevant (see for example Sect. 6.2).

derived from the master equation by introducing a quantum Hamiltonian for the generating function of the n -particle probabilities followed by a semi-classical limit of coordinates. Starting from a general action

$$S = \int dt d^d x [\psi \partial_t \phi + D \nabla \psi \nabla \phi + H_R(\phi, \psi)] , \quad (1.73)$$

where the first term describes the diffusion of particles and $H_R(\phi, \psi)$ is determined by the specific set of reactions one finds an analogy with the action (energy) of an equilibrium system in the following way. From $H_R(\phi, \psi)$ of Eq. (1.73) one can arrive to an effective non-Hermitian ‘‘Hamiltonian’’ $\hat{H}(p_i, q_i)$ operator by making a semi-classical (saddle point approximation) limit.

This operator depends on the canonical coordinates p_i and q_i derived from the generating function set up for n -particle occupancy probabilities $P_n(t)$ at a given site i

$$G(p_i, t) = \sum_{n=0}^{\infty} p_i^n P_n(t) . \quad (1.74)$$

Here the parameter p_i plays the role of a ‘‘canonical momentum’’, such that for $p_i = 1$

$$\rho(t) = \sum_n n P_n(t) = \partial_{p_i} G(p, t)|_{p_i=1} , \quad (1.75)$$

and a conjugated ‘‘coordinate’’ operator can be introduced as

$$q_i = \partial / \partial p_i . \quad (1.76)$$

With these parameters the bosonic Master equation at site i is

$$\partial_t G = \hat{H}_R(p_i, q_i) G . \quad (1.77)$$

and the Keldysh [Keldysh (1964)] ‘‘Hamiltonian’’ looks as

$$\hat{H}(p_i, q_i) = -D \sum_{\langle i, j \rangle} (p_i - p_j)(q_i - q_j) + \sum_i \hat{H}_R(p_i, q_i) \quad (1.78)$$

exhibiting a kinetic part with the diffusion constant D and a reaction part. The equations of motion, conserving the ‘‘energy’’ are

$$\partial_t q_i = \frac{\partial H_R}{\partial p_i} \quad (1.79)$$

$$\partial_t p_i = \frac{-\partial H_R}{\partial q_i} . \quad (1.80)$$

For example in case of ARW the reaction part looks as

$$\hat{H}_R = \lambda/2(1 - p^2)q^2, \quad (1.81)$$

which by $\phi \rightarrow q$, $\psi \rightarrow p$ transcription and a continuum limit looks just as Eq. (1.62). In case more general reactions of type



the non-Hermitian, normally ordered reaction Hamiltonian stays for

$$\hat{H}_R(p, q) = \frac{\lambda}{k!}(p^m - p^k) q^k \quad (1.83)$$

from which one can easily read-off the recipe of writing up such terms.

Due to the resemblance of $H_R(p, q)$ to the effective potential $V(q)$ in equilibrium statistical physics it is plausible that $H_R(p, q)$ encodes the information about possible nonequilibrium transitions as $V(q)$ does in equilibrium.³ In particular the the potential minima (given by $\partial_q V = 0$) might be analogous to the $H_R(p, q) = 0$ zero energy curves in the phase space. According to Elgart and Kamenev these curves and the corresponding topology of the phase portrait classify possible phase transitions of reaction-diffusion models. In other words the **web of these curves** play the role of minima of $V(q)$ of equilibrium statistical mechanics.

Considering now the Hamiltonian system of (1.78) from the normalization condition $G(1, t) = 1$ follows that

$$\hat{H}_R(p, q)|_{p=1} = 0, \quad (1.84)$$

furthermore any Hamiltonian describing a system with the empty absorbing state must satisfy

$$\hat{H}_R(p, q)|_{q=0} = 0. \quad (1.85)$$

Substituting $p = 1$ in (1.79),(1.80) one obtains the mean-field equation for $\rho(t)$, neglecting all fluctuation effects. Therefore the zero “energy” lines of the phase space of this Hamiltonian are the $p = 1$ (corresponding to mean-field) and the $q = 0$ (corresponding to the absorbing phase). These two lines make up the separatrix, i.e. they divide the entire phase space into isolated sectors. All other trajectories described by (1.79),(1.80) cannot intersect them and are confined to one of the sectors (see Fig. 1.3).

