Chapter 1

Mathematical Programming and its Applications in Finance

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Abstract

This article reviews some of the applications of mathematical programming in finance. Of course mathematical programming has long been recognised as a vital modelling approach to solve optimization problems in finance. Markowitz's Nobel Prize winning work on portfolio optimization showed how important a technique it is. Other prominent and well documented applications in long-term financial planning and portfolio problems include asset-liability management for pension plans and insurance companies, integrated risk management for intermediaries, and long-term planning for individuals. Nowadays there is an emphasis on the interaction between optimization and simulation techniques in these problems

There are though many uses of mathematical programming in finance which are not purely about optimizing the return on a portfolio and we will also discuss these applications. For example we discuss how one can use linear programming to estimate the term structure of interest rates for the prices of bonds. In the personal sector finance, where the lending is far greater than the higher profile corporate sector, the use of linear programming as a way of developing credit scorecards is proving extremely valuable.

Key Words: Mathematical programming, optimization problems in finance, portfolio optimization, credit scorecards, linear programming, asset-liability Models

MATHEMATICAL PROGRAMMING AND GAME THEORY FOR DECISION MAKING

1.1 Introduction

Mathematical programming was one of the key tools used in the earliest modelling in finance, namely Markowitz Nobel prize winning work [Markowitz (1952), (1959)] on optimising portfolios of shares or other financial instruments. This lead to a quadratic programming formulation, which has subsequently been extended in many ways. A related problem but from a different area of finance is the asset- liability problem faced by many insurance companies. Despite a long tradition of statistical and actuarial models of the liabilities involved in insurance and variants of portfolio optimisation problems to determine how to hold the assets, it is only in the last decade that these two complementary sides to an insurance company have been put together in one model. This leads to very large scale stochastic programming problems. These are the high profile applications of mathematical programming in finance and continue to be heavily researched not least because of the size of the programmes needed to solve real applications.

There continue though to be new applications of linear programming which are perhaps less well known but equally important to those specific areas of finance. One of these is the way of calculating the yield curve - the markets forecast of what the future of interest rates will be, which are implicit in the prices of bonds.

Another example occurs in consumer credit risk. This area of finance does not receive any of the research attention that corporate lending, equity models and the pricing of equity derivatives has received in the last twenty years. Yet the lending to consumers in most developed countries is much higher than the lending to companies (30% more in the US than the total of business lending). The tool used to assess the risk of lending to customers is to develop a credit scorecard and linear programming has some real advantages in developing such scorecards.

So in this review we briefly outline these four applications and the types of mathematical programming models that can solve them

1.2 Portfolio Optimization

The literature on financial optimization models dates back to the ground breaking application of Markowitz on optimizing a portfolio of financial products by concentrating on the mean return and taking the variance of the return as a measure of the risk. In this basic model one is interested in investing in a single period, there is an initial portfolio available and one of the assets is risk free, i.e. cash.

Assume there are N traded assets labelled $i, i = 1, 2, ..N$ and R_i is the random variable of the return in that one period on asset i, where $Exp(R_i) = r_i$ and the covariance of the returns on the assets is given by the matrix Σ. Assume that w_i^0 is the initial holding of asset i where $w_i^0 \geq 0$. Let x_i be the amount of asset i traded, where positive values means more of the asset is purchased and negative values means some of the asset is sold. The returns and risk (variance) of the resulting portfolio is

$$E[R^T(w^0 + x)] = r^T(w^0 + x); V[R^T(w^0 + x)] = (w^0 + x)^T\Sigma(w^0 + x)$$

Hence if one wants a portfolio with at least an expected return of t but with minimum variance one needs to solve the quadratic programming problem

$$\begin{aligned} \text{Minimise} \quad & (w^0 + x)^T\Sigma(w^0 + x) \\ \text{subject to} \quad & r^T(w^0 + x) \geq t \\ & (1, 1, 1, \ldots)^T x = 0 \end{aligned}$$

where the first constraint ensures the return is at least t (in reality it will always be exactly t) and the second constraint means the trading is self financing. The problem is then solved for different values of t to get a risk return trade-off and the investor chooses the outcome where his utility as a function of risk and return is maximised.