By varying the control parameters of a model the corresponding phase-space trajectories also change and a phase transition is associated to the change of the topology of the that portrait (see example Fig. 1.4).

³However the diffusion has been found to be a relevant in many, mainly fermionic models. See for example [Jensen and Dickman (1993a); Ódor (2000, 2004b,c)] Sect. 4.7.

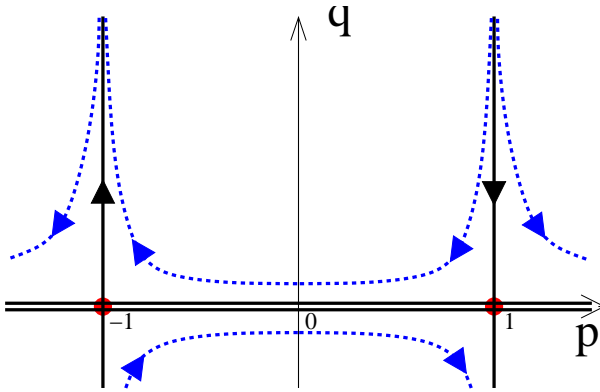


Fig. 1.3 Phase portrait of the ARW system ($2A \xrightarrow{\lambda} \emptyset$). The corresponding classical Hamiltonian is given by $H_R(p, q) = \frac{\lambda}{2}(1 - p^2)q^2$. Solid black lines show zero-energy trajectories: generic lines $p = 1$ and double degenerate $q = 0$ and the “accidental” line $p = -1$. Dashed colored curves indicate trajectories with nonzero energy. The arrows show the evolution direction (from [Elgart and Kamenev (2006)]).

However near to the transition point the saddle point approximation may lose its validity. Using RG one integrates out the small scale fluctuations, but the coupling constants and the functional form of $H_R(p, q)$ changes. Still around the transition the topology may be fully encoded in a relatively simple polynomial, which in turn provides a full characterization of the transition (at least for small $\epsilon = d_c - d$)⁴ Hence considering distinct topologies, stable upon RG transformations one may classify the possible transitions.

This will be a general guideline to be followed in Chapter 4. of this book. Using (1.83) one can easily set up $\hat{H}_R(p, \hat{q})$ for a given family of RD models and find the generic phase space topology corresponding to it. The validity of this approach will be discussed by comparing the predictions of the phase portrait method to numerical results of other approaches.

1.6.2 Ageing and local scale invariance (LSI)

In equilibrium models conformal invariance (CI) has been proven to be a very powerful tool for classifying universality classes [Friedan *et al.* (1984); Cardy (1987)]. This occurs in certain two dimensional critical systems as

⁴We will show later that these polynomials formally look the same as invariants in the effective potential of NPRG [Canet *et al.* (2005)], however there the trajectories of the RG evolve in the momentum space.

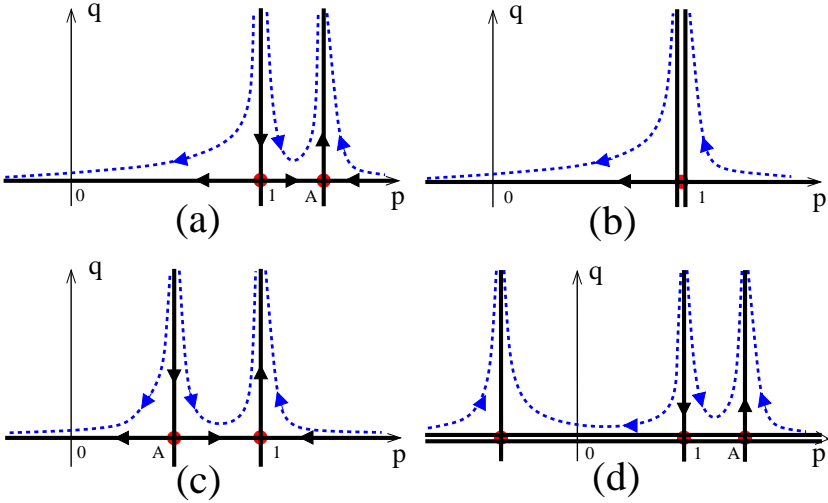


Fig. 1.4 Phase portraits of the system, $kA \xrightarrow{\lambda} (k-l)A, kA \xrightarrow{\mu} (k+m)A$ exhibiting the first-order phase transition. Thick solid lines represent the zero-energy trajectories. (a)–(c) Reaction set with $k = l = m = 1$: (a) absorbing phase, (b) transition point, and (c) unlimited proliferation phase. (d) Reaction set with $k = l = 2$ and $m = 1$ in the extinction phase. (from [Elgart and Kamenev (2006)])

the consequence of a larger group (the CI group) than the mere scale transformations [Belavin *et al.* (1984)]. In principle conformal transformation is a scale transformation: $\mathbf{r} \rightarrow b(\mathbf{r})\mathbf{r}$, with a space dependent rescaling factor, such that angles are kept fixed. In two dimensions CI allows to derive the possible set of critical exponents from the representation theory of the conformal (Virasoro) algebra. Conformal field theory provides us with a simple means of calculating critical exponents as well as the n -point correlation functions of the theory at the critical point [Henkel (1999)].

Recently the generalization of the generators of CI (albeit without invariance under time-translations) are proposed for anisotropic, dynamical models [Henkel (2002); Henkel and Pleimling (2005)]. The corresponding invariance is the so-called local scale-invariance (**LSI**). Since it is supposed to be the extension of the dynamical scale transformations for such systems it may serve as a convenient tool for classifying universality classes of nonequilibrium systems as well.

The quantities of main interest are the two-time autocorrelation function

$$C(t, s) = \langle \phi(t, \vec{r}) \phi(s, \vec{r}) \rangle \quad (1.86)$$

and the auto-response function $R(t, s)$ (1.63), which describe ageing phenomena (for recent reviews see [Bouchaud and Bray (2004); Bray (1994)]). For $t, s \rightarrow \infty$ and $y = t/s > 1$ one expects the scaling forms

$$C(t, s) = s^{-b} f_C(t/s) \quad (1.87)$$

$$R(t, s) = s^{-1-a} f_R(t/s), \quad (1.88)$$

where a and b are ageing exponents and f_C and f_R are scaling functions such that $f_{C,R}(y) \sim y^{-\lambda_{C,R}/Z}$ for $y \gg 1$. Here λ_C and λ_R are the auto-correlation [Fisher and Huse (1988)] and auto-response [Picone and Henkel (2002)] exponents respectively and independent of equilibrium exponents and the dynamical exponent Z (defined as usual $Z = \nu_{\parallel}/\nu_{\perp}$).

As in case of CI one expects that **LSI fully determines the functional form of the scaling functions**. Henkel *et al.* derived $R(t, s)$ in general and the form of $C(t, s)$ for $Z = 2$ by identifying the quasi-primary operators of the theory [Henkel *et al.* (2001, 2006)]. The generalized form of $R(t, s)$ takes into account the difference between physical observable defined in lattice models and the associated quasi-primary scaling operators of the underlying field theory as well. This ansatz, derived for $m_0 = 0$ **disordered** initial order parameter density looks as

$$R(t, s) = s^{-1-a} \left(\frac{t}{s}\right)^{1+a'-\lambda_R/Z} \left(\frac{t}{s} - 1\right)^{-1-a'}, \quad (1.89)$$

where $a' \neq a$ is an independent ageing exponent in general. Some systems with detailed balance symmetry (2.20) has been analyzed recently and found to satisfy (1.89) [Henkel and Pleimling (2005); Berthier *et al.* (1999); Mazenko (2004); Godrèche and Luck (2000a); Lippiello and Zannetti (2000); Henkel and Schütz (2004); Mayer *et al.* (2006); Pleimling and Gambassi (2005)] with $a \neq a'$.

On the other hand renormalization-group results for some important universality classes concluded that $a = a'$ should be hold. In particular explicit two-loop field-theoretical computation of $R(t, s)$ for the $O(N)$ universality class and Model A dynamics at the critical point claim $a = a'$ [Calabrese and Gambassi (2002, 2005)] (see Sect. 2.3.2).