Although this model was fundamental in understanding the portfolio investment problem, it is of limited use in practice because it does not model all the aspects of the real situation. Some of these - limits on short selling, and the need for diversification - can be dealt with by adding appropriate constraints. Short selling is when one sells an asset at the start of the period, which one does not own. At the end of the period the asset has to be bought and passed on to the original buyer. As it stands there is no limit on how much of this can de done but one could put a limit on how much of this can be done by introducing the constraint

$$w_i^0 + x_i \geq -s_i.$$

One can also limit the amounts invested in an asset in three different ways as follows

Limit the amount invested in each asset

$w_i^0 + x_i \leq b_i$

Limit the relative amount invested in each asset

$w_i^0 + x_i \leq \gamma_i \sum_j (w_j^0 + x_j)$

Limit the relative amount invested in a group of assets J

$\sum_{j \in J} (w_j^0 + x_j) \leq \alpha \sum_j (w_j^0 + x_j)$

The original model also ignored the transaction costs involved in trading. If these can be considered to be piece wise linear in the level of the transaction then they can be added to the return constraint without losing linearity. Alternatively as [Mulvey (1993)] suggests one can mimic transaction costs by putting upper limits on the transaction for classes which have high such costs.

There are two real drawbacks to this formulation. The first is that variance is not always the way investors want to measure the risk. In particular it penalises returns which are well above the mean in the same way as those that are below the mean. So other risk measures have been suggested. [Mansini, Ogryczak and Speranza (2003)] reviewed the different measures that could still lead to linear programming formulations. Following [Sharpe (1973)] there have been a number of attempts to linearize the portfolio optimization problem. However if a portfolio is to take advantage of diversification then no risk measure can just be a linear function of the x. The way around it has been to assume there are a number of different scenarios $S_k - k = 1, \ldots, K$ with specific values of the return for each asset in each scenario and a probability p_k of that scenario occurring. In this way one can model other risk measures such as the mean semideviation, which looks at the expected shortfall below the mean value, i.e. for any trading policy $\mathbf{x}$., if $r(\mathbf{x})$ is the mean return and $R(k, \mathbf{x})$ is the actual return under scenario k, the mean semideviation is $sd(x) = E_k[\max\{r(\mathbf{x}) - R(k, \mathbf{x}), 0\}]$.

One can translate that into a convex piecewise linear function of the variables $\mathbf{x}$ by defining the following optimisation problem

$$sd(\mathbf{x}) = \min \sum_{k=1}^{k} d_k p_k$$

$$\text{subject to } d_k \geq r(\mathbf{x}) - R(k, \mathbf{x}), \;\; d_k \geq 0$$

In a similar way one can use other shortfall and stochastic dominance measures of risk such as mean below target deviation, minimise the maximum semideviation and the Gini mean difference which corresponds to the mean worst return. However for other risk measures this may not be possible.

The second issue that brings the standard models into question is that it is a one period model which may be inappropriate for investment problems with long time horizons. This would lead one to using stochastic linear programming models and their application in finance is surveyed in the paper by [Yu, Ji and Wang (2003)]. Thus we would need to extend the models to stochastic programming ones. Instead of doing this, we will consider in the next section the finance problem which has been most modelled as a stochastic programme in the last decade, namely the asset liability problem.

1.3 Asset-liability Models

Asset-Liability management looks at the problem of how to construct a portfolio of securities that will cover the cost of a set of liabilities, which are themselves varying as they depend on external economic conditions. This is exactly the problem that insurance companies have to face. For over a century they have had models which allow them to assess the costs of their liabilities. Fifty years ago the advent of the models in the previous section allowed them to optimise their portfolio of assets. Thus it is surprising that it is only in the last decade or so that they have sought to combine the two sides of their business into one model.

The time scales (many years) involved in such asset-liability problems and the need to allow for the possibility of rebalancing the portfolio at future times in response to new information means one is driven to model these problems as stochastic programming ones. We outline a formulation related to that suggested in [Bradley and Crane (1972)], [Klaassen (1998)] and in the review of [Sodhi (2005)], though other models have been used with considerable success by a number of U.S. and European insurance companies.