Very recently more nontrivial models with $Z \neq 2$ have been found exhibiting LSI. These involve the exactly solved spherical model with $Z = 4$ [Baumann and Henkel (2007)], the nonequilibrium kinetic Ising model in one dimension [Ódor (2006a)] and the phase-ordering kinetics of two dimensional disordered Ising models [Baumann *et al.* (2007)]. For a more detailed discussion of these results see a very recent review [Henkel (2007)]. On the

other hand novel series of numerical studies and argumentation claim [Hinrichsen (2007b,c); Corberi *et al.* (2007)] that the observed accurate fitting with the LSI functional form (1.89) for models with $Z \neq 2$ cannot be considered exact in general, which would mean the existence of a **generalized Galilei-invariance** for these models.

1.7 The effect of disorder

A question of theoretical and experimental interest is whether and how the critical behavior is changed by introducing a small amount of uncorrelated impurity leading to models with quenched disorder. Realistic systems may contain a certain amount of quenched disorder. This disorder can take the form of vacancies or impurity atoms in a crystal lattice, or it can consist of extended defects such as dislocations or grain boundaries. The static critical behavior of equilibrium systems with bond or site disorder is well understood thank to the **Harris criterion** [Harris (1974a)]. It states that the addition of impurities to a system which undergoes a second-order phase transition does not change the critical behavior if

$$\alpha_H < 0 . \quad (1.90)$$

If α_H is positive, the transition is altered. Using the hyper-scaling relation (1.7) this means that the **disorder is irrelevant** for

$$\nu_{\perp} > 2/d . \quad (1.91)$$

This criterion can be derived heuristically by the following scaling arguments. Let's consider the effective local critical temperature in blocks of linear size ξ , which is given by an average of $T + \delta T(\mathbf{x})$ over the volume $V = \xi^d$. A sharp phase transition can only occur if the variation ΔT of these local critical temperatures from block to block is smaller than the global distance from the critical point T . For short-range correlated disorder, the central limit theorem yields $\Delta T \propto \xi^{-d/2} \propto T^{d\nu_{\perp}/2}$. Thus, a clean critical point is perturbatively stable, if the clean critical exponents fulfill the inequality $T^{d\nu_{\perp}/2} < T$ for $T \rightarrow 0$. This implies the exponent inequality (1.91). Critical systems have been divided into three categories according to the response of a pure system for quenched disorder

- Irrelevant disorder : The critical behavior is unchanged, the macroscopic variables are self-averaging (the relative width of their probability distributions vanishes as $L \rightarrow \infty$).

- Relevant (marginal) disorder: New critical behavior fixed point (line) with conventional power-law scaling. Lack of self-averaging (the width of their probability distributions approaches a size dependent constant as $L \rightarrow \infty$).
- Relevant disorder with infinite randomness critical point: Activated scaling behavior. The relative width of their probability distributions diverges with the system size.

The Harris criterion is a *necessary but not sufficient condition for the stability of an impure fixed point*. It only deals with the average behavior of the disorder at large length scales. However, effects due to qualitatively new physics at finite length scales (and finite disorder strength) are not described by the Harris criterion. Nonequilibrium models with quenched spatial disorder were considered first by Kinzel [Kinzel (1985)].

The Harris criterion has been rigorously established for a large class of disordered systems [Chayes *et al.* (1986)]. In nonequilibrium systems the situation is less clear and violations of the Harris criterion are reported (see Sects. 4.2.10, 4.5.6 and 4.6.8). In nonequilibrium phase transitions **rare regions in with large disorder fluctuations** may occur, which can exert much stronger effects, ranging from from strong power-law singularities in the free energy to a complete destruction of the phase transition (for recent reviews see [Iglói and Monthus (2005); Vojta (2006)]). In classical equilibrium systems, they lead to the so-called Griffiths singularity [Griffiths (1969)], of the free energy in the vicinity of the phase transition.