One of the problems in these models is what to do about the requirements at the end of the time horizon. It is reasonable to assume that the company will wish to continue trading thereafter but if one wants to minimise the initial cost of the asset portfolio one needs to cover the liabilities one is drawn to trying to make this residue as close to zero as possible. Alternatively one assumes the initial position is given and tries to maximise the surplus at the end of the time horizon provided all the liabilities have been met. That is the approach we will take here

Consider a T time horizon. Let $s(t)$ be a scenario which ends at time t and gives rise to two new scenarios $s^{(t+1)}$ with equal probability, which share the same history as $s(t)$ until time t. Thus the probabilities of all scenarios at time t are $2^{-(t-1)}$.

The variables and constraints in the resultant stochastic programme are as follows

Decision variables:

$x_{i,s(t)}$ - amount of asset i bought in period t in scenario $s(t)$
$y_{i,s(t)}$ - amount of asset i sold in period in scenario $s(t)$
$l_{s(t)}$ - amount lent at current short rate in period t in scenario $s(t)$
$b_{s(t)}$ - amount borrowed at current short rate in period t in scenario $s(t)$
$x_{i,s(t)}$ - amount of asset i bought in period t in scenario $s(t)$

Costs and Profits:

$c_{i,s(t)}$ - cash flow (dividents etc) from asset i in period t in scenario $s(t)$
$p_{i,s(t)}$ - price (ex-divident) from asset i in period t in scenario $s(t)$
$\nu_{s(t)}$ - present value of a cash flow of 1 in period t in scenario $s(t)$
$\eta_{s(t)}$ - one period interest rate in period t in scenario $s(t)$
$L_{s(t)}$ - liability due in period t in scenario $s(t)$
α_i - transaction cost as proportion of value of trade in asset i.
$h_{i,0}, l_0, b_0$ are initial asset holdings, lendings and borrowings

Model:

$$\text{Maximise } \; 2^{-(T-1)} \sum_{s(T)} \nu_{s(T)} \left(\sum_i p_{i,s(T)} h_{i,s(T)} + l_{sT)} - b_{s(T)} \right)$$

$$\sum_i c_{i,s(t)} h_{i,s(t-1)} + l_{s(t-1)}(1+\eta_{s(t-1)}) + b_{s(t)} + \sum_i (1-\alpha_i) p_{i,s(t)} y_{i,s(t)}$$

$$-\sum_i (1+\alpha_i) p_{i,s(t)} x_{i,s(t)} - l_{s(t)} - b_{s(t-1)}(1+\gamma_{s(t-1)}) = L_{s(t)} \;\; \forall s(t), t = 1, 2, \ldots, T$$

$$h_{i,s(t)} = h_{i,s(t-1)} + x_{i,s(t)} - y_{i,s(t)} \;\; \forall i, s(t), t = 1, 2, \ldots, T$$

$$h_{i,s(t)}, x_{i,s(t)}, y_{i,s(t)}, l_{s(t)}, b_{s(t)} \forall i, s(t), t = 1, 2, \ldots, T$$

One can see from this formulation how critical is the choice of scenarios to represent the uncertainties throughout the whole period of the problem. The scenarios describe the asset prices and also the term structures for the interest rates. Thus scenario generation becomes crucial to building useful models. There are three approaches that are commonly used to do this -i) bootstrapping using historical data ii) modelling the economy and asset returns with vector autoregressive models and iii) using simulations based on multivariate normal distributions of the values at risk from different classes, where the parameters in the normal distribution are obtained using time series analysis.

The other real difficulty is that the size of the scenario tree can make computation almost impossible. This has stimulated even further the work in stochastic programming on how to solve approximately such large problems. Obviously one way is not to have too many stages and so amalgamate together many of the periods towards the end of the time horizon into much larger time periods. However the real advantages come from using aggregation to combine nodes of the tree where appropriate and/or using decomposition approaches such as Benders decomposition, and the more recent interior point methods which can exploit the problem structure. These together with parallel processing of the computation and using object oriented parallel solvers mean that one can solve problems with 1,000,000,000 decision variables [Gondzio and Grothey (2006)].

1.4 Yield Curves

In financial markets the price of bonds can be used to estimate what interest rates will do in the future. This is because bond pricing models really model the current term structure of interest spot rates using both risk free (Treasury) and risky (corporate) securities. The spot rate can be extracted from the prices of zero coupon bonds which would repay only on maturity. However there are very few zero coupon bonds in the market and it is thus necessary to extract the spot interest rates from bonds, both Treasury and corporate, which pay coupon payments throughout their duration as well as making a final repayments.

The standard methods of stripping coupons from bonds are bootstrapping [Fabozzi (1998)] or linear regression [Carleton and Cooper (1976)]. If for each period there is one and only one coupon bond that matures, these techniques generate a unique set of spot interest rates over the period. However if there are periods where no bonds mature or other periods when several bonds mature at the same tine, then there is not a unique solution to the spot rates and in some cases these approaches giver rise to rates with unacceptable features. For example the rates might suggest that receiving one unit later in time is worth more than receiving it earlier in time, which would imply there were negative interest rates between the two times. One could also get results where the price for a high risk zero-coupon bond is higher than for a lower risk zero coupon bond maturing at the same time, which defies logical explanation.

To remedy the mispricing caused by bootstrapping, [Allen, Thomas and Zheng (2000)] suggested using linear programming to strip out the coupons of risk-free and risky bonds in such a way that there are no such difficulties. This approach will produce the same spot interest rates as the bootstrapping technique if there is one and only one coupon bond maturing in each time period.

Suppose there are only risk free bonds, labelled $i, i = 1, \ldots, N_0$ in the market, and bond i has a current price of P_i and $c_i(t)$ is its cash flow at time t. Then one can estimate the pure discounted bond prices $v_0(t)$ of risk free zero-coupon bonds paying 1 at a set of agreed times $t = 0, 1, \ldots, T$ by solving the following linear programming problem

$$\text{Minimize} \sum_{i=1}^{N_0} (a_i + b_i)$$

$$\text{subject to } P_i + a_i = \sum_{i=1}^{T} c_i(t) v_0(t) + b_i$$

$$v_0(t) \geq (1 + m(t)) v_0(t+1)$$

$$a_i, b_i \geq 0$$

$$\text{for } i = 1, \ldots N_0; \text{ and } t = 0, 1, \ldots, T-1$$

where $m(t)$ is the minimum expected forward interest from t to $t + 1$.

The first constraint seeks to match the present value P_i to the discounted cash flows $c_i(t)$ and a_i and b_i are the mispricing errors. a_i is positive and $b_i = 0$ if the price is "too low" and the other way around if the price is "too high". The second constraint ensures there is no mispricing with respect to maturity (if $m(t) = 0$, one has the constraint that bonds of longer maturity should be priced at or below those with shorter maturity).

If one has calculated the values $v_0(t)$ for $t = 0, 1, 2, \ldots T$ one can transform these values into the spot interest rates $i(0, t)$ over the same period by taking

$$v_0(t) = \frac{1}{(1 + i(0,t))^t}$$

One can also use the price of risky bonds not only to determine the term structures of interest rates when applied to bonds of that risk class but also to help determine the term structure of interest rates of all classes including the risk free ones. Suppose bonds are rated according to their riskiness with 1 being the highest quality and M the lowest quality, with 0 remaining the grade ascribed to risk free bonds. Suppose there are N bonds observable in the market . Bond i has current price P_i, maturity date T_i, cashflow $c_i(t)$ for $t = 1, 2, \ldots, T_i$ and credit rating $d(i)$. Suppose for the class of bonds with credit rating $j, j = 0, 1, 2, \ldots, M$ the price of a bond stripped of its coupon paying 1 at t is $v_j(t)$ for $t = 1, 2, \ldots, T$ then we can calculate the best fit for these values from the bond prices given by solving the following Linear Programme

$$\begin{aligned}
&\text{Minimize} \sum_{i=1}^{N} (a_i + b_i) \\
&\text{subject to } P_i + a_i = \sum_{t=1}^{T} c_i(t) v_{d(i)}(t) + b_i \\
&v_0(t) \geq (1 + m(t)) v_0(t+1) \\
&v_j(t+1) - v_{j+1}(t+1) \geq v_j(t+1) - v_{j+1}(t+1) \\
&a_i, b_i \geq 0 \\
&\text{for } i = 1, \ldots N; j = 0, 1, \ldots, M-1, \text{ and } t = 0, 1, \ldots, T-1
\end{aligned}$$