Rare regions occur for example in diluted systems, (see Sect. 4.5.6), which reduces T_c from its clean bulk value T_c^0 . In an infinite system one can find arbitrarily large spatial regions without impurities. For temperatures between T_c^0 and T_c , these regions show local magnetic order, even though the bulk system is globally in the paramagnetic phase. The probability p_r for finding a rare region is exponentially small in its volume V_r and in the impurity concentration p . Up to a pre-exponential factors it is given by

$$p_r \propto e^{-pV_r}. \quad (1.92)$$

Rare regions are thus non-perturbative degrees of freedom that are not accounted for in conventional approaches to phase transitions and critical points that are based on perturbation theory and on PRG. The importance of large rare regions depends on how rapidly the contribution of a single region to observable quantities increases with its size. This is controlled by the relation between d_r (the effective dimensionality of the region) and the

lower critical dimension d_c^- of the phase transition.⁵ One can distinguish three cases:

(a) If $d_r < d_c^-$ an isolated rare region cannot undergo a phase transition, its contribution can at most grow as a power of its linear size, which can't overcome the exponential drop in the rare region density. This causes a weak effect and the critical behavior is of conventional power-law type (see for example the classical, randomly diluted Ising model (Sect. 2.3.7), where $d_r = 0$ and $d_c^- = 1$).

(b) If $d_r = d_c^-$ the rare regions still cannot undergo phase transition independently, but their contribution increases exponentially with size and can overcome the exponential decrease of the rare region density. This results in Griffiths singularities with non-universal, continuously varying exponents as follows.

Inside the Griffiths region ($T_c < T < T_c^0$) the long-time behavior is dominated by these regions and any finite region decays exponentially

$$\rho(t) \propto \int dL_r L_r^d e^{-t/t_R} p_r, \quad (1.93)$$

where t_R is the characteristic decay time of a region of size L_r . The average lifetime of such active regions grows as

$$t_R \propto e^{aV_r} \quad (1.94)$$

because a coordinated fluctuation of the entire region is required to flip it (the constant a vanishes at the clean critical point and increases with decreasing T (see, e.g., [Goldenfeld (1992)]). The saddle point analysis of (1.93) results in power-law decay

$$\rho(t) \propto t^{-p_r/a}, \quad (1.95)$$

hence one gets continuously changing decay exponents (see for example the NEKIM model in 1 + 1 dimension (Sect. 4.6.8)).

At the clean critical temperature $T = T_c^0$ the t_R is determined by finite-size scaling, which diverges as

$$t_R \sim L_r^Z. \quad (1.96)$$

By inserting it to Eq. (1.93) the saddle-point method results in stretched exponential decay of the density.

$$\ln \rho(t) \sim -\tilde{p}^{Z/(d+Z)} t^{d/(d+Z)}, \quad (1.97)$$

where $\tilde{p} = -\ln(1-p)$. The scaling behavior at the **dirty critical point** $T = T_c$ is dominated by the rare regions, resulting in **'activated' scaling**,

⁵where the fluctuations are so strong that they completely destroy the ordered state.

characterized by the ‘**infinite randomness critical**’ point. Here the dynamics is extremely slow. The power-law scaling (1.16) gets replaced by activated dynamical scaling

$$\ln(t_R) \propto \xi^\psi, \quad (1.98)$$

characterized by a new exponent ψ . This exponential relation between time and length scales implies that the dynamical exponent Z is formally infinite.⁶ The activated scaling leads to logarithmic time dependencies at the critical point. For example the density decays as

$$\rho(t) \propto [\ln(t)]^{-\bar{\alpha}}, \quad (1.99)$$

while the average number of particles of a cluster started from a single seed site increases like

$$N(t) \propto [\ln(t)]^{\bar{\eta}} \quad (1.100)$$

with $\bar{\alpha} = \beta/(\nu_\perp \psi)$ and $\bar{\eta} = d/\psi - 2\beta/(\nu_\perp \psi)$ (see for example the directed percolation in $1 + 1$ dimension (Sect. 4.2.10)).

(c) If $d_r > d_c^-$ the rare regions can undergo the phase transition, the dynamics of the locally ordered regions completely freezes, and they develop a truly static order parameter. As a result, the global phase transition is destroyed by smearing [Vojta (2003)] because different spatial parts of the system order at different values of the control parameter.

⁶In contrast, the static scaling behavior remains of power law type.