$m(t)$ is again the minimum expected risk free forward interest rate from t to $t+1$ at time $t = 0$. The third constraint guarantees both that the price of a longer maturity bond is cheaper than that of a shorter maturity bond and that the price of a less risky zero coupon bond is higher than that of a riskier rated one of the same maturity.

Finally note that one could introduce the liquidity of the market into the optimization of the bond price and hence the term structure by recognizing that the issue amounts of different bonds will be quite different. Bonds which have a large amount issued are likely to be more liquid and hence their prices more accurately reflect the market's view than those where far less in value was issued. What is important is the issue value of the bond and if this is w_i for bond i, one can reflect the relative likelihood of the bonds being accurately prices by changing the objective function is the Linear programme above to

$$\text{Minimize} \sum_{i=1}^{N} w_i(a_i + b_i)$$

Whether we use liquidity or not, these linear programmes allow one to calculate the spot price interest rates $i(j,t)$ $j = 0, 1, ..M, t =, 1, T$ for risk free and risky bonds using

$$v_j(t) = \frac{1}{(1 + i(j,t))^t}.$$

1.5 Credit Scorecards

Most financial mathematics courses and text books concentrate exclusively on interest rate models, equities, bonds, their derivatives and corporate lending. However in most first world countries lending to consumers far exceeds lending to companies and yet that area of finance is hardly ever mentioned. Yet at the start of the twenty first century consumer credit is the driving force behind the economies of most of the leading industrial countries. Without it, the phenomenal growth in home ownership and consumer spending of the last fifty tears would not have occurred.

In 2004 the total debt owed by consumers in the US was $10.3 trillion ($10,300,000,000,000) of which $7.5 was on mortgages and $2.2 trillion on consumer credit (personal bank loans, credit cards, overdrafts, motor and retail loans). This is now 30% more than the $7.8 trillion owed by all US industry and almost double the $5.5 trillion of corporate borrowing (the rest being borrowing by small and medium sized companies and agricultural organisations). Figure 1 shows the growth in this borrowing since the 1960s and emphasises how consumer credit has been growing faster than corporate borrowing for most of that period.

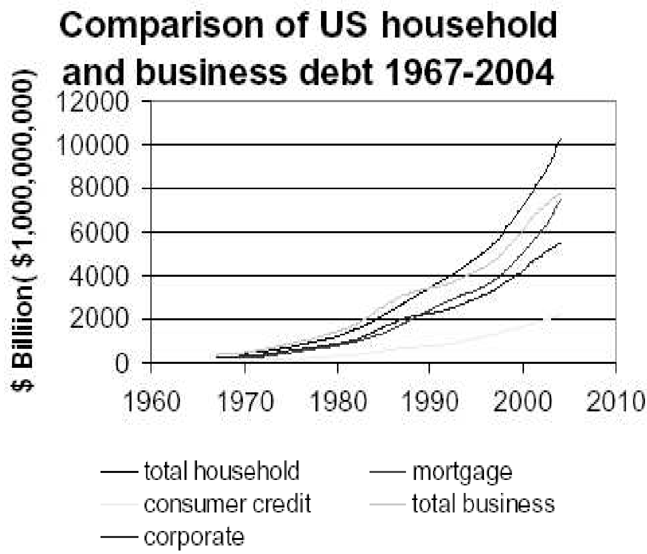

Figure 1: US household and business debt

This growth would not have been possible without credit scoring, the development of automatic risk assessment systems which assess the probability that new and existing customers will default on their loans within a fixed future time period (usually 12 months). The approach to building such credit scorecards is essentially one of classification. A sample of previous customers is taken and each classified as a defaulter or a non-defaulter according to their subsequent performance. The idea then is to identify which combination of application and/or performance attributes of consumers best separate the two groups. This idea of such statistical classification began with [Fisher (1936)] work on discriminant analysis which can be reinterpreted by saying that it is essentially a regression which tries to estimate p, the probability of non-default, as a linear function of the attributes of the consumer $x_1, x_2, \ldots, x_m$ by

$$p = w_0 + w_1 x_1 + w_2 x_2 + \ldots + w_m x_m$$

One of the succesful modifications used in credit scoring is that the x_i are not the orginal characteristics like age, but coarse classified variants of them. So one will split age into a number of age bands and the x_i are then either binary indicator variables of whether consumers are in that band, or weights of evidence transformations so each band is ranked according to the ratio of defaulters to non-defaulters in that band. This is one way of dealing with the non linearity of the relationship between default risk and age.

Of course the probability of non-defaulting in the above equation for those in the sample will be either 0 or 1. When one has used this sample to determine the regression equation one has an equation where the right hand side could take any value from $-\infty$ to $+\infty$ but the left hand side is a probability and so for any new applicants should only take values between 0 and 1. It would be better if the left hand side was a function of p which also could take a wider range of values. One such function is the log of the probability odds. This leads to the logistic regression approach where one matches the log of the probability odds by a linear combination of the consumer attributes, i.e.,

$$\log(p/(1-p)) = s = w_0 + w_1x_1 + w_2x_2 + \quad + w_mx_m$$

The right hand side of the equation is considered the credit score of the individual and ranks the consumers according to their chance of defaulting. One then chooses some cut-off score c, and give loans or credit cards to those applicants with scores above c and refuses it to those with scores below c. For existing customers, the scores are used to determine what changes in credit limit should be allowed and whether one should offer other products to that consumer.

Linear programming can also be used as a classification approach and also ends up with a linear scorecard. [Mangasarian (1965)] was the first to recognise that linear programming could be used for discrimination, but it was the papers by [Freed and Glover (1981a,b)] that sparked off the interest.

Suppose one has a sample of n_G goods and n_B bads and a set of m predictive variables from the application form answers so borrower i has predictive variable values $(x_{i1}, x_{i2}, \ldots, x_{im})$. One seeks to develop a linear scorecard where all the goods will have a value above the cut-off score c and all the bads have a score below the cut-off score. This cannot happen in all cases so we introduce variables a_i which allow for the possible errors - all of which are positive or zero. If we seek to find the weights $(w_1, w_2, \ldots, w_m)$ that minimise the sum of the absolute values of these errors we end up with the following linear programme

$$
\begin{aligned}
&\text{Minimise } a_1 + a_2 + \ldots + a_{n_G+n_B} \\
&\text{subject to} \\
&w_1 x_{i1} + w_2 x_{i2} + \ldots + w_m x_{im} \geq c - a_i,\ 1 \geq i \geq n_G \\
&w_1 x_{i1} + w_2 x_{i2} + \ldots + w_m x_{im} \leq c + a_i,\ n_G + 1 \leq i \leq n_G + n_B \\
&a_i \geq 0 \quad 1 \leq i \leq n_G + n_B
\end{aligned}
$$

In essence this approach is minimising the errors using the l_1 norm, while linear regression minimises the errors using the l_2 norm. One could also use linear programming to minimise the l_∞ norm, i.e., minimise the maximum error, by changing a_i to a in each constraint.

Linear programming is used by several organisations in building their scorecards because it allows one to build the best scorecard with any particular bias. For example, a lender might want to target consumers who are under 25 more than those who are over 25. In the linear programming formulation this can be easily done. For example if x_{25-} is the binary indicator variable that someone is under 25 and x_{25+} is the indicator variable for being over 25, then one requires that the corresponding weights satisfy. $w_{25-} > w_{25+}$. In this way one can construct the scorecard which best classifies the two groups but also has the required bias in it, something which is much harder to do in the standard regression approaches.

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